

Math 337 Winter 2010, Problem set 3  
Due Wed Feb. 24

It is recommended that you try all the problems. Hand in only those which are starred for grading. The hints which are given are just suggestions. You may find a different or better way of doing the problem.

- (1) \*Prove that a “closed” ball  $\{x : d(x, a) \leq r\}$  is a closed set. This justifies the terminology.
- (2) Prove theorem 9.2.4 directly from the sequential definition of compactness, namely that every sequence has a convergent subsequence.

9.2 C, G\*, J\*. Suggestion: for both G and J ignore the hints in the book and work directly from the sequential definition of compactness. Prove G by contradiction, so assume that  $f$  is not uniformly continuous. Show that then there are sequences  $(x_n)$  and  $(y_n)$  and an  $r > 0$  such that  $\rho(x_n, y_n) \rightarrow 0$  but  $d(f(x_n), f(y_n)) > r$ . Now use compactness to find a convergent subsequence of  $(x_n)$  with limit  $a$ . Now use the fact that  $f$  is continuous at  $a$ .

- (3) (a) \*Suppose that  $(x_n)$  and  $(y_n)$  are sequences in a metric space  $X$ ,  $x_n \rightarrow a$  and  $y_n \rightarrow b$ . Show that  $d(x_n, y_n) \rightarrow d(a, b)$ . Hint: show that  $|d(x_n, y_n) - d(a, b)|$  is small by adding and subtracting  $d(x_n, a)$ .
- (b) \*Suppose  $X$  and  $Y$  are closed subsets of  $\mathbb{R}^n$  and  $X$  is bounded. Define  $d(X, Y) = \inf\{d(x, y) : x \in X, y \in Y\}$ . Show that the inf is achieved, i.e. there are  $a \in X$  and  $b \in Y$  such that  $d(a, b) = d(X, Y)$ . Give an example to show that this can fail if neither of  $X$  nor  $Y$  is bounded. Hint: for the proof find  $x_n \in X, y_n \in Y$  such that  $d(x_n, y_n) \rightarrow d(X, Y)$  and then find a sequence  $n_1 < n_2 < \dots$  such that  $x_{n_k}$  converges. Then find a subsequence of  $(n_k)$  along which  $y_n$  also converges. Why is this possible?
- (4) Suppose  $(X, \rho)$  and  $(Y, \sigma)$  are metric spaces and define a metric  $d$  on the Cartesian product  $X \times Y$  by  $d((x_1, y_1), (x_2, y_2)) = \max(\rho(x_1, x_2), \sigma(y_1, y_2))$ . Prove that this is actually a metric and show that  $(x_n, y_n) \rightarrow (a, b)$  in  $X \times Y$  if and only if  $x_n \rightarrow a$  in  $X$  and  $y_n \rightarrow b$  in  $Y$ .
- (5) \*Suppose that  $(x_n)$  is a sequence in a metric space  $X$  such that  $\lim_n x_n = a$  exists. Show that  $\{x_n : n \in \mathbb{N}\} \cup \{a\}$  is a closed subset of  $X$ .
- (6) \*Given a metric space  $X$  and subset  $E \subset X$  define  $\text{int}(E)$  the **interior** of  $E$  to be the set of all  $x \in E$  such that there is an  $\epsilon > 0$  such that  $B_\epsilon(x) \subset E$ . Show that  $\text{int}(E)$  is an open set and that  $(\overline{E})^c = \text{int}(E^c)$ . ( $A^c$  denotes  $X \setminus A$ , the complement of  $A$  in  $X$  and  $\overline{A}$  denotes the closure of  $A$ .)