Math 1000/457 07-08
Problem Set 6
Not for credit

Do the following

Suppose $f \in L_{1(\mathbb{T})}$ and $g \in C^1(\mathbb{T})$. Show that $f * g \in C^1(\mathbb{T})$ and $(f * g)' = f * g'$.

Katznelson Chapt. 1: 1.1, 1.2, 1.4, 1.5, 2.8, 2.9, 2.10, 2.12, 2.13, 2.14, 3.3, 4.2, 4.3, 5.4, 6.1, 6.3

Katznelson Chapt. 2: 1.1

Further hint for 2.13: a priori f' is the convolution of f with a certain trigonometric polynomial $\sum_{|m| \leq n} a_n e_n$. Add further terms to this polynomial outside the range $|m| \leq n$ to make it look related to the Fejer kernel. This will yield the formula given by Katznelson. (Don't try to do this computationally!)

Suppose $f : [a, b] \to X$ is a continuous Banach-valued function and let $F(x) = \int_0^x f(t)dt$. Show that F is differentiable on $[a, b]$ and that $F'(x) = f(x)$, including one-sided derivatives at the endpoints. (Differentiable here means that the difference quotients converge in norm.)

Suppose $f \in L_1(\mathbb{T})$ and $\hat{f}(n) = O(\frac{1}{n^{k+2}(\log n)^\alpha})$ for some $\alpha > 1$. Show that $f \in C^k(\mathbb{T})$.

Suppose $f \in L_1(\mathbb{T})$ and $g \in L_\infty(\mathbb{T})$. Show that $f * g \in C(\mathbb{T})$.

Also look at:

Katznelson Chapt. 1: 1.3, 1.9, 1.12, 1.13, 2.15

If $f \in \mathrm{Lip}_\alpha(\mathbb{T})$ then $\hat{f}(n) = O(\frac{1}{n^\alpha})$. (Hint:

$$\hat{f}(n) = \frac{1}{4\pi} \int_0^{2\pi} (f(t + \frac{\pi}{n}) - f(t))e^{-int} dt$$

An irrational number α is said to be poorly approximable by rationals if there is a constant C such that $|\alpha - m/n| > C/n^2$ for all integers m, n. (For example it is known that any quadratic irrational is poorly approximable.) Suppose α is such a number and suppose $f \in C^2(\mathbb{T})$ with $\int_0^{2\pi} f(x)dx = 0$. Show that f can be expressed as $g - g_\alpha$ for some $g \in L^2(\mathbb{T})$. Show that g is uniquely determined up to addition of a constant. If f is $C^{2+\delta}$ (that is f'' exists everywhere and is Lip(δ) for some $\delta > 0$) show that then $g \in C(\mathbb{T})$ continuous. (Hint: express g as a Fourier series and solve for the coefficients.)

Show that $f \in L^1(\mathbb{T})$ is in $A(\mathbb{T})$ (that is, $\hat{f} \in l_1(\mathbb{Z})$) if and only if $f = g * h$ for some g and h in $L^2(\mathbb{T})$.

If $a_n, n = 1, 2, \ldots$ is a sequence of non-negative numbers and the series $\sum a_n$ is Cesaro summable to S then $\sum a_n$ converges to S. (Cesaro summable means that if we set $s_n = \sum_{k=1}^{n} a_n$ then $\frac{1}{N} \sum_{n=1}^{N} s_n \to S$) If $f \in C(\mathbb{T})$ and $\hat{f}(n) \geq 0$ for every n then $f \in A(\mathbb{T})$.

Recall the ergodic theorem for a unitary operator U on a Hilbert space $\mathcal{H}$:

$$\frac{1}{n+1} \sum_{i=0}^{n} U^i x$$

converges in norm to Px where P denotes the projection on the subspace of U-invariant vectors. Prove it using the spectral theorem for unitary operators.