THICK AFFINE GRASSMANNIAN, ORBITS, TRANSVERSE SLICES AND QUANTIZATIONS

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Abstract. The aim of this note is twofold: to provide an introduction to the scheme-theoretic properties of thick affine Grassmannian, its orbits and transverse slices, and relation to thin affine Grassmannian and Zastava space; to review the work of Kamnitzer-Webster-Weekes-Yacobi [KWWY14] on quantization of transverse slices $\overline{\mathcal{W}}^{\lambda}_{w_0\mu}$ in the thick affine Grassmannian Gr_G for a complex semi-simple G.

All schemes are defined over $\mathbb{C}$ unless specified.
All algebraic groups are assumed to be connected unless specified.

1. Thick Affine Grassmanian

Let G be a complex algebraic group, define the thick affine Grassmanian

$$\mathsf{Gr}_G := G(\!(t^{-1})\!)/G[t].$$

As a functor, it maps an affine $\mathbb{C}$-scheme $S = \operatorname{Spec} R$ to the set of G-torsor $\mathcal{F}_G$ on $\mathbb{P}^1_S$ together with a trivialization $\beta : \mathcal{F}^0_G|_{\mathbb{D}_\infty} \cong \mathcal{F}_G|_{\mathbb{D}_\infty}$, where $\mathbb{D}_\infty$ is the formal disk Spec $R[\![t^{-1}]\!]$ at infinity.

Remark 1.1. Obviously there is a map from the thin affine Grassmanian Gr_G to thick affine Grassmanian Gr_G:

$$G(\!(t)\!)/G[\![t]\!] \to G(\!(t^{-1})\!)/G[t]$$

note that infinite positive power on the LHS is quotient out hence only finite powers remains. From the modular perspective, this maps a trivialization $\beta : \mathcal{F}^0_G|_{\mathbb{P}^1-\{0\}} \cong \mathcal{F}_G|_{\mathbb{P}^1-\{0\}}$ to its restriction on $\mathbb{D}_\infty$. Note that this map realizes Gr_G as a sub-functor of Gr_G, since $G(R[t^{-1}]) \to G(R[\![t^{-1}]\!])$ is injective. However this map is not representable:

Theorem 1.2. *Thick affine Grassmannian Gr_G is represented by a formally smooth and separated scheme.*

Sketch of Proof. Before we start, let's recall that the functor

$$L^+G : R \mapsto G(R[\![t]\!])$$

is a pro-algebraic group, its $\mathbb{C}$-points are just $G(\mathcal{O})$, and $\pi : \mathsf{Gr}_G \to \mathrm{Bun}_G(\mathbb{P}^1)$ is a L^+G-torsor. It follows that Gr_G is a formally smooth functor.

Step 1. GL_n case. We replace the principal bundle by vector bundle of rank n. Define the open substack U_k of $\mathrm{Bun}_n(\mathbb{P}^1)$ by rank n vector bundles $\mathcal{V}$ such that

$$\mathrm{H}^0(\mathbb{P}^1, \mathcal{V}^*(-k)) = \mathrm{H}^0(\mathbb{P}^1, \mathcal{V}(-k)) = 0$$

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Note that U_k is disjoint union of $U_{k,m}$ such that $\dim \mathrm{H}^0(\mathbb{P}^1, \mathcal{V}(k)) = m$. Next we define a stack $\widetilde{U}_{k,m}$ classifying $(\mathcal{V}, \beta)$ where $\mathcal{V} \in U_{k,m}$ and $\beta : \mathcal{V}|_{\operatorname{Spec}\mathbb{C}[t^{-1}]/t^{-2k}} \cong \mathcal{O}^{\oplus n}_{\operatorname{Spec}\mathbb{C}[t^{-1}]/t^{-2k}}$. It follows from definition that $\pi^{-1}(U_{k,m})$ is a $L^{\geq 2k}GL_n$-torsor on $\widetilde{U}_{k,m}$.

Consider the scheme $W_{k,m}$ parametrizing the following data $(\mathcal{M}, \mathcal{V})$ on a test scheme S: A rank m sub-bundle $\mathcal{M}$ of $\mathcal{O}_S^{\oplus 2kn}$, together with a rank n quotient bundle $\mathcal{V}$ of $p^*\mathcal{M}$.

Claim: The map $(\mathcal{V}, \beta) \mapsto (p_*\mathcal{V}(k), \mathcal{V}(k))$ gives rise to a morphism between stacks $\widetilde{U}_{k,m} \to W_{k,m}$, where $p_*\mathcal{V}(k)$ is considered as a subbundle of $\mathcal{O}_S^{\oplus 2kn}$ via composing β with the canonical map $p^*p_*\mathcal{V}(k) \to \mathcal{V}(k)$. In fact, this is an immersion.

Since $W_{k,m}$ is a separated scheme, $\widetilde{U}_{k,m}$ is a separated scheme as well, so $\pi^{-1}(\widetilde{U}_k) = \coprod_m \pi^{-1}(\widetilde{U}_{k,m})$ is a separated scheme. Now $\mathsf{Gr}_{\mathrm{GL}_n}$ is an increasing union of $\pi^{-1}(\widetilde{U}_k)$, hence $\mathsf{Gr}_{\mathrm{GL}_n}$ is a separated scheme.

Step 2. General case. Embed G into GL_n for some n, and we have the following commutative diagram:

$$\begin{tikzcd}
\mathsf{Gr}_G \arrow[r, "i"] \arrow[dr] & \mathrm{Bun}_G \times_{\mathrm{Bun}_{\mathrm{GL}_n}} \mathsf{Gr}_{\mathrm{GL}_n} \arrow[r, "\widetilde{f}"] \arrow[d] & \mathsf{Gr}_{\mathrm{GL}_n} \arrow[d, "\pi"] \\
& \mathrm{Bun}_G \arrow[r, "f"] & \mathrm{Bun}_{\mathrm{GL}_n}
\end{tikzcd}$$

Now f is representable and separated (because GL_n / G is quasi-projective, apply the theory of Hilbert schemes), so its base change $\widetilde{f}$ is representable and separated as well. i is an embedding of torsors so it's representable and separated. As a result, the general case follows from the GL_n case. ■

Corollary 1.3. *Suppose that $H \subset G$ is a subgroup, then the natural map $\mathsf{Gr}_H \to \mathsf{Gr}_G$ is monomorphism, and the diagram is Cartesian*

$$\begin{tikzcd}
\mathrm{Gr}_H \arrow[r] \arrow[d] & \mathrm{Gr}_G \arrow[d] \\
\mathsf{Gr}_H \arrow[r] & \mathsf{Gr}_G
\end{tikzcd}$$

i.e. $\mathrm{Gr}_H \cong \mathsf{Gr}_H \times_{\mathsf{Gr}_G} \mathrm{Gr}_G$.

Proof. We re-state the first statement as following: Given an affine $\mathbb{C}$-scheme $S = \operatorname{Spec} R$, and a G-torsor $\mathcal{F}_G$ on $\mathbb{P}^1_S$ with trivialization β on $\mathbb{D}_\infty$, suppose that there are two sections of $H\backslash\mathcal{F}_G$, denoted by σ_1, σ_2, and they agree on $\mathbb{D}_\infty = \operatorname{Spec} R[\![t^{-1}]\!]$, then $\sigma_1 = \sigma_2$.

Now σ_2 is defined by an ideal sheaf $\mathcal{I}_2$, pull-back of this ideal sheaf along σ_1 is denoted by $\mathcal{I}$, to prove $\sigma_1 = \sigma_2$ is the same as proving $\mathcal{I} = 0$. It's enough to prove that $\mathcal{I}|_{\mathbb{P}^1 - \{0\}}$ is zero. Then the statement follows from $\mathcal{I}|_{\mathbb{D}_\infty} = 0$ and the observation that $R[t^{-1}] \to R[\![t^{-1}]\!]$ is injective.

We re-state the second statement as following: Given an affine $\mathbb{C}$-scheme $S = \operatorname{Spec} R$, a G-torsor $\mathcal{F}_H$ on $\mathbb{P}^1_S$ with trivialization β_H on $\mathbb{D}_\infty$, suppose that its induced G-torsor $\mathcal{F}_G$ has a trivialization $\widetilde{\beta}_G$ on $\mathbb{P}^1 - \{0\}$ whose restriction on $\mathbb{D}_\infty$ agrees with the one induced from β_H, then after an étale base change $S' \to S$, β_H lifts to a trivialization $\widetilde{\beta}_H$ on $\mathbb{P}^1 - \{0\}$.

Decompose H as a reductive H' extended by a unipotent K, then after an affine étale base change $S' \to S$, the induced H' bundle $\mathcal{F}_{H'}$ restricts to a trivial bundle $\mathcal{F}^0_{H'}$ on $\mathbb{P}^1 - \{0\}$ (by Drinfeld-Simpson theorem), and K bundle on $\mathbb{P}^1 - \{0\}$ is trivial because K is a succesive extension of $\mathbb{G}_a$'s. Hence we can assume that $\mathcal{F}_H|_{\mathbb{P}^1-\{0\}}$ is trivial. Fix some trivialization β_0. It remains to prove the following: suppose $g \in G(R[t^{-1}])$ and $g|_{\mathbb{D}_\infty} \in H(R[\![t^{-1}]\!])$, then $g \in H(R[t^{-1}])$.

This is proven in a similar way as the first statement: $H \times (\mathbb{P}^1 - \{0\}) \subset G \times (\mathbb{P}^1 - \{0\})$ is defined by ideal $\mathcal{J}_H$, then the section g pulls back this ideal to an ideal $\mathcal{J}$ on $\mathbb{P}^1 - \{0\}$, then it's equivalent to proving that $\mathcal{J} = 0$. Again this follows from that $\mathcal{J}|_{\mathbb{D}_\infty} = 0$. ■

Remark 1.4. This corollary provides an alternative proof of the first part of [Zhu16, Theorem 1.2.2]. In fact, embed $G \hookrightarrow \mathrm{GL}_n$, then $\mathsf{Gr}_G \to \mathsf{Gr}_{\mathrm{GL}_n}$ is locally finite type (since $\mathrm{Bun}_G \to \mathrm{Bun}_{\mathrm{GL}_n}$ is locally finite type), it follows from the above corollary that $\mathrm{Gr}_G \to \mathrm{Gr}_{\mathrm{GL}_n}$ is representatble and locally finite type, hence Gr_G is an ind-scheme, ind-finite type over $\mathbb{C}$.

Some ind-schemes are schemes, for example, arbitrary disjoint union of schemes is again a scheme. However, this never happens to thin affine Grassmannians (unless $G = 1$):

Corollary 1.5. *For any algebraic group G, thin affine Grassmannian Gr_G is not a scheme, unless G is the trivial group.*

Proof. Assume that for some G, Gr_G is a scheme. If G has unipotent subgroup, then we have $\mathbb{G}_a \subset G$, and it follows from Corollary 1.3 that $\mathrm{Gr}_{\mathbb{G}_a}$ is a scheme. On the other hand, $\mathrm{Gr}_{\mathbb{G}_a}$ is the following functor

$$R \mapsto \varinjlim_k \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{C}[z_1, z_2, \cdots]/(z_k, z_{k+1}, \cdots), R).$$

Now take $U = \operatorname{Spec} R$ to be an open affine subscheme of $\mathrm{Gr}_{\mathbb{G}_a}$, from the description of the functor we know that the embedding map $U \hookrightarrow \mathrm{Gr}_{\mathbb{G}_a}$ is induced from a finite-stage map

$$\phi : \mathbb{C}[z_1, z_2, \cdots, z_n] \to R$$

but $\operatorname{Spec} \mathbb{C}[z_1, z_2, \cdots, z_n]$ embeds into the next stage $\operatorname{Spec} \mathbb{C}[z_1, z_2, \cdots, z_n, z_{n+1}]$ as hyperplane $\{z_{n+1} = 0\}$, which is not open, hence we get a contradiction.

If G has no unipotent subgroup and is non-trivial, then G is a torus, so $\mathbb{G}_m \subset G$ and it follows from Corollary 1.3 that $\mathrm{Gr}_{\mathbb{G}_m}$ is a scheme. On the other hand, it's easy to see that $\mathrm{Gr}_{\mathbb{G}_m}$ has a closed sub-functor

$$R \mapsto \varinjlim_k \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{C}[z]/(z^k), R).$$

Using the same strategy as the first case, we see that this is not representable by scheme. ■

The next corollary concerns with the Picard group of thick affine Grassmannian, parallel to the discussion for $\mathrm{Pic}^e(\mathrm{Gr}_G)$ in [Zhu16, Section 2.4], the conclusion is that under certain assumptions the obvious map $\mathrm{Gr}_G \to \mathsf{Gr}_G$ induces isomorphism of Pic^e via pull-back.

Corollary 1.6. *Same notation as Theorem 1.2, and also assume that G is semi-simple and simply-connected, then*

$$\pi_*\underline{\mathbb{G}_m}_{\acute{e}t} \cong \underline{\mathbb{G}_m}_{\acute{e}t},\ R^1\pi_*\underline{\mathbb{G}_m}_{\acute{e}t} = 0$$

where $\pi : \mathsf{Gr}_G \to \mathrm{Bun}_G$ is the forgeting β map. As a consequence

$$\mathrm{Pic}^e(\mathsf{Gr}_G) \cong \mathrm{Pic}^e(\mathrm{Bun}_G) \cong \mathrm{Pic}^e(\mathrm{Gr}_G)$$

where Pic^e is the Picard group of line bundles with framings at identity.

Proof. The proof in Theorem 1.2 shows that there is a factorization

$$\mathsf{Gr}_G \xrightarrow{q} \widetilde{U} \xrightarrow{h} \mathrm{Bun}_G$$

where h is a G-torsor and q is inverse limit of étale-locally trivial $\mathrm{Lie}(G)$-fibration. Write $q : \mathsf{Gr}_G \to \widetilde{U}$ as the inverse limit of the tower

$$\cdots \to V_k \to V_{k-1} \to \cdots \to V_1 \to \widetilde{U}$$

with transition map $\tau_k : V_k \to V_{k-1}$. Note that τ_k is a étale-locally trivial $\mathrm{Lie}(G)$-fibration and V_{k-1} is smooth (because Bun_G is a smooth stack and all maps we used are smooth), then it follows from the next lemma and the remark below that

$$\tau_{k*}\underline{\mathbb{G}_m}_{\text{ét}} \cong \underline{\mathbb{G}_m}_{\text{ét}},\ R^1\tau_{k*}\underline{\mathbb{G}_m}_{\text{ét}} = 0$$
$$h_*\underline{\mathbb{G}_m}_{\text{ét}} \cong \underline{\mathbb{G}_m}_{\text{ét}},\ R^1h_*\underline{\mathbb{G}_m}_{\text{ét}} = 0$$

Composing these transition maps together, we see that

$$q_{k*}\underline{\mathbb{G}_m}_{\text{ét}} \cong \underline{\mathbb{G}_m}_{\text{ét}},\ R^1q_{k*}\underline{\mathbb{G}_m}_{\text{ét}} = 0,$$

where $q_k : V_k \to \mathrm{Bun}_G$ is the composition. Hence

$$\pi_*\underline{\mathbb{G}_m}_{\text{ét}} = \varinjlim_k q_{k*}\underline{\mathbb{G}_m}_{\text{ét}} \cong \underline{\mathbb{G}_m}_{\text{ét}},\ R^1\pi_*\underline{\mathbb{G}_m}_{\text{ét}} = \varinjlim_k R^1q_{k*}\underline{\mathbb{G}_m}_{\text{ét}} = 0.$$

It follows from the Leray spectral sequence that $\mathrm{Pic}^e(\mathsf{Gr}_G) \cong \mathrm{Pic}^e(\mathrm{Bun}_G)$ via pull-back. The isomorphism $\mathrm{Pic}^e(\mathrm{Bun}_G) \cong \mathrm{Pic}^e(\mathrm{Gr}_G)$ is established in [Fal03, Section 7]. ■

Lemma 1.7. *Suppose that X and Y are connected smooth varieties, then*

- *$\mathcal{O}(X \times Y)^\times$ is generated by $\mathcal{O}(X)^\times$ and $\mathcal{O}(Y)^\times$ via pull-back.*
- *If moreover X is rational, then $\mathrm{Pic}(X \times Y)$ is generated by $\mathrm{Pic}(X)$ and $\mathrm{Pic}(Y)$ via pull-back.*

Remark 1.8. Affine algebraic groups (over algebraically closed field) are always rational. For any algebraic group G we have

$$\mathcal{O}(G)^\times/\mathbb{C}^\times \cong \mathrm{Hom}(G, \mathbb{G}_m)$$

i.e. the character lattice, hence if G is semi-simple, then $\mathcal{O}(G)^\times = \mathbb{C}^\times$. If G is simply-connected, then $\mathrm{Pic}(G) = 0$.

Remark 1.9. If G is almost simple and simply-connected, then

$$\mathrm{Pic}^e(\mathsf{Gr}_G) = \mathbb{Z} \cdot [\pi^*\mathcal{O}(1)]$$

where $\mathcal{O}(1)$ is the generator of $\mathrm{Pic}^e(\mathrm{Bun}_G)$ constructed in [Fal03, Section 7].

Let's make a final remark before we end this section: unlike thin affine Grassmannian Gr_G which is **local**, the thick affine Grassmannian Gr_G is of **global** nature, the definition involves a choice of a global curve. More precisely, we have the following equivalent modular definition for thin affine Grassmannian Gr_G: Fix a smooth proper curve X and a fixed point $x \in X$, Gr_G classifies

- G-torsors $\mathcal{F}_G$ on X together with a trivialization $\beta : \mathcal{F}_G^0|_{X-x} \to \mathcal{F}_G|_{X-x}$.

In this way Gr_G parametrizes modifications of a bundle $\mathcal{F}_G$ at the point x, a.k.a. Hecke modification at x. Note that all "jumps" of bundles happen inside the formal neighborhood of x, Gr_G does not actually "see" the global bundle!

On the other hand, we could also formulate the *thick* version: Let $\mathsf{Gr}_G(X, x)$ be the moduli space classifying

- G-torsors $\mathcal{F}_G$ on X together with a trivialization $\beta : \mathcal{F}_G^0|_{\mathbb{D}_x} \to \mathcal{F}_G|_{\mathbb{D}_x}$, where $\mathbb{D}_x$ is the formal disk at x.

The same idea in Theorem 1.2 also apply to this generalized version and we have

Theorem 1.10. *$\mathsf{Gr}_G(X, x)$ is represented by a formally smooth and separated scheme.*

Unlike the thin affine Grassmannian, $\mathsf{Gr}_G(X, x)$ can actually "see" the global bundle and tell which X it's living on, in fact $G[\![z]\!]$ acts on $\mathsf{Gr}_G(X, x)$ via changing the trivialization β, where z is the coordinate of $\mathbb{D}_x$, and it's easy to see that $G[\![z]\!]$-orbits are one-to-one correspond to equivalence class of G-bundles on X. When X is $\mathbb{P}^1$, orbits are labeled by dominant coweights; if E is an elliptic curve, then $\mathrm{SL}_2[\![z]\!]$-orbits on $\mathsf{Gr}_{\mathrm{SL}_2}(E, 0)$ are uncountable, because semistable SL_2-bundles has coarse moduli space $E/(\mathbb{Z}/2) \cong \mathbb{P}^1$ which has uncountably many points.

2. Orbits

From now on we will assume that G is reductive unless specified. Recall that for dominant coweight $\lambda \in \Lambda_G^+$ there are $G(\mathcal{O})$-orbits $\mathrm{Gr}^\lambda = G(\mathcal{O}) \cdot t^\lambda$ and its closure $\overline{\mathrm{Gr}}^\lambda$ in the thin affine Grassmannian, under the morphism $\mathrm{Gr}_G \to \mathsf{Gr}_G$, they are mapped to $G[t]$-orbits $G[t] \cdot t^\lambda$ and closures. In fact we have

Lemma 2.1. *$\overline{\mathrm{Gr}}^\lambda \to \mathsf{Gr}_G$ is a closed embedding.*

Proof. $\overline{\mathrm{Gr}}^\lambda$ is projective and Gr_G is separated, so $\overline{\mathrm{Gr}}^\lambda \to \mathsf{Gr}_G$ is proper. On the other hand, $\overline{\mathrm{Gr}}^\lambda$ is subfunctor of Gr_G thus $\overline{\mathrm{Gr}}^\lambda \to \mathsf{Gr}_G$ is a monomorphism. The lemma follows from a standard fact that closed embedding $\Longleftrightarrow$ proper monomorphism. ■

Corollary 2.2. *Gr_G is not quasi-compact.*

Proof. Suppose that Gr_G is quasi-compact, then it can be covered by finitely many open affine subschemes. Let the number of open affine be N, then it follows that every $\overline{\mathrm{Gr}}^\lambda$ can be covered by N open affine subschemes, in particular, $\mathrm{H}^i(\overline{\mathrm{Gr}}^\lambda, \mathcal{F}) = 0$ for all $\mathcal{F} \in \mathrm{QCoh}(\overline{\mathrm{Gr}}^\lambda)$ and all $i > N$. Now choose λ such that $\langle 2\rho, \lambda \rangle > N$ and an anti-ample line bundle $\mathcal{L}$ on $\overline{\mathrm{Gr}}^\lambda$, then by Grothendieck duality

$$\mathrm{H}^{\langle 2\rho, \lambda \rangle}(\overline{\mathrm{Gr}}^\lambda, \mathcal{L}) = \mathrm{H}^0(\overline{\mathrm{Gr}}^\lambda, \omega_{\overline{\mathrm{Gr}}^\lambda} \otimes \mathcal{L}^\vee)^*$$

this becomes non-zero after taking sufficiently large tensor power of $\mathcal{L}$, a contradiction. ■

Remark 2.3. Since $\mathrm{Gr}_G = \bigcup_\lambda \overline{\mathrm{Gr}}^\lambda$, we see that Gr_G is an ind-closed subscheme of Gr_G. However the union is not exhaustive, since elements like

$$\begin{bmatrix} \sum_{n\geq 0} t^{-n} & 0 \\ 0 & 1-t^{-1} \end{bmatrix} \in \mathsf{Gr}_{\mathrm{SL}_2}(\mathbb{C})$$

are not in the image of Gr^λ for any λ. A classical analog is the map $\mathrm{Spf}(\mathbb{C}[\![z]\!]) \to \mathrm{Spec}(\mathbb{C}[z])$, in this sense we can regard Gr_G as a "formal spectrum" and Gr_G as the "actual spectrum".

Thanks to Lemma 2.1, when we focus on the $G[t]$-orbits and its subvariaties, we can pretend that we are still dealing with the thin affine Grassmannian. We will denote these orbits by Gr^λ and closures by $\overline{\mathsf{Gr}}^\lambda$. Things becomes different when move on to the infinite dimensional orbits $\mathsf{Gr}_\mu := G[\![t^{-1}]\!] \cdot t^\mu$, they are $\mathbb{A}^\infty$-fibration over G/P_μ, in particular they are actual schemes (unlike Gr_μ which is an ind-scheme).

Analogous to the thin affine Grassmannian case (and the proof is similar), we have

Lemma 2.4 (Birkhoff Decomposition)**.**

$$\mathsf{Gr}_G = \coprod_{\mu\in\Lambda_G^+} \mathsf{Gr}_\mu \tag{2.1}$$

$$\overline{\mathsf{Gr}}_\mu = \coprod_{\nu\geq\mu} \mathsf{Gr}_\nu \tag{2.2}$$

and Gr_μ *is open in* $\overline{\mathsf{Gr}}_\mu$*.*

We can also define the intersections

$$\mathsf{Gr}^\lambda_\mu := \mathsf{Gr}^\lambda \cap \mathsf{Gr}_\mu,\ \overline{\mathsf{Gr}}^\lambda_\mu := \overline{\mathsf{Gr}}^\lambda \cap \mathsf{Gr}_\mu$$

Since Gr_μ is the attracting set of $G\cdot t^\mu$ under the loop rotation $\mathbb{G}_m^{\mathrm{rot}}$-action, and the map $\mathrm{Gr}_G \to \mathsf{Gr}_G$ is $\mathbb{G}_m^{\mathrm{rot}}$-equivaraint, we see that Gr^λ_μ (resp. $\overline{\mathsf{Gr}}^\lambda_\mu$) is image of Gr^λ_μ (resp. $\overline{\mathrm{Gr}}^\lambda_\mu$), in particular Gr^λ_μ is non-empty iff $\lambda\geq\mu$.

Remark 2.5. The following modular definition of $\overline{\mathsf{Gr}}^\lambda_\mu$ is also helpful [FM97, Section 10]: $\overline{\mathsf{Gr}}^\lambda_\mu$ classifies G-torsors $\mathcal{F}_G$ on $\mathbb{P}^1$ with a trivialization $\beta: \mathcal{F}^0_G|_{\mathbb{P}^1-\{0\}} \cong \mathcal{F}_G|_{\mathbb{P}^1-\{0\}}$, such that $\mathcal{F}_G$ has isomorphism type μ and β has defect type $\leq\lambda$, i.e. $\forall \check{\nu}\in\check{\Lambda}^+_G$, the compostion

$$\mathcal{V}^{\check{\nu}}_{\mathcal{F}_G}(-\langle\lambda,\check{\nu}\rangle\cdot\{0\}) \longrightarrow \mathcal{V}^{\check{\nu}}_{\mathcal{F}_G} \xrightarrow{\beta} \mathcal{V}^{\check{\nu}}_{\mathcal{F}^0_G}$$

is regular at $\{0\}$, i.e. has no pole.

Recall that every vector bundle $\mathcal{V}$ on smooth proper curve X has a canonically defined Harder-Narasimhan filtration $0=\mathcal{V}_0\subset\mathcal{V}_1\subset\cdots\subset\mathcal{V}_n=\mathcal{V}$ such that $\mathrm{gr}_i\mathcal{V}$ is a semistable vector bundle and $\mu(\mathrm{gr}_i\mathcal{V}) > \mu(\mathrm{gr}_{i+1}\mathcal{V})$ where μ is the slope (=degree/rank) function. For $X=\mathbb{P}^1$ this filtration is clear: $\mathcal{V}\cong\mathcal{O}(k_1)^{\oplus\ell_1}\oplus\cdots\oplus\mathcal{O}(k_m)^{\oplus\ell_m}$ such that $k_1>\cdots>k_m$, so we take

$$\mathcal{V}_j = \mathcal{O}(k_1)^{\oplus\ell_1}\oplus\cdots\oplus\mathcal{O}(k_j)^{\oplus\ell_j}$$

Note that the filtration defines a **canonical** reduction of structure group from GL_n to the parabolic group P_μ where

$$\mu = (k_1, \cdots, k_1, k_2, \cdots, k_2, \cdots, k_m, \cdots, k_m) \in \Lambda^+_{\mathrm{GL}_n}$$

For general reductive G, take an embedding $r : G \hookrightarrow \mathrm{GL}_n$ for some n, then every G-bundle $\mathcal{F}_G$ has associated rank n vector bundle $\mathcal{V}(\mathcal{F}_G)$. Suppose that $\mathcal{F}_G$ has isomorphism type μ, then $\mathcal{V}(\mathcal{F}_G)$ has isomorphism type $r(\mu)$, then the structure group of $\mathrm{GL}_n \overset{G}{\times} \mathcal{F}_G$ reduces from GL_n to $P_{r(\mu)}$, it follows that $\mathcal{F}_G \cap \mathcal{F}_{P_{r(\mu)}}$ is a principal $G \cap P_{r(\mu)} = P_\mu$-bundle and induces $\mathcal{F}_G$. Denote this P_μ-bundle by $\mathcal{F}_{P_\mu}$.

Claim: $\mathcal{F}_{P_\mu}$ does not depend on the choice of embedding $r : G \hookrightarrow \mathrm{GL}_n$. We call it the *Harder-Narasimhan flag* of $\mathcal{F}_G$.

β sends $\mathcal{F}_{P_\mu}|_{\mathbb{D}_\infty}$ to a flag $\mathcal{F}^0_{P_\mu}|_{\mathbb{D}_\infty}$ of the trivial G-bundle $\mathcal{F}^0_G|_{\mathbb{D}_\infty}$. Notice that sending $t \to \infty$ in the orbit $\mathsf{Gr}_\mu = G[\![t^{-1}]\!] \cdot t^\mu$ is equivalent to just looking at the action of G on the flag at ∞, hence we identify the contraction map $\mathsf{Gr}_\mu \to G \cdot t^\mu = G/P_\mu$ with

$$(\mathcal{F}_G, \beta) \mapsto \mathcal{F}^0_{P_\mu}|_\infty \in G/P_\mu$$

Warning: modular definition given above *a priori* does not give rise to the same scheme structure as those orbits, they may be non-reduced.

Semi-Infinite Orbits. Let U be the positive unipotent, and let $\mathsf{S}_\mu = U(\!(t^{-1})\!) \cdot t^\mu$ $(\mu \in \Lambda_G)$. Similarly let U^- be the negative unipotent, and let $\mathsf{T}_\mu = U^-(\!(t^{-1})\!) \cdot t^\mu$ $(\mu \in \Lambda_G)$. Alternatively, consider maximal torus T, Borel B, and negative Boral B^-, then there are natural maps

$$\begin{tikzcd} & \mathsf{Gr}_B \arrow[dl, "\mathsf{q}"'] \arrow[dr, "\mathsf{i}"] & \\ \mathsf{Gr}_T & & \mathsf{Gr}_G \\ & \mathsf{Gr}_{B^-} \arrow[ul, "\mathsf{q}_-"] \arrow[ur, "\mathsf{i}_-"'] & \end{tikzcd}$$

It's straightforward to see that for any $x \in \mathsf{Gr}_T$, $\mathsf{q}^{-1}(x)$ is the $U(\!(t^{-1})\!)$-orbit through x (considered as a point in Gr_B), hence

$$\mathsf{S}_\mu = \mathsf{i}(\mathsf{q}^{-1}(t^\mu)),\ \mathsf{T}_\mu = \mathsf{i}_-(\mathsf{q}_-^{-1}(t^\mu))$$

There is a similar diagram for the thin affine Grassmannian Gr_G, in fact, we can put these diagrams together:

$$\begin{tikzcd} \mathrm{Gr}_T \arrow[d, "j_T"] & \mathrm{Gr}_B \arrow[l, "q"'] \arrow[r, "i"] \arrow[d, "j_B"] & \mathrm{Gr}_G \arrow[d, "j_G"] \\ \mathsf{Gr}_T & \mathsf{Gr}_B \arrow[l, "\mathsf{q}"] \arrow[r, "\mathsf{i}"'] & \mathsf{Gr}_G \end{tikzcd}$$

Note that the right square is Cartesian by Corollary 1.3, but the left square is not Cartesian: otherwise $\mathrm{Gr}_U \cong q^{-1}(t^\mu) \cong \mathsf{q}^{-1}(t^\mu) \cong \mathsf{Gr}_U$ is a scheme, contradicts with Corollary 1.5.

Nevertheless, we still have

$$S_\mu = i(q^{-1}(t^\mu)) = i(q^{-1}(j_T^{-1}(t^\mu))) = i(j_B^{-1}(\mathsf{q}^{-1}(t^\mu))) = j_G^{-1}(\mathsf{i}(\mathsf{q}^{-1}(t^\mu))) = j_G^{-1}(\mathsf{S}_\mu)$$

Similarly $T_\mu = j_G^{-1}(\mathsf{T}_\mu)$. In particular, the notion of MV-cycles agrees on thin and thick affine Grassmannians:

$$\overline{\mathrm{Gr}}^\lambda \cap S_\mu = \overline{\mathsf{Gr}}^\lambda \cap \mathsf{S}_\mu,\ \overline{\mathrm{Gr}}^\lambda \cap T_\mu = \overline{\mathsf{Gr}}^\lambda \cap \mathsf{T}_\mu$$

It is shown in [MV07a, Theorem 3.2] that $\overline{\mathrm{Gr}}^\lambda \cap S_\mu$ is pure of dimension $\langle \rho, \lambda + \mu \rangle$, and $\overline{\mathrm{Gr}}^\lambda \cap T_\mu$ is pure of dimension $\langle \rho, \lambda - \mu \rangle$.

Another space $T_\mu \cap S_\nu$ is also interesting. Note that the thick analog $\mathsf{T}_\mu \cap \mathsf{S}_\nu \hookrightarrow \mathsf{Gr}_G$ is an immersion subscheme and it's affine. In fact two spaces agree up to nilpotents:

Lemma 2.6. $(T_\mu \cap S_\nu)_{\mathrm{red}} = (\mathsf{T}_\mu \cap \mathsf{S}_\nu)_{\mathrm{red}}$, *and it's a finite type affine variety.*

Proof. Since $T_\mu = t^\mu \cdot T_0$ (similar for others), we can assume that $\mu = 0$ and reset $\nu - \mu$ to ν. We will show that

(*) $\mathsf{T}_0 \cap \mathsf{S}_\nu$ is set-theoretically[1] included in some $\overline{\mathsf{Gr}}^\lambda$.

This implies that $(\mathsf{T}_0 \cap \mathsf{S}_\nu)_{\mathrm{red}}$ is an immersion subscheme of $\overline{\mathsf{Gr}}^\lambda$, so it's a finite type variety, and

$$(\mathsf{T}_0 \cap \mathsf{S}_\nu)_{\mathrm{red}} = (\mathsf{T}_0 \cap \overline{\mathsf{Gr}}^\lambda \cap \mathsf{S}_\nu \cap \overline{\mathsf{Gr}}^\lambda)_{\mathrm{red}} = (T_0 \cap \overline{\mathrm{Gr}}^\lambda \cap S_\nu \cap \overline{\mathrm{Gr}}^\lambda)_{\mathrm{red}} \subset (T_0 \cap S_\nu)_{\mathrm{red}}$$

the reverse inclusion is obvious hence proving all claims.

We proceed by proving GL_n case first. Elements in U (resp. U^-) are the upper (resp. lower) triangular unipotent matrices, coweight $\nu = (\nu_1, \nu_2, \cdots, \nu_n)$. Then (*) is equivalent to the elementary statement: If $A \in U^-(\!(t^{-1})\!), B \in U(\!(t^{-1})\!), C \in \mathrm{GL}_n[t]$ is a solution to equation $A = Bt^\nu C$ then every entry A_{ij} has a finite order pole and the order $\leq -\min(\nu_1, \nu_2, \cdots, \nu_n)$. Proof for this statement is straightforward so omitted. The bound on the order of poles implies that $A \in \overline{\mathsf{Gr}}^\lambda$ for some fixed λ, hence we finish the proof for GL_n case.

The general case is deduced from GL_n case by an embedding $r : G \hookrightarrow \mathrm{GL}_n$, then $\mathsf{T}_{G,0}$ (resp. $\mathsf{S}_{G,\nu}$) is mapped into $\mathsf{T}_{\mathrm{GL}_n,0}$ (resp. $\mathsf{S}_{\mathrm{GL}_n,r(\nu)}$), hence $\mathsf{T}_{G,0} \cap \mathsf{S}_{G,\nu}$ is an immersion subscheme of $\mathsf{T}_{\mathrm{GL}_n,0} \cap \mathsf{S}_{\mathrm{GL}_n,r(\nu)}$, so its reduced scheme is a finite type affine variety, and is contained in the set-theoretic image of $\mathrm{Gr}_G \to \mathsf{Gr}_G$ (using Corollary 1.3). So there exists $\lambda \in \Lambda_G^+$ such that $\overline{\mathsf{Gr}}^\lambda \cap \mathsf{T}_{G,0} \cap \mathsf{S}_{G,\nu}$ contains all generic points of $\mathsf{T}_{G,0} \cap \mathsf{S}_{G,\nu}$ (since the latter only has finitely many generic points), then (*) follows. ■

Remark 2.7. It worths mentioning that there is a modular definition of S_μ as following: let $(\mathcal{F}_G, \beta) \in \mathrm{Gr}_G$, take the standard Borel structure in $\mathcal{F}_G^0$, i.e. the highest weight vector $\mathcal{L}^{\check\lambda}_{\mathcal{F}_G^0} \hookrightarrow \mathcal{V}^{\check\lambda}_{\mathcal{F}_G^0}$ for all $\check\lambda \in \check\Lambda_G^+$, then $(\mathcal{F}_G, \beta)$ is in the closed sub ind-scheme $\overline{S}_\mu$ if and only if the map

$$\mathcal{L}^{\check\lambda}_{\mathcal{F}_G^0}(-\langle \check\lambda, \mu \rangle \cdot \{0\}) \to \mathcal{V}^{\check\lambda}_{\mathcal{F}_G^0} \xrightarrow{(\beta^{\check\lambda})^{-1}} \mathcal{V}^{\check\lambda}_{\mathcal{F}_G}$$

[1]Here by "set-theoretically" we mean all points including generic points, this is done by extending the base field from $\mathbb{C}$ to a large algebraically closed field Ω and the proof works for all such Ω.

is regular, and $(\mathcal{F}_G,\beta)$ is in S_μ if moreover the above map $\mathcal{L}^{\check\lambda}_{\mathcal{F}^0_G}(-\langle\check\lambda,\mu\rangle\cdot\{0\})\to\mathcal{V}^{\check\lambda}_{\mathcal{F}_G}$ has no zero as well. Similarly we replace the Borel structure by the opposite Borel structure and obtain the modular definition for $\overline{T}_\mu$ and T_μ. From this modular description, $\overline{S}_\mu$ and $\overline{T}_\mu$ are stratified[2] as [Zhu16, Corollary 5.3.8]

$$\overline{S}_\mu=\coprod_{\nu\leq\mu}S_\nu,\ \overline{T}_\mu=\coprod_{\lambda\geq\mu}T_\lambda$$

and there are sections σ,σ' of determinant line bundle $\mathcal{L}_{\det}$ such that the vanishing loci

$$V(\sigma)\cap\overline{S}_\mu=\coprod_{\nu<\mu}S_\nu,\ V(\sigma')\cap\overline{T}_\mu=\coprod_{\lambda>\mu}T_\lambda$$

This implies that the variety $\overline{T}_\mu\cap\overline{S}_\nu$ is non-empty if and only if $\mu\leq\nu$, and if it's non-empty then it's projective (since it's closed in some $\overline{\mathrm{Gr}}^\lambda$).

The next proposition is a fact that I learnt from multiple sources, e.g. [FM97, 6.4.1] and [Zhu16, Remark 5.3.12(ii)], unfortunately I couldn't find a reference for a proof, so I worked out a proof and record it here.

Proposition 2.8. *Assume that $\nu\geq\mu$, then $T_\mu\cap S_\nu$ is non-empty, and $\overline{T}_\mu\cap\overline{S}_\nu$, $\overline{T}_\mu\cap S_\nu$ and $T_\mu\cap S_\nu$ are pure of dimension $\langle\rho,\nu-\mu\rangle$.*

Proof. Reset μ to zero and $\nu-\mu$ to ν and use induction on the height of ν. If $\nu=0$ then $\overline{T}_0\cap\overline{S}_0=\overline{T}_0\cap S_0=T_0\cap S_0$ and is a point. Consider three statements:

(P_k) For $\langle\rho,\nu\rangle\leq k$, $\overline{T}_0\cap\overline{S}_\nu$ is pure of dimension $\langle\rho,\nu\rangle$.
(A_k) For $\langle\rho,\nu\rangle\leq k$, $\overline{T}_0\cap S_\nu$ is pure of dimension $\langle\rho,\nu\rangle$.
(A'_k) For $\langle\rho,\nu\rangle\leq k$, $T_0\cap S_\nu$ is non-empty and is pure of dimension $\langle\rho,\nu\rangle$.
The strategy is to show $(P_k)\Longrightarrow(A_k)\Longrightarrow(A'_k)$ and $(A'_k)+(A_k)\Longrightarrow(P_{k+1})$.

$(P_k)\Longrightarrow(A_k)$: because $\overline{T}_0\cap S_\nu$ is a non-empty open subvariety of $\overline{T}_0\cap\overline{S}_\nu$; $(A_k)\Longrightarrow(A'_k)$: if $T_0\cap S_\nu=\emptyset$, then $\overline{T}_0\cap S_\nu=\cup_{0<\nu'\leq\nu}\overline{T}_{\nu'}\cap S_\nu$ has dimension $<\langle\rho,\nu\rangle$, a contradiction, so $T_0\cap S_\nu$ is non-empty, hence is pure of dimension $\langle\rho,\nu\rangle$. Let's show that $(A'_k)+(A_k)\Longrightarrow(P_{k+1})$, so we will set $\langle\rho,\nu\rangle=k+1$. A preliminary observation is that

(*) Every irreducible component of $\overline{T}_0\cap\overline{S}_\nu$ has dimension $\leq k+1$ because $\overline{T}_0\cap(\overline{S}_\nu-S_\nu)$ is the vanishing locus of a section of a line bundle, so the dimension drops by at most one, but every component in $\overline{T}_0\cap(\coprod_{0\leq\nu'<\nu}S_{\nu'})$ has dimension $\leq k$ by (A'_k).

If $k=0$, then $\nu=\alpha_i$ is a simple coroot, $\overline{T}_0\cap\overline{S}_\nu=\{1\}\amalg\{t^{\alpha_i}\}\amalg T_0\cap S_{\alpha_i}$. If $T_0\cap S_{\alpha_i}$ is not empty, then every component has dimension at least one (since it contains a $\mathbb{G}_m$-orbit) and its closure contains $\{1\}$ and $\{t^{\alpha_i}\}$, so it's enough to show that $T_0\cap S_{\alpha_i}\neq\emptyset$. Let $\phi_i:\mathrm{SL}_2\to G$ be the SL_2 associated to the simple coroot α_i, then it suffices to show that $T_0\cap S_\alpha\neq\emptyset$ for the simple coroot α of SL_2, which follows from the equation:

$$\begin{bmatrix}1&0\\t^{-1}&1\end{bmatrix}=\begin{bmatrix}1&-t\\0&1\end{bmatrix}\begin{bmatrix}t&0\\0&t^{-1}\end{bmatrix}\begin{bmatrix}0&-1\\1&t\end{bmatrix}$$

[2] I don't know if similar stratification holds on thick affine Grassmannian Gr_G or not.

If $k>0$, for $x\in\overline{T}_0\cap\overline{S}_\nu$, let $\mathsf{s}(x):=\lim_{z\to 0}\mathrm{Ad}_{2\check{\rho}(z)}(x),\mathsf{t}(x):=\lim_{z\to\infty}\mathrm{Ad}_{2\check{\rho}(z)}(x)$, note that $\mathsf{s}(x),\mathsf{t}(x)\in\{t^{\nu'}\mid 0\le\nu'\le\nu\}$, i.e. the $2\check{\rho}$-torus contracts x to $\mathsf{s}(x)$ and repells x to $\mathsf{t}(x)$. We claim that for all $x\in\overline{T}_0\cap\overline{S}_\nu$ such that either $\mathsf{s}(x)\neq t^\nu$ or $\mathsf{t}(x)\neq 1$, then x lies in an irreducible component of $\overline{T}_0\cap\overline{S}_\nu$ of dimension $\ge k+1$. Assume that $\mathsf{t}(x)\neq 1$ (the other one is similar), then $x\in T_{\mathsf{t}(x)}\cap\overline{S}_\nu$. Observe that there is an $\mathrm{Ad}(G)$-equivariant immersion

$$(\overline{T}_{-\mathsf{t}(x)}\cap S_0)\times(T_0\cap\overline{S}_{\nu-\mathsf{t}(x)})\to\mathrm{Gr}_G$$

induced from the immersion $S_0\times T_0\cong t^{-1}U[t^{-1}]\times t^{-1}U^-[t^{-1}]\to G(\!(t)\!)\to\mathrm{Gr}_G$, and the image of this map is in $\overline{T}_{-\mathsf{t}(x)}\cap\overline{S}_{\nu-\mathsf{t}(x)}$: Let $g\in\overline{T}_{-\mathsf{t}(x)}\cap S_0, g'\in T_0\cap\overline{S}_{\nu-\mathsf{t}(x)}$, then

$$\mathsf{s}(gg')=\lim_{z\to 0}\mathrm{Ad}_{2\check{\rho}(z)}(g)\cdot\mathrm{Ad}_{2\check{\rho}(z)}(g')=\lim_{z\to 0}\mathrm{Ad}_{2\check{\rho}(z)}(g')=\mathsf{s}(g')$$

and similarly $\mathsf{t}(gg')=\mathsf{t}(g)$. It follows from (A_k) that the image of this immersion is pure of dimension $\langle\rho,\mathsf{t}(x)\rangle+\langle\rho,\nu-\mathsf{t}(x)\rangle=\langle\rho,\nu\rangle=k+1$, hence the claim is proven. The claim together with (A_k) implies that

(**) $T_0\cap S_\nu\neq\emptyset$ and every irreducible component of $\overline{T}_0\cap\overline{S}_\nu$ intersects with $T_0\cap S_\nu$ non-trivially.

The last ingredient comes from the proof of [MV07a, Theorem 3.2], in which Mirković and Vilonen actually shown that there is a scheme with support $S_\mu\cap\overline{\mathrm{Gr}}^\lambda$ (resp. $T_\mu\cap\overline{\mathrm{Gr}}^\lambda$) and is locally determined by $\langle\rho,\lambda-\mu\rangle$ (resp. $\langle\rho,\lambda+\mu\rangle$) equations. Now take a large enough λ such that $(T_0\cap\overline{\mathrm{Gr}}^\lambda)\cap(S_\nu\cap\overline{\mathrm{Gr}}^\lambda)=T_0\cap S_\nu$ (non-empty by (**)), so there is a scheme supported on $T_0\cap S_\nu$ and is locally determined by $\langle\rho,2\lambda-\nu\rangle$ equations, thus every irreducible component of $T_0\cap S_\nu$ has dimension $\ge\langle 2\rho,\lambda\rangle-\langle\rho,2\lambda-\nu\rangle=k+1$. This result, together with (*) and (**), implies (P_{k+1}). ■

Corollary 2.9. $\overline{T}_\mu\cap\overline{S}_\nu=\overline{T_\mu\cap S_\nu}$.

Proof. Suppose $Z=\overline{T}_\mu\cap\overline{S}_\nu-\overline{T_\mu\cap S_\nu}$ is not empty, let Z_1 be an irreducible component of $\overline{T}_\mu\cap\overline{S}_\nu-\overline{T_\mu\cap S_\nu}$, then Z_1 is also an irreducible component of $\overline{T}_\mu\cap\overline{S}_\nu$, so $\dim Z_1=\langle\rho,\nu-\mu\rangle$. However,

$$Z_1\subset\left(\coprod_{\mu\le\nu'<\nu}\overline{T}_\mu\cap\overline{S}_{\nu'}\right)\bigcup\left(\coprod_{\mu<\mu'\le\nu}\overline{T}_{\mu'}\cap\overline{S}_\nu\right)$$

whhich has dimension $<\langle\rho,\nu-\mu\rangle$, a contradiction. ■

3. Transverse Slices

Instead of taking $G[\![t^{-1}]\!]$-orbit, we can also consider the $G_1[\![t^{-1}]\!]$-orbit where $G_1[\![t^{-1}]\!]$ is the kernel of evaluation at ∞ map $G[\![t^{-1}]\!]\to G$. Given $\mu\in\Lambda_G$ (note that μ is not required to be dominant), define

$$\mathcal{W}_\mu:=G_1[\![t^{-1}]\!]\cdot t^\mu,\ \mathcal{W}_\mu^\lambda:=\mathcal{W}_\mu\cap\mathsf{Gr}^\lambda,\ \overline{\mathcal{W}}_\mu^\lambda:=\mathcal{W}_\mu\cap\overline{\mathsf{Gr}}^\lambda$$

According to the the modular description in the last subsection, $\mathcal{W}_\mu$ classifies G-torsors $\mathcal{F}_G$ on $\mathbb{P}^1$ with a trivialization $\beta:\mathcal{F}_G^0|_{\mathbb{D}_\infty}\cong\mathcal{F}_G|_{\mathbb{D}_\infty}$, such that $\mathcal{F}_G$ has isomorphism type μ and $\beta|_\infty(\mathcal{F}_{P_\mu}|_\infty)=P_\mu$, where $\mathcal{F}_{P_\mu}$ is the Harder-Narasimhan flag of $\mathcal{F}_G$. With the constrains on the defect type added on, we also have subschemes $\overline{\mathcal{Y}}^\lambda_\mu$ and $\mathcal{Y}^\lambda_\mu$ with their $\mathbb{C}$-points identified with $\overline{\mathcal{W}}^\lambda_\mu$ and $\mathcal{W}^\lambda_\mu$.

The main result in this subsection is the following:

Theorem 3.1. *Let $\mu \in \Lambda_G$, $\mu_0 \in W\mu \cap \Lambda^+_G$ (W is the Weyl group and μ_0 is unique), then $\mathcal{W}_\mu$ is a transverse slice of Gr^{μ_0} in Gr_G, i.e. there is an open subscheme $U \subset \mathsf{Gr}^{\mu_0}$ containing t^μ and an open embedding $j : U \times \mathcal{W}_\mu \hookrightarrow \mathsf{Gr}_G$ such that the diagram commutes:*

$$\begin{tikzcd} U \times \{t^\mu\} \arrow[r] \arrow[d] & U \times \mathcal{W}_\mu \arrow[d, "j"] \\ \mathsf{Gr}^{\mu_0} \times \{t^\mu\} \arrow[r] & \mathsf{Gr}_G \end{tikzcd}$$

Sketch of Proof. Applying a Weyl conjugation on G, we can assume that $\mu = \mu_0 \in \Lambda^+_G$. Let's take a closer look at the schemes Gr^μ and $\mathcal{W}_\mu$. By definition

$$\mathsf{Gr}^\mu = G(\mathcal{O})/G(\mathcal{O}) \cap t^\mu G(\mathcal{O}) t^{-\mu}$$

it has a "big cell" $U = \mathcal{U}/\mathcal{U} \cap t^\mu G(\mathcal{O})t^{-\mu}$ where $\mathcal{U}$ is the preimage of U^-_μ (negative unipotent of P_μ) under the evaluation $G(\mathcal{O}) \to G$. Note that U is isomorphic to $\mathbb{A}^{\langle 2\rho,\mu\rangle}$, and its tangent space at origin is spanned by the image of

$$\bigoplus_{\substack{\alpha_i \in \Phi^+ \\ \langle \alpha_i,\mu\rangle > 0}} \left(\mathbb{C}\cdot f_{\alpha_i} \oplus \mathbb{C}\cdot t f_{\alpha_i} \oplus \cdots \oplus \mathbb{C}\cdot t^{\langle \alpha_i,\mu\rangle -1} f_{\alpha_i}\right)$$

Observe that the $G(\mathcal{O})$-subgroup $V = t^\mu G_1[\![t^{-1}]\!]t^{-\mu} \cap G(\mathcal{O})$ is in fact a subgroup of $\mathcal{U}$ and it intersects with $t^\mu G(\mathcal{O})t^{-\mu}$ trivially and its Lie algebra is exactly the linear space described above. Note that V is also isomorphic to $\mathbb{A}^{\langle 2\rho,\mu\rangle}$. By definiton of the quotient, $V \to U$ is a monomorphism, so it's an open embedding by Zariski Main Theorem. Since V and U are isomorphic to affine space of the same dimension, $V \to U$ is an isomorphism by Ax-Grothendieck theorem.

For $\mathcal{W}_\mu$, the situation is similar. Let $\mathcal{G}$ to be the $G_1[\![t^{-1}]\!]$-subgroup $t^\mu G_1[\![t^{-1}]\!]t^{-\mu} \cap G_1[\![t^{-1}]\!]$, then we claim that $\mathcal{G} \to G_1[\![t^{-1}]\!]/G_1[\![t^{-1}]\!] \cap t^\mu G(\mathcal{O})t^{-\mu} = \mathsf{Gr}_\mu$ is isomorphism, and the proof is similar. Note that we need to truncate $G_1[\![t^{-1}]\!]$ to $G_1(\mathbb{C}[t^{-1}]/t^{-N})$ to apply Zariski Main Theorem and Ax-Grothendieck theorem and then take the $N \to \infty$ limit. Details omitted.

So far we have shown that the following maps are embeddings:

$$V \cdot t^\mu \subset G(\!(t^{-1})\!) \to \mathsf{Gr}_G$$
$$\mathcal{G} \cdot t^\mu \subset G(\!(t^{-1})\!) \to \mathsf{Gr}_G$$

and their images are Gr^μ and $\mathcal{W}_\mu$ respectively. Moreover, the multiplication map $\mathcal{G} \times V \to t^\mu G_1[\![t^{-1}]\!]t^{-\mu}$ is isomorphism (again, apply ZMT and A-G theorem to truncations and take the limit). Since we know that the multiplication map $t^\mu G_1[\![t^{-1}]\!]t^{-\mu} \times t^\mu \to \mathsf{Gr}_G$ is an open embedding (by Birkhoff decomposition), we obtain the desired open embedding as a multiplication map:

$$U \times \mathsf{Gr}_\mu \cong \mathcal{G} \times V \times t^\mu \to \mathsf{Gr}_G$$

The commuativity of the diagram follows from the construction. ∎

It follows from the construction that the we have commutative diagram (assuming $\lambda \geq \mu_0$):

$$\begin{tikzcd}
U \times \{t^\mu\} \arrow[r] \arrow[d] & U \times \overline{\mathcal{W}}^\lambda_\mu \arrow[r] \arrow[d] & U \times \mathcal{W}_\mu \arrow[d] & U \times \mathcal{W}^\lambda_\mu \arrow[l] \arrow[d] \\
\mathsf{Gr}^{\mu_0} \times \{t^\mu\} \arrow[r] & \overline{\mathsf{Gr}}^\lambda \arrow[r] & \mathsf{Gr}_G & \mathsf{Gr}^\lambda \arrow[l]
\end{tikzcd}$$

where all vertical arrows are open embeddings. And we also have a another commutative diagram

$$\begin{tikzcd}
U^-_\mu \times \{t^\mu\} \arrow[r] \arrow[d] & U^-_\mu \times \overline{\mathcal{W}}^\lambda_\mu \arrow[d] & U^-_\mu \times \mathcal{W}^\lambda_\mu \arrow[l] \arrow[d] \\
\mathsf{Gr}^{\mu_0}_{\mu_0} \times \{t^\mu\} \arrow[r] & \overline{\mathsf{Gr}}^\lambda_{\mu_0} & \mathsf{Gr}^\lambda_{\mu_0} \arrow[l]
\end{tikzcd}$$

where all vertical arrows are open embeddings and the middle one is compatible with the contraction map $\mathsf{Gr}^\lambda_{\mu_0} \to G \cdot t^\mu = G/P_\mu$, i.e. the composition $U^-_\mu \times \{t^\mu\} \to U^-_\mu \times \overline{\mathcal{W}}^\lambda_\mu \to \mathsf{Gr}^\lambda_{\mu_0} \to G \cdot t^\mu$ is the open embedding $U^-_\mu \cdot t^\mu$. This shows that the $\overline{\mathsf{Gr}}^\lambda_\mu$ (resp. Gr^λ_μ) is a Zariski-locally trivial fibration over G/P_μ with typical fiber $\overline{\mathcal{W}}^\lambda_\mu$ (resp. $\mathcal{W}^\lambda_\mu$).

Corollary 3.2. *We list some properties of $\mathcal{W}^\lambda_\mu$ and $\overline{\mathcal{W}}^\lambda_\mu$ here:*

1. $\mathcal{W}^\lambda_\mu$ *is non-empty if and only if $\lambda \geq \mu$.*
2. $\mathcal{W}^\lambda_\mu$ *is smooth and connected (if non-empty).*
3. $\overline{\mathcal{W}}^\lambda_\mu$ *is affine, normal, Gorenstein and has rational singularities (if non-empty).*
4. $\dim \mathcal{W}^\lambda_\mu = \dim \overline{\mathcal{W}}^\lambda_\mu = \dim \overline{\mathsf{Gr}}^\lambda - \dim \overline{\mathsf{Gr}}^{\mu_0} = \langle 2\rho, \lambda - \mu_0 \rangle$ *(if non-empty).*

Proof. Since $\lambda \geq \mu \Leftrightarrow \lambda \geq \mu_0$, (1) follows from the corresponding statement for $\mathsf{Gr}^\lambda_{\mu_0}$. Since U is smooth and $U \times \mathcal{W}^\lambda_\mu$ is isomorphic to an open subscheme of Gr^λ, and the latter is smooth and connected, this implies (2). $\overline{\mathcal{W}}^\lambda_\mu$ is a closed subscheme of affine scheme $\mathcal{W}_\mu$ so $\overline{\mathcal{W}}^\lambda_\mu$ is affine, $U \times \overline{\mathcal{W}}^\lambda_\mu$ is isomorphic to an open subscheme of $\overline{\mathsf{Gr}}^\lambda$, and the latter is normal and has rational singularities [Fal03, Theorem 8] and Gorenstein [Zhu14, Theorem 6.11], this implies (3). $\dim \mathcal{W}^\lambda_\mu = \dim \mathsf{Gr}^\lambda_\mu - \dim U = \dim \overline{\mathsf{Gr}}^\lambda - \dim \overline{\mathsf{Gr}}^{\mu_0} = \langle 2\rho, \lambda - \mu_0 \rangle$, and for $\lambda' < \lambda$, $\dim \mathcal{W}^{\lambda'}_\mu < \dim \mathcal{W}^\lambda_\mu$, so $\dim \mathcal{W}^\lambda_\mu = \dim \overline{\mathcal{W}}^\lambda_\mu$, this proves (4). ■

Corollary 3.3. *For $\lambda, \mu \in \Lambda^+_G$, $\lambda \geq \mu$, Gr^λ_μ is smooth and connected and $\overline{\mathsf{Gr}}^\lambda_\mu$ is normal, Gorenstein and has rational singularities, moreover*

$$\dim \mathsf{Gr}^\lambda_\mu = \dim \overline{\mathsf{Gr}}^\lambda_\mu = \langle 2\rho, \lambda - \mu \rangle + \dim G/P_\mu$$

Let's a take a closer look at the stabilizer t^μ in $G_1[\![t^{-1}]\!]$, its Lie algebra is spanned by

$$\bigoplus_{\substack{\alpha_i \in \Phi^+ \\ \langle \alpha_i, \mu \rangle > 0}} \left(\mathbb{C} \cdot t^{-1} f_{\alpha_i} \oplus \mathbb{C} \cdot t^{-2} f_{\alpha_i} \oplus \cdots \oplus \mathbb{C} \cdot t^{-\langle \alpha_i, \mu \rangle} f_{\alpha_i} \right)$$

Suppose μ is chosen such that $\exists n \in \mathbb{Z}_{>0}$ and $\forall \alpha_i \in \Phi^+, \langle \alpha_i, \mu \rangle \geq n$, then the image of the stablizer St_μ in the quotient $G_1[\![t^{-1}]\!] \to G_1[t^{-1}]/(t^{-n})$ equals to $U_1^-[t^{-1}]/(t^{-n})$, this gives rise to a surjective map

$$\pi_{\mu,n} : \mathcal{W}_\mu \to \left(G_1[t^{-1}]/(t^{-n})\right) / \left(U_1^-[t^{-1}]/(t^{-n})\right)$$

Note that the target space is the fiber over the identity element of the projection from the n-th order loop space of G/U^- to G/U^-, so we will denote it by $L_1^n(G/U^-)$. The map $\pi_{\mu,n}$ is $\mathbb{G}_m^{\mathrm{rot}}$-equivariant by construction.

Lemma 3.4. *Same notation as above, then $\forall k < n$, $\pi_{\mu,n}^* : \mathcal{O}(L_1^n(G/U^-)) \to \mathcal{O}(\mathcal{W}_\mu)$ induces isomorphism on homogeneous degree k elements with respect to the $\mathbb{G}_m^{\mathrm{rot}}$-action.*

Proof. Since $\pi_{\mu,n}$ is dominant, $\pi_{\mu,n}^*$ is injective on functions. For the surjectivity, consider the pushout diagram

$$\begin{tikzcd} L_1G \arrow[r] \arrow[d, "\pi_n"'] & L_1G/\mathrm{St}_\mu \arrow[d, "\pi_{\mu,n}"] \\ L_1^nG \arrow[r] & L_1^n(G/U^-) \end{tikzcd}$$

note that all maps are $\mathbb{G}_m^{\mathrm{rot}}$-equivariant, then π_n^* induces isomorphism on homogeneous degree k elements. Moreover, $\forall \phi \in \mathcal{O}(\mathcal{W}_\mu)$ of degree k, the pull-back of ϕ in L_1G comes from a degree k element in L_1^nG, i.e. there are two maps $\mathcal{W}_\mu \to \mathbb{A}^1, L_1^nG \to \mathbb{A}^1$ which agree on L_1G, hence by the property of pushout, it comes form a map $L_1^n(G/U^-) \to \mathbb{A}^1$, in other word, ϕ is in the image of $\pi_{\mu,n}^*$. ∎

There is a relation between different slices: suppose that $\mu, \mu' \in \Lambda_G^+$ are dominant coweights, and $\mu' - \mu \in \Lambda_G^+$ as well, then we have an inclusion

$$G_1[\![t^{-1}]\!] \cap t^\mu G[t] t^{-\mu} \subset G_1[\![t^{-1}]\!] \cap t^{\mu'} G[t] t^{-\mu'}$$

this defines a surjective map $\mathsf{Gr}_\mu \to \mathsf{Gr}_{\mu'}$ by $gt^\mu \mapsto gt^{\mu'}$.

Lemma 3.5. *Same notation as above, and also let $\nu \in \Lambda_G^{\mathrm{pos}}$, i.e. the positive coroot cone. Assume that $\mu + \nu \in \Lambda_G^+$, then the map defined above sends $\overline{\mathcal{W}}_\mu^{\mu+\nu}$ to $\overline{\mathcal{W}}_{\mu'}^{\mu'+\nu}$ and is birational.*

Sketch of Proof. Recall the modular definition of $\overline{\mathsf{Gr}}^\lambda$, now $gt^\mu \in \overline{\mathcal{W}}_\mu^{\mu+\nu}$ has a pole of order $\leq \mu + \nu$, and $t^{\mu'-\mu}$ has a pole of order $\leq \mu' - \mu$, so $gt^{\mu'} = gt^\mu \cdot t^{\mu'-\mu}$ has a pole of order $\leq (\mu + \nu) + (\mu' - \mu) = \mu' + \nu$, hence $gt^{\mu'} \in \overline{\mathcal{W}}_{\mu'}^{\mu'+\nu}$.

In Remark A.16 we construct a birational morphism $s_\mu^{\mu+\nu} : \overline{\mathcal{W}}_\mu^{\mu+\nu} \to \mathcal{Z}^{-w_0\nu}$ (the Zastava space of degree $-w_0\nu$) and it fits into a commutative diagram

$$\begin{tikzcd} \overline{\mathcal{W}}_\mu^{\mu+\nu} \arrow[r] \arrow[dr, "s_\mu^{\mu+\nu}"'] & \overline{\mathcal{W}}_{\mu'}^{\mu'+\nu} \arrow[d, "s_{\mu'}^{\mu'+\nu}"] \\ & \mathcal{Z}^{-w_0\nu} \end{tikzcd}$$

This implies that $\overline{\mathcal{W}}^{\mu+\nu}_{\mu} \to \overline{\mathcal{W}}^{\mu'+\nu}_{\mu'}$ is birational. ■

Example 3.6. As we have seen in Vasya and Anne's talks, slices $\overline{\mathcal{W}}^{\lambda}_{\mu}$ in type A has nice description in terms of Mirković-Vybornov isomorphism [MV07b]. We write down the map in the Lemma 3.5 in the case $G = \mathrm{SL}_2$ for $\mu = n\check{\alpha}$, $\mu' = m\check{\alpha}$ and $\nu = \check{\alpha}$ explicitly:

$$\overline{\mathcal{W}}^{(n+1)\check{\alpha}}_{n\check{\alpha}} = \left\{ \begin{bmatrix} 1 + wt^{-1} + \cdots + w^{n+1}t^{-(n+1)} & ut^{-1} \\ vt^{-(n+1)} & 1 - wt^{-1} \end{bmatrix} \cdot t^{n\check{\alpha}} \;\middle|\; uv + w^{n+2} = 0 \right\}$$

$$\overline{\mathcal{W}}^{(n+1)\check{\alpha}}_{n\check{\alpha}} \to \overline{\mathcal{W}}^{(m+1)\check{\alpha}}_{m\check{\alpha}} : (u, v, w) \mapsto (u, vw^{m-n}, w)$$

In fact the Zastava space $Z^{\check{\alpha}} \cong \mathbb{A}^2$ and the map $s^{(n+1)\check{\alpha}}_{n\check{\alpha}} : (u, v, w) \mapsto (u, w)$.

Example 3.7. Suppose that G is a reductive group, $\mu \in \Lambda^+_G$ and $\check{\alpha}_i$ is a simple coroot, then we have a SL_2 associated with $\check{\alpha}_i$ and a map $\phi_i : \mathrm{SL}_2 \to G$ sending the standard coroot of SL_2 (denoted by $\check{\alpha}$) to $\check{\alpha}_i$. Then $\mathrm{SL}_2[\![t^{-1}]\!]_1$ acts on t^{μ} with stabilizer

$$\mathrm{SL}_2[\![t^{-1}]\!]_1 \cap \phi_i^{-1}(t^{\mu}G[t]t^{-\mu}) = \mathrm{SL}_2[\![t^{-1}]\!]_1 \cap t^{n\check{\alpha}}\,\mathrm{SL}_2[t]t^{-n\check{\alpha}}$$

where $n = \langle \mu, \alpha_i \rangle$, so ϕ_i induces a monomorphism

$$\Phi_i : \mathcal{W}_{\mathrm{SL}_2, n\check{\alpha}} \to \mathcal{W}_{G,\mu}$$

Now assume that $\mu + \check{\alpha}_i \in \Lambda^+_G$, i.e. dominant, then we have the following:

Lemma 3.8. *Φ_i induces an isomorphism*

$$\Phi_i : \overline{\mathcal{W}}^{(n+1)\check{\alpha}}_{\mathrm{SL}_2, n\check{\alpha}} \cong \overline{\mathcal{W}}^{\mu+\check{\alpha}_i}_{G,\mu} \tag{3.1}$$

Proof. We first show that $\Phi_i(\overline{\mathcal{W}}^{(n+1)\check{\alpha}}_{\mathrm{SL}_2, n\check{\alpha}}) \subset \overline{\mathcal{W}}^{\mu+\check{\alpha}_i}_{G,\mu}$. To see why this is the case, we use the Lemma 3.5 to represent elements of a dense open subscheme $U \subset \overline{\mathcal{W}}^{(n+1)\check{\alpha}}_{\mathrm{SL}_2, n\check{\alpha}}$ as $gt^{n\check{\alpha}}$ where $g \in \overline{\mathcal{W}}^{\check{\alpha}}_{\mathrm{SL}_2, 0}$, so $\Phi_i(gt^{n\check{\alpha}}) = \phi_i(g)t^{\mu}$. Since $\phi_i(g)$ has pole $\leq \check{\alpha}_i$ and t^{μ} has pole $\leq \mu$ so we see that $\phi_i(g)t^{\mu}$ has pole $\leq \mu + \check{\alpha}_i$, i.e. $\Phi_i(U) \subset \overline{\mathcal{W}}^{\mu+\check{\alpha}_i}_{G,\mu}$. This is equivalent to $U \subset \Phi_i^{-1}(\overline{\mathcal{W}}^{\mu+\check{\alpha}_i}_{G,\mu})$, but $\Phi_i^{-1}(\overline{\mathcal{W}}^{\mu+\check{\alpha}_i}_{G,\mu})$ is a closed subscheme of $\mathcal{W}_{\mathrm{SL}_2, n\check{\alpha}}$ and U is dense in $\overline{\mathcal{W}}^{(n+1)\check{\alpha}}_{\mathrm{SL}_2, n\check{\alpha}}$, thus we have $\Phi_i(\overline{\mathcal{W}}^{(n+1)\check{\alpha}}_{\mathrm{SL}_2, n\check{\alpha}}) \subset \overline{\mathcal{W}}^{\mu+\check{\alpha}_i}_{G,\mu}$.

Note that $\dim \overline{\mathcal{W}}^{(n+1)\check{\alpha}}_{\mathrm{SL}_2, n\check{\alpha}} = \langle \check{\alpha}, \alpha \rangle = 2$ and $\dim \overline{\mathcal{W}}^{\mu+\check{\alpha}_i}_{G,\mu} = \langle 2\rho, \check{\alpha}_i \rangle = 2$, so Φ_i is a monomorphism between normal varieties of the same dimension, thus it's an open embedding by the ZMT. Since Φ_i is $\mathbb{G}_m^{\mathrm{rot}}$-equivarant, the complement of $\Phi_i(\mathcal{W}_{\mathrm{SL}_2, n\check{\alpha}})$ is a $\mathbb{G}_m^{\mathrm{rot}}$-invaraint closed subscheme, if it's not empty then it must contain t^{μ} (because the action contracts $\overline{\mathcal{W}}^{\mu+\check{\alpha}_i}_{G,\mu}$ to t^{μ}), but that's impossible. So Φ_i is also surjective. ■

It follows from the above Lemma and Mirković-Vybornov isomorphism that $\overline{\mathcal{W}}^{\mu+\check{\alpha}_i}_{G,\mu}$ is a Kleinian singularity $\mathbb{A}^2/(\mathbb{Z}/n+2)$.

Remark 3.9. Although this note is written under the assumption of charateristic zero, all results at this point hold in positive characteristics as well. However, the characteristic zero assumption is essential for later sections (except for the appendix A).

4. Symplectic Structure on Gr_G

Recall the notion of Possion-Lie group [CP95, 1.2]:

Definition 4.1. A Poisson-Lie group is an algebraic group G with a Poisson structure $\{-,-\}$ such that the multiplication $m : G \times G \to G$ is a Poisson map, i.e. push-forward of Poisson bivector on $G\times G$ agrees with the one on G. A Poisson-Lie homomorphism between Poisson-Lie groups G, H is a Lie group homomorphism $G \to H$ such that it's compatible with the Poisson structures.

From the definition we see that there is a Lie bracket on $\mathfrak{g}^*$ defined by

$$[\mathrm{d}f_1|_e, \mathrm{d}f_2|_e] := \mathrm{d}\{f_1, f_2\}|_e$$

and it's compatible with the Lie bracket in the sense that its dual $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ satisfies

$$\delta([X, Y]) = (\mathrm{ad}_X \otimes 1 + 1 \otimes \mathrm{ad}_X)\delta(Y) - (\mathrm{ad}_Y \otimes 1 + 1 \otimes \mathrm{ad}_Y)\delta(X)$$

Such structure $(\mathfrak{g}, \delta)$ is called a Lie bialgebra.

Remark 4.2. The Poisson struture on a Poisson-Lie group is never symplectic, since the compatibility of Poisson bivector Π with multiplication implies that

$$\Pi_{gg'} = R_{g'*}\Pi_g + L_{g*}\Pi_{g'}$$

in particular $\Pi_e = 2\Pi_e = 0$.

A way to produce Lie bialgebras is through Manin triples, let's recall the definition:

Definition 4.3. A Manin triple is a triple of Lie algebras $(\mathfrak{g}, \mathfrak{l}_+, \mathfrak{l}_-)$ with a non-degenerate invariant symmetric bilinear form $(-,-)$ on $\mathfrak{g}$, such that $\mathfrak{l}_\pm$ are isotropic Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{l}_+ \oplus \mathfrak{l}_-$ (as linear space).

From the definition, we see that $\mathfrak{l}_\pm$ are dual to each other through the bilinear form $(-,-)$. Now we can define a Lie cobracket δ_+ on $\mathfrak{l}_+$ by the dual of Lie bracket on $\mathfrak{l}_+$, it's straightforward to check that $(\mathfrak{l}_+, \delta_+)$ is a Lie bialgebra (similar for $(\mathfrak{l}_-, \delta_-)$).

The case we will be interested in is $(\mathfrak{g}(\!(t^{-1})\!), t^{-1}\mathfrak{g}[\![t^{-1}]\!], \mathfrak{g}[t])$, where $\mathfrak{g}$ is a complex semisimple Lie algebra with a Killing form $\langle -,- \rangle$, and the bilinear form on $\mathfrak{g}(\!(t^{-1})\!)$ is defined in term of residue:

$$(f(t), g(t)) := -\mathrm{Res}_{t\to 0}\langle f(t), g(t)\rangle$$

This gives rise to a Lie bialgebra structure on $t^{-1}\mathfrak{g}[\![t^{-1}]\!] = \mathrm{Lie}(G_1[\![t^{-1}]\!])$.

Suppose that G is a algebraic group with subgroups L_+ and L_-, such that $\mathfrak{g} = \mathrm{Lie}(G)$ has a non-degenerate invariant symmetric bilinear form $(-,-)$ and $(\mathfrak{g}, \mathfrak{l}_+, \mathfrak{l}_-)$ forms a Manin triple, then define the *r-matrix* by

$$r = \frac{1}{2}\sum_i e_i^* \wedge e_i \in \wedge^2\mathfrak{g}$$

where $\{e_i\}$ and $\{e_j^*\}$ are basis of $\mathfrak{l}_+$ and $\mathfrak{l}_-$, respectively, such that $(e_i, e_j^*) = \delta_{ij}$. It's easy to see that r doesn't depend on the choice of basis. The r-matrix has the property that the Schouten bracket $[[r,r]] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \in \wedge^3\mathfrak{g}$ is given by

$$([[r,r]], a \wedge b \wedge c) = (a, [b,c]),\ \forall a,b,c \in \mathfrak{g}$$

Define a tensor $\Pi \in \Gamma(G, \wedge^2 T_G)$ by

$$\Pi = \widetilde{r}_R - \widetilde{r}_L$$

where $\widetilde{r}_R$ (resp. $\widetilde{r}_L$) is the right (resp. left) invarant tensor generated by r, in other word, $\Pi_g = L_{g*}(r) - R_{g*}(r)$.

Proposition 4.4 (2.2, 2.3, and 2.9 of [LY08])**.** *Π induces a Poisson structure which makes (G, Π) a Poisson-Lie group, and $L_\pm$ are Poisson-Lie subgroups. Moreover,*

($\mathfrak{g}$) *The induced Lie bialgebra structures on $\mathfrak{g}$ is given by*

$$\delta(X) = (\mathrm{ad}_X \otimes 1 + 1 \otimes \mathrm{ad}_X) r$$

($\mathfrak{l}$) *The induced Lie bialgebra structures on $\mathfrak{l}_\pm$ are $(\mathfrak{l}_+, \delta_+)$ and $(\mathfrak{l}_-, -\delta_-)$.*

The action $G \curvearrowright G/L_-$ induces a map $\wedge^2\mathfrak{g} \to \Gamma(G/L_-, \wedge^2 T_{G/L_-})$ and let Π_{G/L_-} be the image of r, then Π_{G/L_-} is a Poisson structure on G/L_-. Moreover,

(P) *All L_+ and L_- orbits are Poisson subvariaties of G/L_-.*

(S) *Intersections of L_+ and L_- orbits are symplectic leaves of G/L_-.*

Remark 4.5. When we work with group ind-scheme, for example $G(\!(t^{-1})\!)$, the notion of r-matrix needs modification such that it lives in certain completion. For the Manin triple $(\mathfrak{g}(\!(t^{-1})\!), t^{-1}\mathfrak{g}[\![t^{-1}]\!], \mathfrak{g}[t])$, the r-matrix is

$$r = -\sum_{n=0}^{\infty}\sum_a J_a u^n \otimes J_a v^{-n-1} = \frac{\sum_a J_a \otimes J_a}{u - v} \in \mathfrak{g}(\!(u^\pm)\!) \otimes \mathfrak{g}(\!(v^\pm)\!)$$

where $\{J_a\}$ is an orthonormal basis of $\mathfrak{g}$.

Now back to the main setting, we take the triple in Proposition 4.4 to be $(G(\!(t^{-1})\!), G_1[\![t^{-1}]\!], G[t])$ for a semi-simple algebraic group G. From the proposition we deduce that

- There is a natural Poisson structure on Gr_G, and is invariant under the $G(\!(t^{-1})\!)$-action.
- For $\mu \in \Lambda_G$ and $\lambda \in \Lambda_G^+$, $\mathcal{W}_\mu$ and Gr^λ are Poisson subschemes.
- $\mathcal{W}_\mu^\lambda$ are symplectic leaves.

Lemma 4.6. *$\mathsf{S}_\mu \cap \overline{\mathcal{W}}_\mu^\lambda$ and $\mathsf{T}_\mu \cap \overline{\mathcal{W}}_\mu^\lambda$ are coisotropic subvarieties of $\overline{\mathcal{W}}_\mu^\lambda$. Moreover, if $\mu \in \Lambda_G^+$, then*

- *$\overline{\mathsf{Gr}}^\lambda \cap \mathsf{S}_{w_0\mu}$ is a Lagrangian subvariety of $\overline{\mathcal{W}}_{w_0\mu}^\lambda$.*
- *$\overline{\mathsf{Gr}}^\lambda \cap \mathsf{T}_\mu$ is a Lagrangian subvariety of $\overline{\mathcal{W}}_\mu^\lambda$.*

Proof. We prove the S-part, then the T-part follows from a w_0-conjugation.

Note that S_μ is the locus contracting to t^μ under the action of $2\check{\rho}$-torus, and $\overline{\mathcal{W}}_\mu^\lambda$ is affine, so $S_\mu \cap \overline{\mathcal{W}}_\mu^\lambda$ is the closed subvariety which contracts to t^μ, and its defining ideal is generated by all negative weights elements under the action of $2\check{\rho}$-torus. Since the Poisson bracket is invariant under $G(\!(t^{-1})\!)$-action, in particular, the bracket of any pair of elements of negative weights is still of negative weight, hence $S_\mu \cap \overline{\mathcal{W}}^\lambda_\mu$ is coisotropic.

When $\mu \in \Lambda_G^+$, $S_{w_0\mu} = U_1[\![t^{-1}]\!] \cdot U[t] \cdot t^{w_0\mu}$, note that $U[t] \subset t^{w_0\mu} G[t] t^{-w_0\mu}$, so $S_{w_0\mu} = U_1[\![t^{-1}]\!] \cdot t^{w_0\mu} \subset \mathcal{W}_{w_0\mu}$. Hence $\overline{\mathsf{Gr}}^\lambda \cap S_{w_0\mu}$ is a coisotropic subvariety of $\overline{\mathcal{W}}^\lambda_{w_0\mu}$. We also know that $\dim \overline{\mathsf{Gr}}^\lambda \cap S_{w_0\mu} = \langle \rho, \lambda - \mu \rangle$ [Zhu16, Theorem 5.3.9], i.e. half-dimensional, thus the second statement follows from the first one. ∎

5. $\overline{\mathcal{W}}^\lambda_\mu$ Has Symplectic Singularities

Recall that a normal varaity is said to have symplectic singularities if there is a symplectic form Ω on the smooth locus X^{reg}, and locally on X, there are resolutions of singularities $p : V \to U$ such that $p^*\Omega$ extends to U (not required to be non-degenerate). This implies that for all resolution of singlarities $q : Y \to X$, $q^*\Omega$ extends to Y. $q : Y \to X$ is called a **symplectic resolution** if $q^*\Omega$ is non-degenerate. The following general fact on symplectic singularities is proven in [Nam00, Theorem 6].

Theorem 5.1 (Namikawa)**.** *A normal variety X has symplectic singularities if and only if there is a symplectic form Ω on the smooth locus X^{reg}, and X has rational Gorenstein singularities.*

Recall that a normal variety is said to have terminal singularity if the canonical sheaf K_X is $\mathbb{Q}$-factorial and has positive discrepancies, i.e. for a resolution $f : Y \to X$, $K_Y = f^*(K_X) + \sum_i a_i E_i \in \mathrm{Cl}(Y)_{\mathbb{Q}}$ and $\forall i, a_i > 0$. The following facts are related to the terminality of the singularities.

Theorem 5.2 (Namikawa-BCHM)**.** *Suppose that X has symplectic singularities, then X has a $\mathbb{Q}$-factorial terminalization $p : \widetilde{X} \to X$ (i.e. $p^*K_X = K_{\widetilde{X}}$), and if X is conical then $\widetilde{X}$ has a $\mathbb{G}_m$-action and p is equivariant. Moreover the singularity of X is terminal if and only if*

$$\mathrm{codim}_X(\mathrm{Sing}(X)) \geq 4$$

The first statement is in [BCHM10, Corollary 1.4.3], the second statement is in [Nam01, Corollary 1]. By proposition 3.2, $\overline{\mathcal{W}}^\lambda_\mu$ has rational Gorenstein singularities and its smooth locus $\mathcal{W}^\lambda_\mu$ has a symplectic form, thus we have:

Corollary 5.3. *$\overline{\mathcal{W}}^\lambda_\mu$ has symplectic singularities.*

In the work of Kamnitzer *et.al.*, they found a $\mathbb{Q}$-factorial terminalization of $\overline{\mathcal{W}}^\lambda_\mu$, in the case that G is of adjoint type. Their construction is the following: Suppose that $\lambda = \sum_{i=1}^n \lambda_i$, where λ_i are fundamental coweights, then consider the convolution

$$m : \mathrm{Gr}^{\vec{\lambda}} = \mathrm{Gr}^{\lambda_1} \widetilde{\times} \cdots \widetilde{\times} \mathrm{Gr}^{\lambda_n} \subset \overline{\mathrm{Gr}}^{\vec{\lambda}} = \overline{\mathrm{Gr}}^{\lambda_1} \widetilde{\times} \cdots \widetilde{\times} \overline{\mathrm{Gr}}^{\lambda_n} \to \overline{\mathrm{Gr}}^\lambda$$

and define

$$\mathcal{W}^{\vec{\lambda}}_\mu := m^{-1}(\mathcal{W}^\lambda_\mu),\ \overline{\mathcal{W}}^{\vec{\lambda}}_\mu := m^{-1}(\overline{\mathcal{W}}^\lambda_\mu)$$

Then the convolution map $m : \overline{\mathcal{W}}^{\vec{\lambda}}_\mu \to \overline{\mathcal{W}}^{\lambda}_\mu$ is proper birational and Poisson. Since $G_1[\![t^{-1}]\!]$-orbits are tranverse to $G[t]$-orbits, we have equality

$$\mathrm{codim}_{\overline{\mathcal{W}}^{\vec{\lambda}}_\mu}(\mathrm{Sing}(\overline{\mathcal{W}}^{\vec{\lambda}}_\mu)) = \mathrm{codim}_{\overline{\mathrm{Gr}}^{\vec{\lambda}}}(\mathrm{Sing}(\overline{\mathrm{Gr}}^{\vec{\lambda}}))$$

and the latter is at least 4, since for any $\nu < \lambda_i$, $\lambda_i - \nu$ has at least two simple coroots in its decomposition (because $\lambda_i - \alpha_j$ is never dominant), so $\langle 2\rho, \lambda_i - \mu\rangle \geq 4$. By Theorem 5.2, $\overline{\mathcal{W}}^{\vec{\lambda}}_\mu$ has terminal singularities. $\mathbb{Q}$-factoriality and discrepancies are local properties, which boils down to simple cases like $\mathbb{Q}$-factoriality of $\overline{\mathrm{Gr}}^{\lambda_i}$ and discrepancies of minimal resolution of $\overline{\mathcal{W}}^{\lambda}_{\lambda-\alpha_i}$ (Kleinian singularity). In this way they prove that

Proposition 5.4. $m : \overline{\mathcal{W}}^{\vec{\lambda}}_\mu \to \overline{\mathcal{W}}^{\lambda}_\mu$ *is a* $\mathbb{G}_m^{\mathrm{rot}}$*-equivariant* $\mathbb{Q}$*-factorial terminalization.*

In fact, Kamnitzer *et.al.* proved the following finer result concerning with the condition when $m : \overline{\mathcal{W}}^{\vec{\lambda}}_\mu \to \overline{\mathcal{W}}^{\lambda}_\mu$ is a symplectic resolution:

Proposition 5.5. *The following are equivalent:*

(1) $\overline{\mathcal{W}}^{\lambda}_\mu$ *possesses a symplectic resolution.*
(2) $\overline{\mathcal{W}}^{\vec{\lambda}}_\mu$ *is smooth.*
(3) $\mathcal{W}^{\vec{\lambda}}_\mu = \overline{\mathcal{W}}^{\vec{\lambda}}_\mu$.
(4) $\nexists\, \nu_1, \cdots, \nu_n \in \Lambda_G$ *such that* $\nu_1 + \cdots + \nu_n = \mu$*, for all* k*,* ν_k *is a weight of* $\mathcal{V}^{\lambda_k}_{{}^LG}$ *and for some* k*,* ν_k *is not an extremal weight of* $\mathcal{V}^{\lambda_k}_{{}^LG}$*.*

Some explanations: (1)$\Longleftrightarrow$(2) comes from a theorem of Namikawa [Nam11, 5.6] stating that if a variety with symplectic singularities possesses a symplectic resolution, then all $\mathbb{Q}$-factorial terminalizations of it are smooth. (2)$\Longleftrightarrow$(3) is because the smooth locus of $\overline{\mathsf{Gr}}^{\lambda}$ is exactly Gr^{λ}. (3)$\Longrightarrow$(4): Otherwise $(t^{\nu_1}, t^{\nu_1+\nu_2}, \cdots, t^{\mu}) \in \overline{\mathcal{W}}^{\vec{\lambda}}_\mu - \mathcal{W}^{\vec{\lambda}}_\mu$. (4)$\Longrightarrow$(3): Consider the torus $\mathbb{G}_m \to \mathbb{G}_m^{\mathrm{rot}} \times T$ where the first factor is identity and the second factor is 2ρ, then any point $(L_1, L_2, \cdots, L_n)$ contracts to a point $(t^{\mu_1}, t^{\mu_2}, \cdots, t^{\mu})$ under the action of this torus, if $(L_1, L_2, \cdots, L_n) \in \overline{\mathcal{W}}^{\vec{\lambda}}_\mu - \mathcal{W}^{\vec{\lambda}}_\mu$, then $d(\mu_{k-1}, \mu_k) \leq \lambda_k$ for all k and there exists k such that $d(\mu_{k-1}, \mu_k) < \lambda_k$, then the set $(\mu_1, \cdots, \mu_k - \mu_{k-1}, \cdots, \mu - \mu_{n-1})$ is a set $(\nu_1, \cdots, \nu_n)$ in (4). Here $d(\mu, \nu) := (W \cdot (\mu - \nu)) \cap \Lambda_G^+$.

6. Yangian Quantizes $\mathcal{W}_0$

Before we start, let's first rewrite the modular definition of $\overline{\mathsf{Gr}}^{\lambda}$ in terms of equations on $G(\!(t^{-1})\!)$. The condition that

$$\mathcal{V}^{\check{\nu}}_{\mathcal{F}_G}(-\langle \lambda, \check{\nu}\rangle \cdot \{0\}) \longrightarrow \mathcal{V}^{\check{\nu}}_{\mathcal{F}_G} \xrightarrow{\beta} \mathcal{V}^{\check{\nu}}_{\mathcal{F}^0_G}$$

is regular at $\{0\}$ can be checked by writing down the matrix entry: let $v \in \mathcal{V}^{\check{\nu}}$ and $\beta \in (\mathcal{V}^{\check{\nu}})^*$, the matrix element $\Delta_{\beta,v}$ valued at $g \in G(\!(t^{-1})\!)$ is $\langle \beta, gv\rangle$ and can be expanded as power series

$$\Delta_{\beta,v}(g) = \sum_{s=-\infty}^{\infty} \Delta_{\beta,v}^{(s)}(g)t^{-s}$$

and this expression can not have pole of order greater than $\langle \lambda, \check{\nu} \rangle$, meaning that

Proposition 6.1. *Same notation as above, then* $[g] \in \overline{\mathsf{Gr}}^{\lambda}$ *if and only if* $\forall \check{\nu} \in \check{\Lambda}_G^+$, $\forall v \in \mathcal{V}^{\check{\nu}}, \beta \in (\mathcal{V}^{\check{\nu}})^*$ *and* $\forall s > \langle \lambda, \check{\nu} \rangle$, $\Delta_{\beta,v}^{(s)}(g) = 0$.

Remark 6.2. In the definition of $\overline{\mathrm{Gr}}^{\lambda}$, we make a choice of the direction of isomorphism $\beta : \mathcal{F}_G^0 \to \mathcal{F}_G$. Switching the direction corresponds to an involution on Gr_G, sending t^{μ} to $t^{-w_0\mu}$. Denote the element $-w_0\mu$ by μ^*. In this way the notation in [KWWY14] is restored.

Using this notation, we can write down the Poisson structure on $G_1[\![t^{-1}]\!]$ explicitly. We have power series $\Delta_{\beta,v} = \sum_{s=0}^{\infty} \Delta_{\beta,v}^{(s)} u^{-s}$, note that $\Delta_{\beta,v}^{(0)}(g) = \langle \beta, v \rangle$ is a constant function.

Proposition 6.3. *The Poisson bracket* $\{\Delta_{\beta_1,v_1}(u_1), \Delta_{\beta_2,v_2}(u_2)\} \in \mathcal{O}(G_1[\![t^{-1}]\!])[\![u_1^{-1}, u_2^{-1}]\!]$ *equals*

$$\frac{1}{u_1 - u_2} \sum_a \Delta_{\beta_1, J_a v_1}(u_1)\Delta_{\beta_2, J_a v_2}(u_2) - \Delta_{J_a\beta_1, v_1}(u_1)\Delta_{J_a\beta_2, v_2}(u_2) \tag{6.1}$$

Equivalently, the unpacked version is

$$\{\Delta_{\beta_1,v_1}^{(r+1)}, \Delta_{\beta_2,v_2}^{(s)}\} - \{\Delta_{\beta_1,v_1}^{(r)}, \Delta_{\beta_2,v_2}^{(s+1)}\} = \sum_a \Delta_{\beta_1,J_a v_1}^{(r)} \Delta_{\beta_2,J_a v_2}^{(s)} - \Delta_{J_a\beta_1,v_1}^{(r)} \Delta_{J_a\beta_2,v_2}^{(s)} \tag{6.2}$$

Proof. The Poisson bracket valued at g is given by

$$\begin{aligned}\{\Delta_{\beta_1,v_1}(u_1), \Delta_{\beta_2,v_2}(u_2)\}_g &= \langle \mathrm{d}\Delta_{\beta_1,v_1}(u_1)_g \otimes \mathrm{d}\Delta_{\beta_2,v_2}(u_2)_g, \Pi_g \rangle \\ &= \langle \mathrm{d}\Delta_{\beta_1,v_1}(u_1)_g \otimes \mathrm{d}\Delta_{\beta_2,v_2}(u_2)_g, L_{g*}r - R_{g*}r \rangle\end{aligned}$$

and the r-matrix is given by

$$\frac{\sum_a J_a \otimes J_a}{u_1 - u_2}$$

so it equals

$$\begin{aligned}&\frac{1}{u_1 - u_2} \sum_a \left(\langle \beta_1, g(u_1) J_a v_1 \rangle \langle \beta_2, g(u_2) J_a v_2 \rangle - \langle \beta_1, J_a g(u_1) v_1 \rangle \langle \beta_2, J_a g(u_1) v_2 \rangle \right) \\ =&\frac{1}{u_1 - u_2} \sum_a \left(\langle \beta_1, g(u_1) J_a v_1 \rangle \langle \beta_2, g(u_2) J_a v_2 \rangle - \langle J_a\beta_1, g(u_1) v_1 \rangle \langle J_a\beta_2, g(u_1) v_2 \rangle \right) \\ =&\frac{1}{u_1 - u_2} \sum_a \Delta_{\beta_1, J_a v_1}(u_1)\Delta_{\beta_2, J_a v_2}(u_2) - \Delta_{J_a\beta_1, v_1}(u_1)\Delta_{J_a\beta_2, v_2}(u_2)\end{aligned}$$

■

In order to proceed, we need to introduce more notations:

- The Drinfeld generators e_i, f_i, h_i for $\mathfrak{g} = \mathrm{Lie}(G)$ where
$$[h_i, e_j] = b_{ij}e_j,\ [h_i, f_j] = -b_{ij}f_j,\ [e_i, f_j] = \delta_{ij}h_j$$
and symmetric matrix (b_{ij}) is related to the Cartan matrix (a_{ij}) in the sense that $b_{ij} = d_i a_{ij}$ where d_i are coprime positive integers (such d_i is unique).
- The Chevalley generators e'_i, f'_i, h'_i are related to Drinfeld generators by
$$e_i = -d_i^{1/2}e'_i,\ f_i = -d_i^{1/2}f'_i,\ h_i = d_i h_i$$
- A lift of Weyl group is defined via
$$\overline{s}_i = \exp(f'_i)\exp(-e'_i)\exp(f'_i)$$
- If $w_1, w_2 \in W$ and $\tau \in \check{\Lambda}_G^+$, we define
$$\Delta_{w_1\tau, w_2\tau}(g) = \langle \overline{w}_1 v_{-\tau}, \overline{w}_2 v_\tau\rangle$$
where $\overline{w}_i$ is the lift defined above, v_τ is the highest weight vector of $\mathcal{V}^\tau$ and $v_{-\tau}$ is the lowest weight vector of $(\mathcal{V}^\tau)^*$.

The Yangian Y is a $\mathbb{C}[\hbar]$-algebra, with generators $E_i^{(s)}, F_i^{(s)}, H_i^{(s)}$ $(s \geq 1)$ packed in power series
$$E_i(u) = \sum_{s=1}^{\infty} E_i^{(s)}u^{-s},\ H_i(u) = 1 + \sum_{s=1}^{\infty} H_i^{(s)}u^{-s},\ F_i(u) = \sum_{s=1}^{\infty} F_i^{(s)}u^{-s}$$
with a bunch of relations (for type A, see Surya's note). Note that one of these relations is
$$[E_i(u), F_j(v)] = \delta_{ij}\frac{\hbar}{u-v}(H_i(u) - H_i(v)) \tag{6.3}$$

Theorem 6.4. *There is an isomorphism of $\mathbb{N}$-graded Poisson-Hopf algebras $\phi : Y/\hbar Y \cong \mathcal{O}(G_1[\![t^{-1}]\!])$ such that*
$$\phi(H_i(u)) = \prod_j \Delta_{\omega_j,\omega_j}(u)^{-a_{ji}}$$
$$\phi(F_i(u)) = d_i^{-1/2}\frac{\Delta_{\omega_i, s_i\omega_i}(u)}{\Delta_{\omega_i,\omega_i}(u)}$$
$$\phi(E_i(u)) = d_i^{-1/2}\frac{\Delta_{s_i\omega_i, \omega_i}(u)}{\Delta_{\omega_i,\omega_i}(u)}$$
where ω_i are fundamental weights, and $\mathcal{O}(G_1[\![t^{-1}]\!])$ is graded by $\mathbb{G}_m^{\mathrm{rot}}$-action.

7. Shifted Yangian Quantizes $\mathcal{W}_{w_0\mu}$

By definition, $\mathcal{O}(\mathcal{W}_{w_0\mu}) = (\mathcal{O}(G_1[\![t^{-1}]\!]))^{G_1[\![t^{-1}]\!] \cap t^{w_0\mu}G[t]t^{-w_0\mu}}$, and in fact it's a Poisson subalgebra

Proposition 7.1. *$\mathcal{O}(\mathcal{W}_{w_0\mu})$ is Poisson generated by*

- *$\Delta^{(s)}_{s_i\omega_i,\omega_i}$, for all $i \in I$ and $s > 0$;*
- *$\Delta^{(s)}_{\omega_i,\omega_i}$, for all $i \in I$ and $s > 0$;*
- *$(\Delta_{\omega_i,s_i\omega_i}/\Delta_{\omega_i,\omega_i})^{(s)}$, for all $i \in I$ and $s > \langle\mu^*, \alpha\rangle$.*

Definition 7.2. The shifted Yangian Y_μ is the subalgebra of Y generated by $H_i^{(s)}$ for all $i \in I, s > 0$, $E_\alpha^{(s)}$ for all $\alpha \in \Phi^+, s > 0$, and $F_\alpha^{(s)}$ for all $\alpha \in \Phi^+, s > \langle \mu^*, \alpha \rangle$.

Theorem 7.3. *The isomorphism $\phi : Y/\hbar Y \cong \mathcal{O}(G_1[\![t^{-1}]\!])$ restricts to an isomorphism between Possion algebras: $\phi : Y_\mu/\hbar Y_\mu \cong \mathcal{O}(\mathcal{W}_{w_0\mu})$.*

8. Shifted Truncated Yangian and $\overline{\mathcal{W}}^\lambda_{w_0\mu}$

Recall that we have a modular definition of the transverse slice, denoted by $\overline{\mathcal{Y}}^\lambda_\mu$, it is proven in [KMW18] that

Theorem 8.1. *The ideal defining $\overline{\mathcal{Y}}^\lambda_{w_0\mu}$ in $\mathcal{W}_{w_0\mu}$ is Poisson generated by $\Delta^{(s)}_{\omega_i,\omega_i}$, for all $i \in I$ and $s > \langle \lambda - \mu, \omega_i \rangle$.*

On the Yangian side, there is a notion called the shifted truncated Yangian Y^λ_μ constructed as following:

(1) First define a $\mathbb{C}[\![\hbar]\!]$-algebra D^λ_μ with generators $z_{i,k}, (z_{i,k} - z_{i,l})^{-1}, \beta_{i,k}, \beta_{i,k}^{-1}$, for $i \in I$ and $1 \leq k, l \leq m_i := \langle \lambda - \mu, \omega_i \rangle$ and relations that all generators commutes except that $\beta_{i,k} z_{i,k} = (z_{i,k} + d_i\hbar)\beta_{i,k}$.

(2) Let $\lambda_i = \langle \lambda, \alpha_i \rangle$, and fix some numbers $c_i^{(r)} \in \mathbb{C}$ for $i \in I$ and $1 \leq r \leq \lambda_i$. Define polynomials

$$C_i(x) = x^{\lambda_i} + c_i^{(1)} x^{\lambda_i - 1} + \cdots + c_i^{(\lambda_i)},\ Z_i(x) = \prod_{k=1}^{m_i}(x - z_{i,k}),\ Z_{i,k}(x) = \prod_{\ell \neq k}(x - z_{i,\ell})$$

Let $\mu_i = \langle \mu, \alpha_i \rangle$ and set $F_{\mu,i}(u) = \sum_{s=1}^\infty F_i^{s+\mu_i} u^{-s}$ and set

$$r_i(u) = u^{-\lambda_i} C_i(u) \frac{\prod_{j \neq i} \prod_{p=1}^{-a_{ji}} (1 - u^{-1}\hbar(b_{ij}/2 + d_j p))^{m_j}}{(1 - \hbar d_i u^{-1})^{m_i}}$$

(3) Deform the Yangian to $Y(\mathbf{r})$ with the same set of generators but with the relation 6.3 replaced by

$$[E_i(u), F_j(v)] = \delta_{ij} \frac{\hbar}{u - v}(r_i(u)H_i(u) - r_i(v)H_i(v))$$

and similarly for the shifted version $Y_\mu(\mathbf{r})$.

(4) Define the series $A_i(u) = 1 + \sum_{s \geq 1} A_i^{(s)} u^{-s}$ by the equation

$$H_i(u) = \frac{\prod_{j \neq i} \prod_{p=1}^{-a_{ji}} A_j(u - \frac{\hbar}{2}(\alpha_i + p\alpha_j, \alpha_j))}{A_i(u) A_j(u - \frac{\hbar}{2}(\alpha_i, \alpha_i))}$$

Then it follows that there is a map of $\mathbb{C}[\![\hbar]\!]$-algebras $\Psi^\lambda_\mu : Y_\mu(\mathbf{r}) \to D^\lambda_\mu$, called the GKLO representation [GKLO05], defined by

$$\begin{aligned} A_i(u) &\mapsto u^{-m_i} Z_i(u) \\ E_i(u) &\mapsto d_i^{1/2} \sum_{k=1}^{m_i} \frac{\prod_{j\neq i}\prod_{p=1}^{-a_{ji}} Z_j(z_{i,k} - \hbar(b_{ij}/2 + d_j p))}{(u - z_{i,k}) Z_{i,k}(z_{i,k})} \beta_{i,k}^{-1} \\ F_{\mu,i}(u) &\mapsto -d_i^{1/2} \sum_{k=1}^{m_i} C_i(z_{i,k} + \hbar d_i) \frac{\prod_{j\neq i}\prod_{p=1}^{-a_{ji}} Z_j(z_{i,k} - \hbar(b_{ij}/2 - d_i) - \hbar d_j p)}{(u - z_{i,k} - \hbar d_i) Z_{i,k}(z_{i,k})} \beta_{i,k} \end{aligned}$$

and denote the image by $Y^\lambda_\mu(\mathbf{c})$.

Theorem 8.2. *There is an isomorphism of Poisson algebras* $Y^\lambda_\mu(\mathbf{c})/\hbar Y^\lambda_\mu(\mathbf{c}) \cong \mathcal{O}(\overline{\mathcal{Y}}^\lambda_{w_0\mu})$.

Conjecture 8.3. $\overline{\mathcal{Y}}^\lambda_{w_0\mu}$ *is reduced.*

Corollary 8.4. *Assuming the Conjecture 8.3,* $Y^\lambda_\mu(\mathbf{c})$ *is a quantization of* $\overline{\mathcal{W}}^\lambda_{w_0\mu}$.

Remark 8.5. It is shown [KMWY18] that the Conjecture 8.3 is true for type A. This is also true for all simply-laced case (according to Joel).

Appendix A. Zastava Spaces

In this appendix we define the Zastava space, explain some of its basic properties and construct the promised birational morphism $\overline{\mathcal{W}}^{\lambda+\mu}_\lambda \to Z^{-w_0\mu}$ in the Lemma 3.5. We start with the general notion of quasimap space.

Quasimap Space. A general theorem [FGI05, 5.23] asserts that we can make sense of the mapping space $\underline{\mathrm{Map}}(X, Y)$ between a projective variety X and a quasi-projective variety Y, in terms of subvariety of $\underline{\mathrm{Hilb}}_{X\times Y}$. However, (components of) this space is usually not proper, even assuming Y is projective. For instance, let $X = Y = \mathbb{P}^1$, components of $\underline{\mathrm{Map}}(\mathbb{P}^1, \mathbb{P}^1)$ are labelled by degrees, degree one maps are automorphisms, which is PSL_2, so it's not proper.

To compactify the mapping space, the notion of "maps" needs to be extended. One of extension is called the quasimap. The basic observation is the following: suppose $\iota : Y \hookrightarrow \mathbb{P}^n$ is a fixed immersion, then a map $f : X \to Y$ is the same as a sub line-bundle of the rank $n + 1$ trivial bundle on X (determines a map $X \to \mathbb{P}^n$), together with relations restricting the image to land in Y. The essential point is the fuctorial interpertation of $\mathbb{P}^n$.

In this note we will be interested in the flag scheme G/B for a reductive group G. A well-known functorial definition of G/B can be describe as: an S-point of G/B is a T-torsor $\mathcal{F}_T$ with a collection of line subbundles $\kappa^{\check\lambda} : \mathcal{L}^{\check\lambda}_{\mathcal{F}_T} \hookrightarrow \mathcal{O}_S \otimes \mathcal{V}^{\check\lambda}$, where $\check\lambda$ runs through $\check\Lambda^+_G$, $\mathcal{V}^{\check\lambda}$ is the irreducible G-representation of highest weight $\check\lambda$, and $\mathcal{L}^{\check\lambda}_{\mathcal{F}_T}$ is the induced line bundle from the character $\check\lambda : T \to \mathbb{G}_m$. Moreover, they should satisfy the Plücker relations:

- For $\check\lambda = 0$, κ^0 is the identity map $\mathcal{O}_S \to \mathcal{O}_S$.
- For $\check\lambda, \check\mu \in \check\Lambda^+_G$, the diagram commutes

$$\begin{tikzcd} \mathcal{L}^{\check{\lambda}}_{\mathcal{F}_T} \otimes \mathcal{L}^{\check{\mu}}_{\mathcal{F}_T} \arrow[r, "\kappa^{\check{\lambda}} \otimes \kappa^{\check{\mu}}"] \arrow[d] & \mathcal{O}_S \otimes \mathcal{V}^{\check{\lambda}} \otimes \mathcal{V}^{\check{\mu}} \arrow[d] \\ \mathcal{L}^{\check{\lambda}+\check{\mu}}_{\mathcal{F}_T} \arrow[r, "\kappa^{\check{\lambda}+\check{\mu}}"] & \mathcal{O}_S \otimes \mathcal{V}^{\check{\lambda}+\check{\mu}} \end{tikzcd}$$

This motivates the following definition:

Definition A.1. For a base scheme S, a quasimap from $\mathbb{P}^1$ to G/B of degree $\alpha \in \Lambda_G^{\mathrm{pos}}$ (positive coroot cone) consists of a T-torsor $\mathcal{F}_T$ with a collection of maps $\kappa^{\check{\lambda}} : \mathcal{L}^{\check{\lambda}}_{\mathcal{F}_T} \hookrightarrow \mathcal{O}_{\mathbb{P}^1_S} \otimes \mathcal{V}^{\check{\lambda}}$ ($\check{\lambda} \in \check{\Lambda}^+_G$) with respect to the conditions that

(1) Plücker relations are satisfied.
(2) $\kappa^{\check{\lambda}}$ is injective when pull-back to fiber $\mathbb{P}^1_s, \forall s \in S$.
(3) Degree of $\mathcal{L}^{\check{\lambda}}_{\mathcal{F}_T}$ is $-\langle \alpha, \check{\lambda} \rangle$.

The set of quasimaps over S of degree α is denoted by $\underline{\mathcal{QM}}^{\alpha}_S$.

Proposition A.2. *The functor $S \mapsto \underline{\mathcal{QM}}^{\alpha}_S$ is represented by a projective scheme, denoted by $\underline{\mathcal{QM}}^{\alpha}$. Moreover, there is an open embedding $\underline{\mathrm{Map}}^{\alpha}(\mathbb{P}^1, G/B) \hookrightarrow \underline{\mathcal{QM}}^{\alpha}$.*

The quasimap scheme $\underline{\mathcal{QM}}^{\alpha}$ is usually not reduced, we will be focused on its underlying variety $\mathcal{QM}^{\alpha} = (\underline{\mathcal{QM}}^{\alpha})_{\mathrm{red}}$. The following is an easy exercise in deformation theory:

Lemma A.3. *The open subscheme $\mathring{\mathcal{QM}}^{\alpha} := \underline{\mathrm{Map}}^{\alpha}(\mathbb{P}^1, G/B)$ is smooth of dimension* $\dim G/B + \langle 2\rho, \alpha \rangle$.

Proof. The obstruction for deforming a map $f : \mathbb{P}^1 \to G/B$ lives in $\mathrm{H}^1(\mathbb{P}^1, f^*T_{G/B})$, but $T_{G/B}$ is a succesive extension by non-negative line bundles, so $\mathrm{H}^1(\mathbb{P}^1, f^*T_{G/B}) = 0$, whence $\underline{\mathrm{Map}}^{\alpha}(\mathbb{P}^1, G/B)$ is smooth. The dimension is computed by Grothendieck-Riemann-Roch:

$$\begin{aligned} \dim \mathrm{H}^0(\mathbb{P}^1, f^*T_{G/B}) = \chi(f^*T_{G/B}) &= \int_{\mathbb{P}^1} (\dim G/B + f^*c_1(T_{G/B}))(1 + c_1(\mathcal{O}(1))) \\ &= \dim G/B + \sum_{\check{\alpha}_i \in \Phi^+} \langle \check{\alpha}_i, \alpha \rangle \\ &= \dim G/B + \langle 2\rho, \alpha \rangle \end{aligned}$$

■

We will see shortly that $\mathring{\mathcal{QM}}^{\alpha}$ is connected and dense in $\mathcal{QM}^{\alpha}$, so $\mathcal{QM}^{\alpha}$ is an integral variety of dimension $\dim G/B + \langle 2\rho, \alpha \rangle$.

Remark A.4. In the definition of quasimaps, the G-torsor is taken to be trivial, if we allow the G-torsor $\mathcal{F}_G$ to vary, then this is exactly the definiton of $\overline{\mathrm{Bun}}_B$ in [BG99], and there is a pullback diagram

$$\begin{tikzcd} \underline{\mathcal{QM}} \arrow[r] \arrow[d] & \overline{\mathrm{Bun}}_B \arrow[d, "\bar{\mathfrak{p}}"] \\ \mathrm{Spec}\, \mathbb{C} \arrow[r, "\mathrm{triv}"] & \mathrm{Bun}_G \end{tikzcd}$$

Let the open subvariety $\mathcal{Q}^\alpha \subset \mathcal{QM}^\alpha$ be the locus where quasimap defines a map in an open neighborhood of $\infty \in \mathbb{P}^1$, i.e. $\kappa^{\check{\lambda}} : \mathcal{L}^{\check{\lambda}}_{\mathcal{F}_T} \hookrightarrow \mathcal{O}_S \otimes \mathcal{V}^{\check{\lambda}}$ is a line subbundle nearby ∞. Note that the subbundle condition only need to be checked for finitely many $\check{\lambda} \in \check{\Lambda}^+_G$, e.g. a basis of $\check{\Lambda}_G$, since other representions can be generated from tensor product of those basis. It follows form the definition that $(\mathcal{F}_T, \kappa) \mapsto (\mathcal{F}_T|_\infty, \kappa|_\infty)$ (sending a quasimap to the image of ∞) gives rise to a Zariski-locally trivial fibration $\mathcal{Q}^\alpha \to G/B$.

Definition A.5. The Zastava variety $\mathcal{Z}^\alpha$ is the fiber over $[B^-]$ in the aforementioned fibration $\mathcal{Q}^\alpha \to G/B$.

Stratification. We introduce some notations first. Let the set of simple coroots be $\Delta = \{\alpha_1, \alpha_2, \cdots, \alpha_r\}$, and let $\alpha = \sum_{i=1}^r a_i\alpha_i \in \Lambda^{\text{pos}}_G$, a Δ-colored divisor on a smooth proper curve X of multidegree α is an r-tuple of effective divisors $\{D_1, D_2, \cdots, D_r\}$ such that $\deg D_i = a_i$. Apparently, the moduli space of Δ-colored divisors on X of multidegree α is

$$X^\alpha := \prod_{i=1}^r X^{(a_i)}$$

where $X^{(a_i)}$ is the symmetric product X^{a_i}/S_{a_i}. We use $\mathbb{A}^\alpha$ (resp. $\mathbb{P}^\alpha$) to denote $(\mathbb{A}^1)^\alpha$ (resp. $(\mathbb{P}^1)^\alpha$). For a Δ-colored divisor D and a T-torsor $\mathcal{F}_T$, define the twist $\mathcal{F}_T(-D)$ by $\mathcal{F}_T(-D)^{\check{\lambda}} := \mathcal{F}^{\check{\lambda}}_T(-\sum_i \langle \check{\lambda}, \alpha_i\rangle \cdot D_i)$, for $\check{\lambda} \in \check{\Lambda}_G$.

Given a decomposition $\alpha = \beta + \gamma$, $\beta, \gamma \in \Lambda^{\text{pos}}_G$, then there is a proper morphism $\sigma_{\beta,\gamma} : \mathcal{QM}^\beta \times \mathbb{P}^\alpha \to \mathcal{QM}^\alpha$, defined by

$$(\mathcal{F}_T, \kappa) \times D \mapsto (\mathcal{F}_T(-D), \kappa)$$

$\sigma_{\beta,\gamma}$ is a monomorphism because any invertible subsheaf of a free sheaf is determined by its restriction to the open loucs on which it's a line subbundle. Thus $\sigma_{\beta,\gamma}$ is a closed embedding.

Proposition A.6. $\mathcal{QM}^\alpha = \coprod_{0\le\beta\le\alpha} \sigma_{\beta,\alpha-\beta}\left(\mathring{\mathcal{QM}}^\beta \times \mathbb{P}^{\alpha-\beta}\right)$.

Proof. Every invertible subsheaf $\mathcal{L}$ of a locally free sheaf $\mathcal{V}$ is contained in a *unique* line subbundle $\widetilde{\mathcal{L}}$ (called the normalization of $\mathcal{L}$) defined as the preimage of the torsion subsheaf of $\mathcal{V}/\mathcal{L}$ under the quotient map $\mathcal{V} \to \mathcal{V}/\mathcal{L}$. Now for any $\kappa^{\check{\lambda}} : \mathcal{L}^{\check{\lambda}}_{\mathcal{F}_T} \hookrightarrow \mathcal{O}_{\mathbb{P}^1_S} \otimes \mathcal{V}^{\check{\lambda}}$, take the mormalization of $\mathcal{L}^{\check{\lambda}}_{\mathcal{F}_T}$ and we have $\widetilde{\kappa}^{\check{\lambda}} : \widetilde{\mathcal{L}}^{\check{\lambda}}_{\mathcal{F}_T} \hookrightarrow \mathcal{O}_{\mathbb{P}^1_S} \otimes \mathcal{V}^{\check{\lambda}}$, and they satisfy conditions (1) and (2) in the Definition A.1, and the degree is $\beta \le \alpha$ for some β. Hence $(\widetilde{\mathcal{F}}_T, \widetilde{\kappa})$ is a point in $\mathring{\mathcal{QM}}^\beta$, and $\widetilde{\mathcal{L}}^{\check{\lambda}}_{\mathcal{F}_T}/\mathcal{L}^{\check{\lambda}}_{\mathcal{F}_T}$ determines a Δ-colored divisor D, such that $\mathcal{F}_T = \widetilde{\mathcal{F}}_T(-D)$. ∎

Remark A.7. The divisor D determined by $\widetilde{\mathcal{L}}^{\check{\lambda}}_{\mathcal{F}_T}/\mathcal{L}^{\check{\lambda}}_{\mathcal{F}_T}$ is called the **defect** of the quasimap $(\mathcal{F}_T, \kappa)$, it measures the distance of a quasimap from being an actual map.

Corollary A.8. $\mathcal{Z}^\alpha = \coprod_{0\le\beta\le\alpha} \sigma_{\beta,\alpha-\beta}\left(\mathring{\mathcal{Z}}^\beta \times \mathbb{A}^{\alpha-\beta}\right)$.

Remark A.9. The construction in the proposition also imply that

$$\mathcal{QM}^\alpha = \mathring{\mathcal{QM}}^\alpha \cup \bigcup_{\alpha_i \le \alpha} \sigma_{\alpha-\alpha_i,\alpha_i}\left(\mathcal{QM}^{\alpha-\alpha_i} \times \mathbb{P}^1\right)$$

Thus in order to show that $\mathfrak{Q}\mathcal{M}^\alpha$ is irreducible, it's enough to use induction and show that $\mathfrak{Q}\mathring{\mathcal{M}}^\alpha$ is connected and $\Sigma_{\alpha-\alpha_i,\alpha_i} := \sigma_{\alpha-\alpha_i,\alpha_i}\left(\mathfrak{Q}\mathring{\mathcal{M}}^{\alpha-\alpha_i} \times \mathbb{P}^1\right)$ is in the closure of $\mathfrak{Q}\mathring{\mathcal{M}}^\alpha$ for all $\alpha_i \le \alpha$. This will be done in the next subsection.

Convolution. Fix a point $x \in \mathbb{P}^1$, define the **Hecke stack** at x (denoted by $\mathcal{H}_x$) classifying following data: $(\mathcal{F}_G, \mathcal{F}'_G, \beta)$ where $\mathcal{F}_G, \mathcal{F}'_G$ are G-torsors on $\mathbb{P}^1$ and $\beta : \mathcal{F}'_G|_{\mathbb{P}^1 - x} \cong \mathcal{F}_G|_{\mathbb{P}^1-x}$ is an isomorphism. For $\lambda \in \Lambda^+_G$, also define the closed substack $\overline{\mathcal{H}}^\lambda_x$ as those $(\mathcal{F}_G, \mathcal{F}'_G, \beta)$ such that $\forall \check{\lambda} \in \check{\Lambda}^+_G$, $\beta^{\check{\lambda}} : \mathcal{V}^{\check{\lambda}}_{\mathcal{F}_G}(-\langle \check{\lambda}, \lambda\rangle \cdot x) \to \mathcal{V}^{\check{\lambda}}_{\mathcal{F}'_G}$ is regular.

There are two obvious maps $h^{\leftarrow}_G$ and $h^{\rightarrow}_G$ from $\mathcal{H}_x$ to Bun_G: sending $(\mathcal{F}_G, \mathcal{F}'_G, \beta)$ to $\mathcal{F}_G$ and $\mathcal{F}'_G$ respectively. Then $h^{\leftarrow}_G : \overline{\mathcal{H}}^\lambda_x \to \mathrm{Bun}_G$ is an étale-locally trivial $\overline{\mathrm{Gr}}^\lambda$ fibration; and $h^{\rightarrow}_G : \overline{\mathcal{H}}^\lambda_x \to \mathrm{Bun}_G$ is an étale-locally trivial $\overline{\mathrm{Gr}}^{-w_0\lambda}$ fibration. Note that $-w_0$ shows up because here $\mathcal{F}'_G$ plays the role of $\mathcal{F}^0_G$ in the modular definiton of Gr_G, now projection along $h^{\rightarrow}_G$ swap the role of $\mathcal{F}'_G$ and $\mathcal{F}_G$, which is equivalent to replacing β by β^{-1} in the definition of Gr_G, hence sending any representation to its dual, i.e. $\lambda \mapsto -w_0\lambda$. We will use λ^* to denote $-w_0\lambda$.

Let's denote by Conv_x the fiber product $\mathcal{H}_x \times_{\mathrm{Bun}_G} \mathrm{Bun}_B$, and similarly denote by $\overline{\mathrm{Conv}}^\lambda_x$ the fiber product $\overline{\mathcal{H}}^\lambda_x \times_{\mathrm{Bun}_G} \mathrm{Bun}_B$. In other word, Conv_x is the stack classifying $(\mathcal{F}_G, \mathcal{F}'_G, \beta, \mathcal{F}'_T, \kappa')$, where $(\mathcal{F}_G, \mathcal{F}'_G, \beta) \in \mathcal{H}_x$ and $\kappa'^{\check{\lambda}} : \mathcal{L}^{\check{\lambda}}_{\mathcal{F}'_T} \to \mathcal{V}^{\check{\lambda}}_{\mathcal{F}'_G}$ is a collection of line subbundles satisfying Plücker relations.

Construction A.10. *There is a commutative diagram*

$$\begin{tikzcd}
\overline{\mathrm{Bun}}_B \arrow[d, "\bar{\mathfrak{p}}"] & \overline{\mathrm{Conv}}^\lambda_x \arrow[l, "\phi"'] \arrow[r, "\widetilde{h}^{\rightarrow}_G"] \arrow[d, "\mathfrak{p}"] & \mathrm{Bun}_B \arrow[d, "\mathfrak{p}"] \\
\mathrm{Bun}_G & \overline{\mathcal{H}}^\lambda_x \arrow[l, "h^{\leftarrow}_G"'] \arrow[r, "h^{\rightarrow}_G"] & \mathrm{Bun}_G
\end{tikzcd}$$

The right square is the pull-back diagram for defining $\overline{\mathrm{Conv}}^\lambda_x$, and ϕ is defined by

$$(\mathcal{F}_G, \mathcal{F}'_G, \beta, \mathcal{F}'_T, \kappa') \mapsto (\mathcal{F}_G, \mathcal{F}'_T(w_0\lambda \cdot x), \kappa)$$

where $\kappa^{\check{\lambda}}$ is the composition $\mathcal{L}^{\check{\lambda}}_{\mathcal{F}'_T}(\langle\check{\lambda}, w_0\lambda\rangle \cdot x) \xhookrightarrow{\kappa'^{\check{\lambda}}} \mathcal{V}^{\check{\lambda}}_{\mathcal{F}'_G}(\langle\check{\lambda}, w_0\lambda\rangle \cdot x) \xrightarrow{\beta^{\check{\lambda}}} \mathcal{V}^{\check{\lambda}}_{\mathcal{F}_G}$. Note that $\beta^{\check{\lambda}}$ in the second arrow has no pole.

Next, for $\nu \in \Lambda^{\mathrm{pos}}_G$, denote by ${}_{x,\nu}\mathrm{Bun}_B$ the locally closed substack of $\overline{\mathrm{Bun}}_B$ classifying $(\mathcal{F}_G, \mathcal{F}_T, \kappa) \in \overline{\mathrm{Bun}}_B$ with defect $\nu \cdot x$. Define $\overline{\mathrm{Conv}}^\lambda_{x,\nu} := \phi^{-1}({}_{x,\nu}\mathrm{Bun}_B)$, then it follows from the modular description of S_μ in Remark 2.7 that

Lemma A.11. *$\phi : \overline{\mathrm{Conv}}^\lambda_{x,\nu} \to {}_{x,\nu}\mathrm{Bun}_B$ is an étale-locally trivial fibration of typical fiber $\overline{\mathrm{Gr}}^\lambda \cap S_{\nu+w_0\lambda}$, and $\widetilde{h}^{\rightarrow}_G : \overline{\mathrm{Conv}}^\lambda_{x,\nu} \to \mathrm{Bun}_B$ is an étale-locally trivial fibration of typical fiber $\overline{\mathrm{Gr}}^{\lambda^*} \cap S_{-\nu+\lambda^*}$.*

Remark A.12. This is a special case of [BG99, Lemma 3.3.6].

Now we specialize the map $\overline{\mathrm{Bun}}_B \xrightarrow{\bar{\mathfrak{p}}} \mathrm{Bun}_G$ to the quasimap space, as described in the Remark A.4, namely we pull back the whole left square in the Construction A.10 along the trivial bundle $\mathrm{Spec}\,\mathbb{C} \to \mathrm{Bun}_G$, and obtain the following diagram (here we ignore all nilpotents):

$$\mathcal{QM} \xleftarrow{\phi} C_x^\lambda \xrightarrow{\mathsf{p}} \overline{\mathrm{Gr}}^\lambda$$

For $\alpha \in \Lambda_G^{\mathrm{pos}}$, define $C_x^{\alpha,\lambda} := \phi^{-1}(\mathcal{QM}^\alpha)$, then

Lemma A.13. *Restriction of ϕ on the preimage of $\mathring{\mathcal{QM}}^\alpha$ is isomorphism. Moreover assume that $\lambda - \alpha^* \in \Lambda_G^+$, then the image of $\mathsf{p} : C_x^{\alpha,\lambda} \to \overline{\mathrm{Gr}}^\lambda$ is $\overline{\mathrm{Gr}}^\lambda_{\lambda-\alpha^*}$, and p is a Zariski-locally trivial fibration with typical fiber $P_{\lambda-\alpha^*}/B$.*

Sketch of Proof. The first statement is a direct application of Lemma A.11 by setting $\nu = 0$. Assume that $\lambda - \alpha^* \in \Lambda_G^+$, for a point $(\mathcal{F}_T, \kappa)$ in $\mathcal{QM}^\alpha$, and suppose that $(\mathcal{F}_G^{\mathrm{triv}}, \mathcal{F}'_G, \beta, \mathcal{F}'_T, \kappa')$ is a point in its preimage under ϕ, then we have $\deg(\mathcal{F}'_T) = \deg(\mathcal{F}_T) - w_0\lambda = \lambda^* - \alpha$, by Construction A.10. Since $\deg(\mathcal{F}'_T) \in \Lambda_G^+$ by assumption, its induced G-torsor is of type $\deg(\mathcal{F}'_T)^* = \lambda - \alpha^*$, whence the image of $\mathsf{p} : C_x^{\alpha,\lambda} \to \overline{\mathrm{Gr}}^\lambda$ lies in $\overline{\mathrm{Gr}}^\lambda_{\lambda-\alpha^*}$. In fact, $C_x^{\alpha,\lambda}$ classifies $(\mathcal{F}_G^{\mathrm{triv}}, \mathcal{F}'_G, \beta, \mathcal{F}'_T, \kappa')$ such that $(\mathcal{F}'_G, \beta^{-1}) \in \overline{\mathrm{Gr}}^\lambda_{\lambda-\alpha^*}$ and $\mathcal{F}'_T$ has degree $\lambda^* - \alpha$.

Let $P := P_{\lambda-\alpha^*}$, R be its solvable radical and $H := P/R$, consider the intermediate variety $C_{x,P}^{\alpha,\lambda}$ classifying $(\mathcal{F}_G^{\mathrm{triv}}, \mathcal{F}'_G, \beta, \mathcal{F}'_P, \kappa'_P)$ such that $(\mathcal{F}'_G, \beta^{-1}) \in \overline{\mathrm{Gr}}^\lambda_{\lambda-\alpha^*}$, $\mathrm{Ind}_P^H \mathcal{F}'_P$ is trivial, and the R-torsor $\mathcal{F}'_R$ determined by $\mathcal{F}'_P$ has degree $\lambda^* - \alpha$. Then $C_{x,P}^{\alpha,\lambda} \to \overline{\mathrm{Gr}}^\lambda_{\lambda-\alpha^*}$ is isomorphism with inverse map sending $\mathcal{F}'_G$ to its Harder-Narasimhan flag.

Then the map $C_x^{\alpha,\lambda} \to C_{x,P}^{\alpha,\lambda}$ is clear: it's the relative moduli space of choices of reducing the P-torsor $\mathcal{F}'_P$ to a B-torsor of degree $\lambda^* - \alpha$, or equivalently, the choices of reducing the H-torsor $\mathrm{Ind}_P^H\mathcal{F}'_P$ (which is trivial) to a $B(H)$-torsor of degree 0, where $B(H)$ is the Borel of H. This is reprensented by a $H/B(H)$-fibration, or equivalently, a P/B-fibration. ∎

Proposition A.14. $\mathcal{QM}^\alpha$ *is an irreducible variety.*

Proof. By Remark A.9, it suffices to show that $\mathring{\mathcal{QM}}^\alpha$ is connected and $\Sigma_{\alpha-\alpha_i,\alpha_i}$ is in the closure of $\mathring{\mathcal{QM}}^\alpha$ for all $\alpha_i \leq \alpha$. Take any point $(\mathcal{F}_T, \kappa)$ in the stratum $\Sigma_{\alpha-\alpha_i,\alpha_i}$, then it has defect of type $\alpha_i \cdot x$ for some $x \in \mathbb{P}^1$. Now form the convolution diagram at x as above, and choose $\lambda \in \Lambda_G^+$ such that $\lambda - \alpha^* \in \Lambda_G^+$, then Lemma A.13 asserts that $\mathring{\mathcal{QM}}^\alpha$ is in the image of $C_x^{\alpha,\lambda}$ which is an integral variety, whence $\mathring{\mathcal{QM}}^\alpha$ is connected. Moreover, Lemma A.11 also tells us that $(\mathcal{F}_T, \kappa)$ is in the image of ϕ, by taking $\nu = \alpha_i$, thus $(\mathcal{F}_T, \kappa)$ is in the closure of $\mathring{\mathcal{QM}}^\alpha$. This holds for all points in $\Sigma_{\alpha-\alpha_i,\alpha_i}$, whence $\Sigma_{\alpha-\alpha_i,\alpha_i}$ is in the closure of $\mathring{\mathcal{QM}}^\alpha$. ∎

Remark A.15. As an open subvariety of $\mathcal{QM}^\alpha$, $\mathcal{Q}^\alpha$ is irreducible as well, and since $\mathcal{Q}^\alpha \to G/B$ is a locally trivial fibration, it follows that $\mathcal{Z}^\alpha$ is irreducible. This is stated in [FM97, 6.4.3], unfortunately I couldn't follow the argument there.

Remark A.16. Setting $x = 0$, then the image of ϕ is contained in $\mathcal{Q}^\alpha$, and all maps in the diagram are $G \times \mathbb{G}_m^{\mathrm{rot}}$-equivariant

$$\mathcal{Q}^\alpha \xleftarrow{\phi} C_0^{\alpha,\lambda} \xrightarrow{\mathsf{p}} \overline{\mathrm{Gr}}^\lambda_{\lambda-\alpha^*}$$

where G acts on the G-torsor $\mathcal{F}_G$, and $\mathbb{G}_m^{\mathrm{rot}}$ acts on $\mathbb{P}^1$ by scaling. Moreover, the diagram is also compatible with taking flag at ∞ (compare with Remark 2.5), i.e. the following commutes:

$$\begin{tikzcd} \mathcal{Q}^\alpha \arrow[d, "q"] & C_0^{\alpha,\lambda} \arrow[l, "\phi"'] \arrow[r, "\mathsf{p}"] \arrow[d, "q'"] & \overline{\mathrm{Gr}}^\lambda_{\lambda-\alpha^*} \arrow[d] \\ G/B & G/B \arrow[l, "\mathrm{id}"'] \arrow[r, "\mathsf{p}_\infty"] & G\cdot t^{\lambda-\alpha^*} \end{tikzcd}$$

and p_∞ sends $[B^-]$ to $t^{\lambda-\alpha^*}$. In fact, $\mathsf{p} : C_0^{\alpha,\lambda} \to \overline{\mathrm{Gr}}^\lambda_{\lambda-\alpha^*}$ is the pullback of $\mathsf{p}_\infty : G/B \to G\cdot t^{\lambda-\alpha^*} = G/P_{\lambda-\alpha^*}$ along the contraction map $\overline{\mathrm{Gr}}^\lambda_{\lambda-\alpha^*} \to G\cdot t^{\lambda-\alpha^*}$, in other word, a choice of reducing Harder-Narasimhan flag to a Borel flag of degree $\lambda^*-\alpha$ is equivalent to a reduction at ∞, since a reduction of trivial H-torsor to a $B(H)$-torsor of degree zero is completely determined at ∞ (see the notation in the proof of Lemma A.13).

Finally we can construct the birational morphism $s^\lambda_{\lambda-\alpha^*} : \overline{\mathcal{W}}^\lambda_{\lambda-\alpha^*} \to \mathcal{Z}^\alpha$ used in Lemma 3.5. $\mathcal{Z}^\alpha$ is the fiber of $q : \mathcal{Q}^\alpha \to G/B$ at the point $[B^-]$, and the fiber of $q' : C_0^{\alpha,\lambda} \to G/B$ at $[B^-]$ is isomorphic to the fiber of contraction $\overline{\mathrm{Gr}}^\lambda_{\lambda-\alpha^*} \to G\cdot t^{\lambda-\alpha^*}$ at $t^{\lambda-\alpha^*}$, which is exactly $\overline{\mathcal{W}}^\lambda_{\lambda-\alpha^*}$. Identifying $\overline{\mathcal{W}}^\lambda_{\lambda-\alpha^*}$ with $q'^{-1}([B^-])$ and projection along ϕ gives rise to the desired morphism $s^\lambda_{\lambda-\alpha^*}$, it's birational because ϕ is isomorphism when restricted to $\phi^{-1}(\mathring{\mathcal{Z}}^\alpha)$. Note that $s^\lambda_{\lambda-\alpha^*}$ is $\mathbb{G}_m^{\mathrm{rot}}$-equivariant.

Plücker Sections. In this subsection, we rewrite the definition of Zastava space $\mathcal{Z}^\alpha$ in terms of combinatorial data. First of all, let's introduce some notations: Fix a set of highest weight vectors $w_{\check\lambda^*} \in (\mathcal{V}^{\check\lambda})^*$ for all $\check\lambda \in \check\Lambda^+_G$ such that they are compatible with tensor products, i.e. $\omega_0 = 1$, and the image of $\omega_{\check\lambda^*+\check\mu^*}$ under the tensor product map $(\mathcal{V}^{\check\lambda+\check\mu})^* \to (\mathcal{V}^{\check\lambda})^* \otimes (\mathcal{V}^{\check\lambda+})^*$ is $\omega_{\check\lambda^*} \otimes \omega_{\check\mu^*}$, such set can be built up from a basis of $\check\Lambda_{ZG^\circ}$ together with $\check\Delta$.

Let $(\mathcal{F}_T, \kappa)$ be a quasimap of degree α, i.e. a collection of sections $\mathcal{L}^{\check\lambda} \subset \mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{V}^{\check\lambda}$ satisfying Plücker relations. Suppose $(\mathcal{F}_T, \kappa) \in \mathcal{Z}^\alpha$, then $\mathcal{L}^{\check\lambda}$ has no pole at ∞ thus it's represented by a polynomial $f_{\check\lambda} \in \mathcal{V}^{\check\lambda}[t]$ of degree $\langle\check\lambda, \alpha\rangle$ (up to normalization). Moreover, $\mathcal{L}^{\check\lambda}|_\infty \subset \mathcal{V}^{\check\lambda}$ is the reduction to B^- by definition of $\mathcal{Z}^\alpha$, so $\lim_{t\to\infty} f_{\check\lambda}(t)/t^{\langle\check\lambda,\alpha\rangle}$ is a lowest weight vector in $\mathcal{V}^{\check\lambda}$. This is summarized in the following

Lemma-Definition A.17. *$\mathcal{Z}^\alpha$ has the equivalent modular definition by the Plücker sections $f_{\check\lambda}[t] \in \mathcal{V}^{\check\lambda}[t]$, such that for the decomposition $f_{\check\lambda}[t] = v_{-\check\lambda^*}[t] \oplus g_{\check\lambda}[t]$, where $v_{-\check\lambda^*}$ is the lowest weight component of $f_{\check\lambda}$ and $g_{\check\lambda}$ are other weights components, one has*

(1) *$\langle\omega_{\check\lambda^*}, v_{-\check\lambda^*}[t]\rangle$ is a monic polynomial of degree $\langle\check\lambda, \alpha\rangle$.*
(2) $\deg g_{\check\lambda} < \langle\check\lambda, \alpha\rangle$.
(3) *$f_{\check\lambda}[t]$ are compatible with tensor products, i.e. $f_0[t] = 1$, and the image of $f_{\check\lambda} \otimes f_{\check\mu}$ under the tensor product map $\mathcal{V}^{\check\lambda} \otimes \mathcal{V}^{\check\lambda+} \to \mathcal{V}^{\check\lambda+\check\mu}$ is $f_{\check\lambda+\check\mu}$.*

Proof. We have constructed Plücker sections from quasimaps in Zastava space, note that the normalization ambiguity is fixed by the monicity in the condtion (1). The converse is also direct: given a set of Plücker sections $f_{\check\lambda}$, then they generate invertible subsheaves $\mathcal{L}^{\check\lambda}$ of free sheaves $\mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{V}^{\check{\lambda}}$, and $\mathcal{L}^{\check{\lambda}} \subset \mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{V}^{\check{\lambda}}$ obeys the Plücker relations because of condition (3), this shows that $\kappa^{\check{\lambda}} : \mathcal{L}^{\check{\lambda}} \to \mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{V}^{\check{\lambda}}$ is a quasimap of degree α; and $\lim_{t\to\infty} f_{\check{\lambda}}(t)/t^{\langle \check{\lambda},\alpha\rangle}$ is a lowest weight vector in $\mathcal{V}^{\check{\lambda}}$ because of conditions (1) and (2), which implies that the image of ∞ under the quasimap is $[B^-]$, hence the quasimap determined by $f_{\check{\lambda}}[t]$ is in the Zastava $\mathcal{Z}^\alpha$. ■

Proposition A.18. *$\mathcal{Z}^\alpha$ is an affine variety.*

Proof. Using the Plücker section description of $\mathcal{Z}^\alpha$, we see that $\mathcal{Z}^\alpha$ is a closed subscheme of the affine space of coefficients of polynomials in $\mathcal{V}^{\check{\lambda}}$, with defining ideal generated by the conditions (1), (2) and (3), hence $\mathcal{Z}^\alpha$ is an affine scheme. It's also finite type, so $\mathcal{Z}^\alpha$ is an affine variety. ■

Proposition A.19. *Suppose μ is chosen such that $\exists n \in \mathbb{Z}_{>0}$ and $\forall \alpha_i \in \Phi^+, \langle \alpha_i, \mu\rangle \geq n$, then the $\mathbb{G}_m^{\mathrm{rot}}$-equivariant birational morphism $s_\mu^{\mu+\alpha^*} : \overline{\mathcal{W}}_\mu^{\mu+\alpha^*} \to \mathcal{Z}^\alpha$ induces isomorphism on homogeneous elements of degree k, for all $k < n$.*

Proof. Recall that in Lemma 3.4, we have a morphism $\pi_{\mu,n} : \mathcal{W}_\mu \to L_1^n(G/U^-)$ which induces isomorphism on homogeneous elements of degree k, for all $k < n$. We claim that there is a map $q_n^\alpha : \mathcal{Z}^\alpha \to L_1^n(G/U^-)$ which makes the diagram commutes

$$\begin{tikzcd}
\overline{\mathcal{W}}_\mu^{\mu+\alpha^*} \arrow[r, "s_\mu^{\mu+\alpha^*}"] \arrow[dr, "\pi_{\mu,n}"'] & \mathcal{Z}^\alpha \arrow[d, "q_n^\alpha"] \\
 & L_1^n(G/U^-)
\end{tikzcd}$$

Then $\pi_{\mu,n}^* = (s_\mu^{\mu+\alpha^*})^* \circ (q_n^\alpha)^*$ is surjective on homogeneous elements of degree k, for all $k < n$, and $(s_\mu^{\mu+\alpha^*})^*$ is injective because it's birational, whence the isomorphism holds. Let's move on to the proof of the claim.

The map q_n^α is constructed as following: Since $\kappa|_\infty$ gives rise to a canonical reduction from G to B^-, and the G-torsor $\mathcal{F}_G$ is trivial, this equips $\mathcal{F}_T$ with a *canonical* trivialization at ∞, hence it extends to $\mathbb{P}^1$ to canonical isomorphism $\psi : \mathcal{F}_T \cong \mathcal{O}(-\alpha \cdot \{0\})$, and the restriction of ψ on $\mathbb{P}^1 - \{0\}$ is a canonical isomorphism $\psi : \mathcal{F}_T \cong \mathcal{F}_T^{\mathrm{triv}}$. In other word, on the locus $V \subset \mathbb{P}^1$ where the quasimap is an actual map, there exists a unique lift $V - \{0\} \to G/U^-$ such that the image of ∞ is 1 and composition with projection to G/B^- agrees with the quasimap. Since $\infty \in V$, restrction of the map $V - \{0\} \to G/U^-$ to the n'th formal neighborhood of ∞ gives rise to a point in $L_1^n(G/U^-)$. The construction is functorial (i.e. generalizes to S-points for test scheme S), so it defines a map $q_n^\alpha : \mathcal{Z}^\alpha \to L_1^n(G/U^-)$. q_n^α is $\mathbb{G}_m^{\mathrm{rot}}$-equivariant by construction.

We still need to explain the commutativity of the diagram. Observe that the definition of $\pi_{\mu,n}$ has the same flavor of origin as q_n^α: Given a point $(\mathcal{F}_G, \beta) \in \mathsf{Gr}_\mu$, it carries a Harder-Narasimhan flag $\mathcal{F}_{P_\mu}$ and under the framing $\beta : \mathcal{F}_G^0|_{\mathbb{D}_\infty} \cong \mathcal{F}_G|_{\mathbb{D}_\infty}$, $\mathcal{F}_{P_\mu}$ goes to the standard P_μ at ∞, furthermore, $\mathcal{F}_{P_\mu}$ reduces to a Borel $\mathcal{F}_B$ such that under the framing $\beta : \mathcal{F}_G^0|_{\mathbb{D}_\infty} \cong \mathcal{F}_G|_{\mathbb{D}_\infty}$, $\mathcal{F}_B$ goes to the standard B^- at ∞ (this is described in Remark A.16). Run the same argument as the construction of q_n^α, $(\mathcal{F}_G, \beta)$ gives rise to a map $\mathbb{D}_\infty \to G/U^-$, i.e. a point in $L_1(G/U^-)$, and this is functorial thus it defines a map $\pi_\mu : \mathsf{Gr}_\mu \to L_1(G/U^-)$. Moreover, since $G_1[\![t^{-1}]\!]$ acts on Gr_μ by composing with β, it acts on the Borel reduction $\beta^{-1}(\mathcal{F}_B) \subset \mathcal{F}_G^0|_{\mathbb{D}_\infty}$ by composing as well, which is compatible with $G_1[\![t^{-1}]\!]$-action on $L_1(G/U^-)$, thus π_μ is $G_1[\![t^{-1}]\!]$-equivariant. The Borel structure $\beta^{-1}(\mathcal{F}_B)$ determined by t^μ is the constant B^- on $\mathbb{D}_\infty$, so $\pi_\mu(t^\mu) = 1 \in L_1(G/U^-)$, thus π_μ agrees with the natural quotient map $G_1[\![t^{-1}]\!]/\operatorname{St}_\mu \to G_1[\![t^{-1}]\!]/U_1[\![t^{-1}]\!]$. Finally $s_\mu^{\mu+\alpha^*}$ maps the Borel reduction $\beta^{-1}(\mathcal{F}_B)$ to the Borel structure of $(\mathcal{F}_T, \kappa)$ on $\mathbb{D}_\infty$ in the Zastava side, whence the commutativity of diagram follows. ■

Corollary A.20. *$\mathcal{O}(\mathcal{Z}^\alpha) = \varprojlim_{\mu} \mathcal{O}\left(\overline{\mathcal{W}}_\mu^{\mu+\alpha^*}\right)$ and $\mathcal{Z}^\alpha$ is normal.*

Proof. The first statement follows immediately from Lemma 3.5 and Propoistion A.19 as $n \to \infty$. The second statement is the consequence of the first one, if $f(T) \in \mathcal{O}(\mathcal{Z}^\alpha)[T]$ is a monic polynomial with a root x in the field of fractions of $\mathcal{O}(\mathcal{Z}^\alpha)$, then by the normality of $\overline{\mathcal{W}}_\mu^{\mu+\alpha^*}$ (Corollary 3.2), $x \in \mathcal{O}\left(\overline{\mathcal{W}}_\mu^{\mu+\alpha^*}\right)$ for all μ, passing to the limit and we conclude that $x \in \mathcal{O}(\mathcal{Z}^\alpha)$. ■

Appendix B. Limit of Y_μ^λ and Zastava Spaces

Definition B.1. The **Borel Yangian** Y_∞ is the subalgebra of Y geberated by $H_i^{(s)}$ for all $i \in I, s > 0$, $E_\alpha^{(s)}$ for all $\alpha \in \Phi^+, s > 0$.

From the definition of Y_μ, we see that the Borel Yangian is the limit of Y_μ in the direct system $(\Lambda_G^+, \leq)$.

Fix a $\nu \in \Lambda_G^{\mathrm{pos}}$, and dominant $\mu \geq \mu_0$ such that $\mu_0 + \nu$ and $\mu + \nu$ are dominant as well. Then the generators of $Y_\mu^{\mu+\nu}(\mathbf{c})$ is a subset of the generators of $Y_\mu^{\mu_0+\nu}(\mathbf{c})$ and the relations are the same, so there is a map $Y_\mu^{\mu+\nu}(\mathbf{c}) \to Y_\mu^{\mu_0+\nu}(\mathbf{c})$. Moreover it's an isomorphism on the N-th filtered piece if $N \leq \langle \nu, \alpha_i \rangle$ for all $\alpha_i \in \Phi^+$. A direct system is formed in this way and it stabilizes to a quotient of the Borel Yangian Y_∞ by the two-sided ideal $A_i^{(s)}$ for $s > \langle \nu, \alpha_i \rangle$, denoted by $Y_\infty^{\infty+\nu}$.

Combine Corollary A.20 with Corollary 8.4, we obtain

Corollary B.2. *Assuming the Conjecture 8.3, $Y_\infty^{\infty+\nu}$ is a quantization of $\mathcal{Z}^\nu$.*

References

[BCHM10] Caucher Birkar, Paolo Cascini, Christopher D Hacon, and James McKernan. Existence of minimal models for varieties of log general type. *Journal of the American Mathematical Society*, 23(2):405–468, 2010.

[BG99] Alexander Braverman and Dennis Gaitsgory. Geometric eisenstein series. *arXiv preprint math/9912097*, 1999.

[CP95] Vyjayanthi Chari and Andrew N Pressley. *A guide to quantum groups.* Cambridge university press, 1995.

[Fal03] Gerd Faltings. Algebraic loop groups and moduli spaces of bundles. *Journal of the European Mathematical Society*, 5(1):41–68, 2003.

[FGI05] Barbara Fantechi, Lothar Gottsche, and Luc Illusie. *Fundamental algebraic geometry: Grothendieck's FGA explained.* Number 123. American Mathematical Soc., 2005.

[FM97] Michael Finkelberg and Ivan Mirković. Semiinfinite Flags. I. Case of global curve P^1. *arXiv e-prints*, pages alg–geom/9707010, July 1997.

[GKLO05] Anton Gerasimov, Sergei Kharchev, Dimitri Lebedev, and Sergey Oblezin. On a class of representations of the yangian and moduli space of monopoles. *Communications in mathematical physics*, 260(3):511–525, 2005.

[KMW18] Joel Kamnitzer, Dinakar Muthiah, and Alex Weekes. On a reducedness conjecture for spherical schubert varieties and slices in the affine grassmannian. *Transformation Groups*, 23(3):707–722, 2018.

[KMWY18] Joel Kamnitzer, Dinakar Muthiah, Alex Weekes, and Oded Yacobi. Reducedness of affine grassmannian slices in type a. *Proceedings of the American Mathematical Society*, 146(2):861–874, 2018.

[KWWY14] Joel Kamnitzer, Ben Webster, Alex Weekes, and Oded Yacobi. Yangians and quantizations of slices in the affine grassmannian. *Algebra & Number Theory*, 8(4):857–893, 2014.

[LY08] Jiang-Hua Lu and Milen Yakimov. Group orbits and regular partitions of poisson manifolds. *Communications in Mathematical Physics*, 283(3):729–748, 2008.

[MV07a] Ivan Mirković and Kari Vilonen. Geometric langlands duality and representations of algebraic groups over commutative rings. *Annals of mathematics*, 166(1):95–143, 2007.

[MV07b] Ivan Mirković and Maxim Vybornov. Quiver varieties and beilinson-drinfeld grassmannians of type a. *arXiv preprint arXiv:0712.4160*, 2007.

[Nam00] Yoshinori Namikawa. Extension of 2-forms and symplectic varieties. *arXiv preprint math/0010114*, 2000.

[Nam01] Yoshinori Namikawa. A note on symplectic singularities. *arXiv preprint math/0101028*, 2001.

[Nam11] Yoshinori Namikawa. Poisson deformations of affine symplectic varieties. *Duke Mathematical Journal*, 156(1):51–85, 2011.

[Zhu14] Xinwen Zhu. On the coherence conjecture of pappas and rapoport. *Annals of Mathematics*, pages 1–85, 2014.

[Zhu16] Xinwen Zhu. An introduction to affine grassmannians and the geometric satake equivalence. *arXiv preprint arXiv:1603.05593*, 2016.