Locally convex spaces

Suppose X is a real vector space and U is a convex subset containing 0. U is called *absorbing* if $\bigcup_{r>0} rU = X$. Given such a U define $p = p_U$ by $p(x) = \inf\{r \geq 0 : x \in rU\}$. p is called the *gauge* of U.

Proposition 1. *p is a positive Minkowksi functional. If $U = -U$ it is a semi-norm.*

Proof: That $p(ax) = ap(x)$ for $a > 0$ is clear. For the sub-additivity, given x and y in X we may write $x = (p(x) + \epsilon)u$ and $y = (p(y) + \epsilon)v$ with $u, v \in U$. Thus $x + y = (p(x) + \epsilon + p(y) + \epsilon)w$, where

$$w = \frac{1}{p(x) + \epsilon + p(y) + \epsilon}(p(x) + \epsilon)u + (p(y) + \epsilon)v) \in U.$$

Thus $p(x+y) \leq p(x)+\epsilon+p(y)+\epsilon$. Since $\epsilon > 0$ is arbitrary we get subaddivity. Finally if $U = -U$ we get $p(x) = p(-x)$ so p is a semi-norm.

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Recall that a topological vector space (TVS) is a vector space over K (equals $\mathbb{R}$ or $\mathbb{C}$) with a Hausdorff topolgy such that the functions $(\lambda, x) \mapsto \lambda x$ from $K \times X$ to X and $(x, y) \mapsto x + y$ from $X \times X$ to X are continuous. If X is a TVS and U is a convex symmetric neighborhood of 0 then $n^{-1}x \in U$ for all large n, by the continuity of scalar multiplication, hence $x \in nU$). Thus p_U is defined. Note that for all $x \in U$ we have $(1 + \epsilon)x \in U$ for all small enough ϵ (why?) and this implies that $p_U < 1$ on U. Evidently $p_U \geq 1$ on U^c so we have $U = \{x : p(x) < 1\}$, that is U is the open unit ball for the seminorm p_U.

Recall that a TVS X is called a locally convex space (LCS) if there is a neighborhood base $\mathcal{U}$ at 0 consisting of open convex sets. Explicitly, this means that every neighborhood of 0 contains some element of $\mathcal{U}$. We can always assume that such a neighborhood base consists of symmetric sets, by replacing each $U \in \mathcal{U}$ by $U \cap -U$, which is again an open convex neighborhood of 0.

Theorem 1. *A real TVS is an LCS iff its topology is generated by a family of semi-norms.*

Proof: We have already discussed (or see Folland) the "if" of "iff". For the converse suppose $\mathcal{U}$ is a neighborhood base at 0 consisting of symmetric convex sets. Consider the semi-norms $p_U, U \in \mathcal{U}$. I claim they generate the toplogy of X. For this it is enough to show that a net x_α converges to 0 iff $p_U(x_\alpha) \to 0$. If $x_\alpha \to 0$ and $U \in \mathcal{U}$ then x_α is eventually in ϵU so $p_U(x_\alpha) < \epsilon$ eventually. On the other hand if $p_U(x_\alpha) \to 0$ for all $U \in \mathcal{U}$ then eventually $p_U(x_\alpha) < 1$, that is $x_\alpha \in U$.

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Suppose X and Y are TVS and $T : X \to Y$ is linear. Then T is continuous if and only if T is continuous at 0. We have seen this is true when X and Y are normed and we leave the proof here as an exercise. If X is a TVS we will denote by X^* the space of continuous linear functionals on X.

Theorem 2. *Supppose X is a real TVS, U is an open convex subset of X and $x \notin U$. Then there are $f \in X^*$ and $a \in \mathbb{R}$ such that $f(U) \subset (-\infty, a)$ and $f(x) \geq a$.*

Proof: Wolog $0 \in U$. Let $p = p_U$, a positive Minkowski functional. Define f on $\mathbb{R}x$ by $f(tx) = tp(x)$. Then $f \leq p$ on $\mathbb{R}x$ ($f(tx) = p(tx)$ for $t > 0$, $f(tx)$ is negative for $t < 0$) so f has an extension, also denoted f, to X with $f \leq p$. Then $f \leq p < 1$ on U since U is open and $p(x) \geq 1$.

Now we check that f is continuous. Let $N = U \cap -U$. Then $f < 1$ on N so $f > -1$ on N. Thus $|f| < 1$ on N so $|f| < \epsilon$ on ϵN, giving continuity of f at 0.

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Remark: If X is a complex TVS then the real- and the complex-valued continuous linear functionals on X are in one-one correspondence as in the case of normed vector spaces: if f is real-linear then $F(x) = f(x) - if(ix)$ is complex-linear, every complex F is of this form and one is continuous iff the other is. (Recall also that if X is a normed vector space then $\|F\| = \|f\|$.) As a consequence the weak toplogy on X, that is the topology generated by X^*, is the same whether it is considered as a real or complex space TVS.

Theorem 3. *Suppose X is a real LCS, C is a closed convex subset and $x \notin C$. Then there is an $f \in X^*$ and $a \in \mathbb{R}$ such that $f \leq a$ on C and $f(x) > a$. In particular any closed convex subset C of a real* or *complex X is weakly closed, i.e. closed in the topology generated by X^*.*

Proof: Wolog $x = 0$. Let U be a symmetric convex neighborhood of 0 such that $U \cap C$ is empty. Then $U + C = \bigcup\{U + c : c \in C\}$ is open, convex and does not contain 0. By the previous theorem there is $f \in X^*$ such that $f < 0$ on $U + C$ and of course $f(0) = 0$. f is not identically 0 on U since for each x there is an n such that $\frac{1}{n}x \in U$. Thus we may choose $u \in U$ such that $f(u) > 0$. Then for any $c \in C$ $f(u + c) < 0$ so $f(c) < -f(u) < 0 = f(0)$.

The claim about closed convex subsets follows immediately in case X is real. The complex case follows by the remarks preceding the proof of Theorem 3.

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We can apply Theorem 3 in the complex case as well: the conclusion is that there is $f \in X^*$ whose real part $\Re f$ separates C and x.

Corollary 1. *In a normed vector space X, any norm-closed convex subset is weakly closed.*

Lemma 1. *Suppose X is a vector space over K, E is a subspace of the algebraic dual $L(X, K)$ (linear maps from X to K) and let τ be the topology it generates. Then any τ-continuous linear functional on X belongs to E.*

Proof: The toplogy here is generated by the semi-norms $|f|, f \in E$. As we have seen (or see Folland) the sets $\bigcap_{i=1}^{j}\{x : |f_i(x)|\}$, $f_1,\ f_k \in E$, $\epsilon > 0$ form a neighborhood base at 0. Since f is continuous at 0 there are $f_1, \ldots, f_k \in E$ such that $|f_i(x)| < ep$ for all i implies $|f(x)| < 1$. Let $F = (f_1, \ldots f_k)$, a map from X to K^k. Evidently $F(x) = 0$ implies $f(x) = 0$ so we may define $\varphi : F(X) \to K^k$ by $\varphi(F(x)) = f(x)$ and this will be a well-defined linear map. Then φ has the form $\varphi(y) = \sum_{i=1}^{k} \lambda_i y_i$ for some $\lambda_1, \ldots \lambda_k \in K$, so $f(x) = \sum_{i=1}^{k} \lambda_i f_i(x)$. Since E is a subspace it follows that $f \in E$.

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Here is an application to Banach spaces (which can also be proved more directly). Recall that a Banach space X may be considered as a subspace of X^{**} via the natural embedding $x \mapsto \hat{x}$ ($\hat{x}(f) = f(x)$).

Theorem 4. *If X is a Banach space then $B_1(X)$ is weak-star dense in $B_1(X^{**})$. X is reflexive iff $B_1(X)$ is compact in the weak toplogy. A norm-closed subspace of a reflexive Banach space is reflexive.*

Proof:

Note that, by Lemma 1, the weak-* continuous functionals on X^{**} are those of the form $\varphi \mapsto (\varphi, F), f \in X^*$. Thus by theorem 3, if $\varphi \in B_1(X^{**})$ is not in the weak-star closure of $B_1(X)$ there is an element $f \in X^*$ such that $\Re(x, f) = \Re(f(x)) \leq 1$ for $x \in B_1(X)$ but $\Re(\varphi, f) = \Re(\varphi(f)) > 1$. The fact that $\Re(f(x)) \leq 1$ for $\|x\| \leq 1$ means that the real linear functional $\Re f$ has norm $\|\Re f \leq 1\|$, so $\|f\| = \|\Re f\| \leq 1$. But then, since $\|\varphi\| \leq 1$, it follows that $|(\varphi, f)| \leq 1$, which contradicts $\Re(\varphi, f) > 1$ and establishes the desired density.

Note that the the weak topology on X is the same as its weak-star topology as a subset of X^{**}. If X is reflexive then $B_1(X) = B_1(X^{**})$ is compact in this topology. Conversely if $B_1(X)$ is compact in this topology then it is closed in X^{**}. But it is also dense in $B_1(X^{**})$ hence $B_1(X) = B_1(X^{**})$ which implies $X = X^{**}$.

If X is reflexive and E is a norm-closed subspace then the Hahn-Banach theorem implies that the weak topology on E, considered as an abstract Banach space, co-incides with the weak topology on E inherited from X. $B_1(E) = B_1(X) \cap E$, $B_1(X)$ is compact in the weak topology and E is closed in the weak topology (why?). Thus $B_1(E)$ is compact in the weak topology, hence E is reflexive.

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The Krein-Milman theorem

Suppose X is a real vector space (no topology for the moment) and C is a convex subset. An *extreme point* of C is a point $z \in C$ with the property: if a line segment l contained in E has z in its interior then $l = \{z\}$. By a line segment we mean any set of the form $\{tx + (1-t)y : t \in [0,1]\}$ and by the interior of l we mean $\{tx + (1-t)y : t \in (0,1)\}$. For example, in $\mathbb{R}^2$, the extreme points of a closed triangle are its vertices, the extreme points of a closed disc are the points on the boundary and an open disc has no extreme points.

More generally a convex subset $E \subset C$ is called *extreme* (in C) in case: if the interior of a line segement l intersects E then $l \subset E$. We note that any intersection of extreme subsets of C is again extreme. Moreover if E is extreme in C and $F \subset E$ is extreme in E then F is extreme in C. We will use the elementary fact that if f is any linear functional on E, $x \in C$ and $m = f(x) \geq f(y)$ for all $y \in C$ then the set $\{x \in C : f(x) = m\}$ is extreme in C (supply a proof).

Note that the closure of a convex set in a TVS X is convex. If E is any subset of X the *closed convex hull* of E is the closure of the convex hull and we write it as $\overline{\text{co}}\ E$. It is the smallest closed convex set containing E.

Theorem 5 (Krein-Milman)**.** *Suppose C is a compact convex subset of a LCS X over $\mathbb{R}$ or $\mathbb{C}$. Then C has extreme points. Moreover C is the closed convex hull of its extreme points.*

Proof: The complex case follows immediately from the real case, so we assume X is real. Let $\mathcal{E}$ denote the set of all non-empty compact extreme subsets of C, ordered by inclusion. Use Zorn's lemma to find a minimal element of $E \in \mathcal{E}$. (Justification: if $\mathcal{C} \subset \mathcal{E}$ is a totally ordered subset and $E = \cap\mathcal{C}$ then E is compact, extreme and non-empty, by the finite intersection property of compact sets, so $E \in \mathcal{E}$ is a lower bound for $\mathcal{C}$.) We claim E is a singleton. Suppose not, so let x, y be distinct elements of E. Then there is an $f \in X^*$ such that $f(x) \neq f(y)$. Since f is continuous on the compact set E it achieves a maximum value m on E. Let M be the subset of E where this maximum value is achieved. Then M is a compact extreme subset of E, not equal to E since f is not constant on E. Now M extreme in E and E extreme in C implies M extreme in C, contradicting minimality of E. This contradiction means that E is a singleton. Note that we have shown a little more than claimed: any compact extreme subset of C contains an extreme point of C.

Now let E be the set of extreme points of C and, aiming for a contradiction, suppose $x \in C \backslash \overline{\text{co}}\ E$. Then there is an $f \in X^*$ such that $f(\overline{\text{co}}\ E) \leq c < f(x)$. Let M be the set of points in C where f achieves its max on C. As in the first part M is a non-empty, compact, extreme subset of C so it contains an extreme point of C. But M is disjoint from $\overline{\text{co}}\ E$, a contradiction!

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