Math 337 Winter 2010, Problem set 2
Due Wed Feb. 9

It is recommended that you try all the problems. Hand in only those which are starred for grading. The hints which are given are just suggestions. You may find a different or better way of doing the problem.

§2.8 G*, I   In I also prove that the continued fraction converges by showing that it is Cauchy, as in example 2.8.8.

(1) Suppose that f is a contraction mapping on a closed interval $I = [a, b] \subset \mathbb{R}$, that is, a function $f : I \to I$ such that there is a constant $c \in (0, 1)$ with the property that $|f(x) - f(y)| \leq c|x - y|$. Show that f is continuous on I. Let x_0 be any point of I and define (x_n) recursively by $x_{n+1} = f(x_n)$. Show that the sequence x_n is Cauchy and that $a = \lim x_n$ belongs to I and is a fixed point of f (that is, $f(a) = a$). Show that f has only one fixed point.

(2) Let $f(x) = \sqrt{5 - 2x}$ and $I = [1/2, 2]$. Show that $f(I) \subset I$ and that f is a contraction mapping on I (use the mean value theorem). Let $x_0 = 0$ and $x_{n+1} = f(x_n)$. Check that that $x_2 \in I$ (without using a calculator!) and deduce that x_n converges to a fixed point $a \in I$. Find the value of a.

§4.1 A, C, D*, F (use C)

§9.1 A, E*, J, N*

(3) If X is a metric space and E and F are subsets of X define $d(E, F) = \inf\{d(x, y) : x \in E, y \in F\}$. Does this define a metric on the collection of all subsets of X? Explain. If X is compact, E and F are closed and $E \cap F = \varnothing$ show that $d(E, F) > 0$. Give an example to show that this conclusion may be false if X is not required to be compact.

(4) *Suppose that X is a compact metric space, $a \in X$ and (x_n) is a sequence with property that for every subsequence (y_k) of (x_n) (so $y_k = x_{n_k}$) there is a subsequence z_j of (y_k) such that $\lim_j z_j = a$. Prove that $\lim_n x_n = a$. Hint: suppose not and find a subsequence of (x_n) which remains bounded away from a, that is, $|x_n - a| > \epsilon$ for some $\epsilon > 0$.