§ Beilinson–Ginzburg–Soergel Koszul duality

G = connected reductive group$/\mathbb{C}$ $\supset \underset{\cup\atop U}{B} \supset T$, Langlands dual $G^\vee \supset \underset{\cup\atop U^\vee}{B^\vee} \supset T^\vee$ $/\mathbb{C}$

Symplectic duality $T^*(G/B) \longleftrightarrow T^*(G^\vee/B^\vee)$, (principal block):

$\mathcal{O}_0 \simeq P(U\backslash G/B)$ = U-eqvt (i.e. B-constructible) $\mathbb{C}$-perv sheaves on G/B

$G\supset B\supset T/\mathbb{F}_q$ $\quad\uparrow$ For ("degrading functor")

$\mathcal{O}_0^{mix} \simeq P^{mix}(U\backslash G/B)$ = certain U-eqvt mixed $\overline{\mathbb{Q}}_\ell$-perv sheaves on G/B

$\circlearrowleft$ $\langle 1\rangle = (-\frac{1}{2})$ Tate twist

$D^{mix}(U\backslash G/B) := D^b P^{mix}(U\backslash G/B)$

Thm (Koszul duality [BGS]) $\exists$ triangulated equiv.

$$\kappa : D^{mix}(U\backslash G/B) \xrightarrow{\sim} D^{mix}(G^\vee\backslash B^\vee/U^\vee)$$

$$\Delta_w^{mix} \longmapsto \overset{\vee}{\Delta}{}_w^{mix} \qquad \text{"standard sheaves"}$$

$$IC_w^{mix} \longmapsto T_w^{\vee, mix} \qquad \text{"indec. tilting"}$$

$$\vdots$$

$\forall\, w\in W$ = Weyl group

and intertwining

$$[1] \longleftrightarrow [1]$$

$$\langle 1\rangle \longleftrightarrow [1]\langle -1\rangle \qquad (\Rightarrow \text{"mix" is essential})$$

shuffling $\quad Sh_s \longleftrightarrow Tw_s^\vee$

twisting $\quad Tw_s \longleftrightarrow Sh_s^\vee \qquad \forall\, s\in S = \{\text{simple reflections}\}$

$\Rightarrow$ exchanging the commuting braid group actions

§ Monoidal Koszul duality, char 0 [Bezrukavnikov–Yun]

Grothendieck gp: as $\mathbb{Z}[v,v^{-1}]$-modules

$$[D^{mix}(U\backslash G/B)]_\Delta \simeq H = \text{Hecke alg}/\mathbb{Z}[v,v^{-1}]$$

$$[\Delta_w^{mix}] \longleftrightarrow \delta_w \quad \text{"standard basis"}$$

$$[IC_w^{mix}] \longleftrightarrow \text{Kazhdan–Lusztig basis}$$

$$\langle 1\rangle \longleftrightarrow v$$

κ induces involution

$$\iota : H \longrightarrow H$$

$$\delta_w \longmapsto \delta_w$$

$$v \longmapsto -v^{-1} \qquad (\Rightarrow \text{"mix" is essential ... can't set } v=1)$$

In fact, ι is a <u>ring</u> involution!

Q Upgrade κ to monoidal equiv?

$D^{mix}(U\backslash G/B) \circlearrowleft (D^{mix}(B\backslash G/B), \underset{B}{*})$ "Hecke category"

$\quad\wr \qquad\qquad\qquad \wr$

$H \circlearrowleft H$ right regular rep of H

$$\left(D^{mix}(B\backslash G/B), \underset{B}{*}\right) \ni \nabla_s^{mix}$$

$$Sh_s \circlearrowright D^{mix}(U\backslash G/B) \circlearrowleft Tw_s \simeq (-) \underset{B}{*} \nabla_s^{mix}$$

$$\wr \big| \kappa$$

$$\left(D^{mix}(B^\vee\backslash G^\vee/B^\vee), \underset{B^\vee}{*}\right)$$

$$Tw_s^\vee \circlearrowright D^{mix}(B^\vee\backslash G^\vee/U^\vee) \circlearrowleft Sh_s^\vee$$

Seek monoidal category in $\boxed{}$.

<u>Thm</u> (Monoidal Koszul duality [BY])

$\exists$ triangulated monoidal cat $\left(\widehat{D^{mix}(U\backslash G/U)}, \underset{U}{*}\right)$ of <u>free-monodromic cx</u>

(Completion involves "free unipotent monodromy" local systems)

and monoidal equivalence

$$\left(D^{mix}(B\backslash G/B), \underset{B}{*}\right) \xrightarrow{\sim} \left(\widehat{D^{mix}(U^\vee\backslash G^\vee/U^\vee)}, \underset{U}{*}\right)$$

such that BGS Koszul duality is recovered as quotient.

(Even G Kac-Moody)

§ <u>Positive char coeffs: parity cx [Juteau-Mautner-Williamson], mixed modular derived cat [Achar-Riche]</u>

For application to modular rep thry, want BGS Koszul duality with positive char coeff (especially for affine flag var). Need suitable analogue of D^{mix}.

$G \supset B \supset T_{/\mathbb{C}}$, (W, S) Weyl gp, $\Bbbk$ field.

$D^b(B\backslash G/B, \Bbbk)$ = B-eqvt derived cat of sheaves of $\Bbbk$-v.s.

$D^b(U\backslash G/B, \Bbbk)$ = B-constr. ——— " ———

$\{1\}$ = cohomological shift (usually denoted by $[1]$)

$s \in S$, $\pi_s : G/B \longrightarrow G/P^s$ minimal partial flag var

$$\rightsquigarrow \quad D^b(G/B, \Bbbk) \underset{\pi_s^*}{\overset{\pi_{s*}}{\rightleftarrows}} D^b(G/P^s, \Bbbk) \qquad (B\text{-eqvt or } B\text{-constr.})$$

Given $w \in W$, choose reduced expression $\underline{w} = (s_1, \dots, s_{\ell(w)})$,

$$\mathcal{E}_{\underline{w}} := \pi^*_{s_{\ell(w)}} \pi_{s_{\ell(w)}*} \cdots \pi^*_{s_1} \pi_{s_1 *} \underbrace{\Bbbk_{BidB/B}}_{\text{skyscraper at point stratum}} \{\ell(w)\}.$$

$\operatorname{supp} \mathcal{E}_{\underline{w}} = \overline{BwB/B}$ = Schubert var $\overline{X_w}$, $\mathcal{E}|_{X_w} \simeq \Bbbk_{X_w}\{\ell(w)\}$

$\Rightarrow \exists!$ direct summand $\mathcal{E}_w$ (in $D^b(B\backslash G/B, \Bbbk)$ or $D^b(U\backslash G/B, \Bbbk)$) with full support.

Facts - indep of choice of $\underline{w}$

- if char $\Bbbk = 0$, then $\mathcal{E}_w \simeq IC_w$ ($\Rightarrow$ 0-canonical basis = KL basis)
- in general, agrees with parity sheaf (characterized by parity vanishing of stalk/costalk cohomologies)
- $(\mathrm{Parity}(B\backslash G/B, \Bbbk), \overset{*}{B})$ categorifies Hecke alg, class of $\mathcal{E}_w =:$ p-canonical basis, $p = \mathrm{char}\,\Bbbk$.
  $\uparrow$ $\oplus$ of $\{\}$ of $\mathcal{E}_w$'s
- in char 0,

$$\overbrace{K^b \mathrm{SemiSimple}(U\backslash G/B)}^{\oplus \text{ of } \{\} \text{ of } IC_w\text{'s}} \xrightarrow{\sim} D^{mix}(U\backslash G/B)$$
$$[1] \longleftrightarrow [1]$$
$$[1]\{-1\} \longleftrightarrow \langle 1 \rangle$$

Def [Achar-Riche] For any field $\Bbbk$, mixed modular derived category

$D^{mix}(B\backslash G/B, \Bbbk) := K^b \mathrm{Parity}(B\backslash G/B, \Bbbk)$

$D^{mix}(U\backslash G/B, \Bbbk) := K^b \mathrm{Parity}(U\backslash G/B, \Bbbk)$

Tate twist $\langle 1 \rangle := [1]\{-1\}$

Thm [AR] $\exists$ recollement structure $\rightsquigarrow$ perverse t-structure

all defined as complexes of parity complexes! $\Delta_w^{mix}, \nabla_w^{mix}, IC_w^{mix}, \ldots \in P^{mix}(U\backslash G/B, \Bbbk)$ and $P^{mix}(B\backslash G/B, \Bbbk)$

$T_w^{mix} \in P^{mix}(U\backslash G/B, \Bbbk)$ graded highest weight

Thm (Modular Koszul duality [AR]) If $\Bbbk$ is a field whose char is good for G, then $\exists$ triangulated equiv.

$$D^{mix}(U\backslash G/B, \Bbbk) \xrightarrow{\sim} D^{mix}(B^\vee\backslash G^\vee/U^\vee, \Bbbk)$$
$$\Delta_w^{mix} \longmapsto \Delta_w^{\vee, mix}$$
$$\mathcal{E}_w^{mix} \longmapsto T_w^{\vee, mix}$$
$$\vdots \qquad \forall w \in W$$

and intertwining

$$[1] \longleftrightarrow [1]$$
$$\langle 1 \rangle \longleftrightarrow \{1\} = [1]\langle -1 \rangle.$$

§ New definition of free-monodromic cx

Achar-Riche's approach is only for finite flag var. For modular KD for affine or Kac-Moody flag var, want to first show modular analogue of Bezrukavnikov-Yun monoidal KD.
Problem: $D^{mix}(U\backslash G/U)$ involves infinite rank nonsemisimple local system, not obvious how to deal with in $D^{mix} := K^b\mathrm{Parity}$.

Philosophy: replace geometric complication with homological algebra / category theoretic one

Let $G \supset B \supset T_{/\mathbb{C}}$ Kac-Moody, $\Bbbk$ field of char $\neq 2$.

Thm (Achar-M-Riche-Williamson)

$\exists$ category $D^{mix}(U\mathbin{/\!\!/}G\mathbin{/\!\!/}U, \Bbbk)$ of free-monodromic cx, monoidal additive subcat $(\mathrm{Tilt}(U\mathbin{/\!\!/}G\mathbin{/\!\!/}U, \Bbbk), \hat{*})$ of free-monodromic tilting cx.

Thm [AMRW]
$\exists$ additive monoidal equivalence
$$(\mathrm{Parity}(B\backslash G/B, \Bbbk), \underset{B}{*}) \xrightarrow{\sim} (\mathrm{Tilt}(U^\vee \backslash\!\backslash G^\vee /\!/ U^\vee, \Bbbk), \hat{*}).$$

Application (for G = loop group) char formula for tilting modules of reductive gp involving p-Kazhdan-Lusztig polynomials.

"Defn" A free-mon cx is a pair $(\mathcal{F}, \delta)$, $\mathcal{F} = (\mathcal{F}^i)_{i\in\mathbb{Z}}$ is sequence of $\mathcal{F}^i \in \mathrm{Parity}(B\backslash G/B, \Bbbk)$

$\delta \in A \otimes_R \underbrace{\mathrm{End}(\mathcal{F})}_{R\text{-bimodule}} \otimes_{\Bbbk} R^\vee$ satisfying ...

R-bimodule, $R = \mathrm{Sym}^\bullet(\Bbbk \otimes_{\mathbb{Z}} X^*(T)) \simeq \mathbb{H}^\bullet_B(pt)$

A = Koszul cx (resolution of $\Bbbk$ = triv mod of R) by free R-modules),

so $A \otimes_R - \simeq \Bbbk \overset{L}{\underset{R}{\otimes}} -$ homological substitute for passing from B-eqvt to U-eqvt.

$R^\vee$ forces Langlands dual $R^\vee \simeq \mathbb{H}^\bullet_{B^\vee}(pt)$ action on the right.

Ex
$$\widetilde{T}_s^{mix} := \begin{tikzcd} \mathcal{E}_{id} \arrow[loop right] \arrow[d, bend left, "1\otimes\delta\otimes\alpha_s^\vee"] \\ \mathcal{E}_s \arrow[u, "\partial"] \arrow[loop right] \\ \mathcal{E}_{id} \arrow[u, "d"] \arrow[uu, bend left=60, "-\alpha_s\otimes id\otimes 1"] \end{tikzcd}$$

Rmk (1) Showing that $\hat{*}$ is bifunctorial takes ~50 pages, technical rk 2 calculation
(2) monoidal functor defined by generators and rels (Elias-Williamson calculus)

Rmk (In preparation with Hogancamp) Much better proof of bifunctoriality of $\hat{*}$ and more conceptual definition of $D^{mix}(U\backslash\!\backslash G/\!/U, \Bbbk)$: it is the derived cat of A-bimodules in $D^{mix}(B\backslash G/B, \Bbbk)$, where A = Koszul cx, viewed as an algebra object in $D^{mix}(B\backslash G/B, \Bbbk)$.

Conj (derived A-bimodules in $\underbrace{D^{mix}(B\backslash G/B, \Bbbk)}_{\text{Hecke cat.}}) \simeq \underbrace{D^{mix}(B^\vee\backslash G^\vee/B^\vee, \Bbbk)}_{\text{Langlands dual Hecke cat.}}$

§ Soergel bimodules

$G \supset B \supset T_{/\mathbb{C}}$ connected reductive, $\Bbbk$ field of char $\neq 2, 3$.

$W \curvearrowright \mathfrak{h} := \Bbbk \otimes_{\mathbb{Z}} X_*(T) \rightsquigarrow W \curvearrowright R := \mathrm{Sym}_{\Bbbk}(\mathfrak{h}^*)$, $\mathbb{Z}$-graded with $\deg \mathfrak{h}^* = 2$.

$R\text{-gmod-}R$:= $\mathbb{Z}$-graded R-bimodules (and deg 0 R-bim homs)
$\circlearrowleft$
(1) = shift grading down

For $w \in W$, choose reduced exp $\underline{w} = (s_1, \ldots, s_{\ell(w)})$,

$B_{\underline{w}} := R \otimes_{R^{s_1}} \cdots \otimes_{R^{s_{\ell(w)}}} R\,(\ell(w)) \quad \in R\text{-gmod-}R$

$\cup \oplus$

B_w = unique "largest" direct summand

<u>Facts</u> [Soergel]
- indep of $\underline{w}$
- $(\mathrm{SBim}(\mathfrak{h}, W), \otimes_R)$ categorifies Hecke alg
  $\uparrow$ $\oplus$ of ( ) of B_w's
- Monoidal equiv.

$$\mathbb{H}^\bullet : (\mathrm{Parity}(B\backslash G/B, \Bbbk), \overset{\ast}{B}) \xrightarrow{\sim} (\mathrm{SBim}(\mathfrak{h}, W), \otimes_R)$$
$$\mathcal{E}_w \longmapsto B_w$$
$$\{1\} \longleftrightarrow (1)$$

<u>Ex</u> $(G = SL_2)$ $\quad W = S_2 \circlearrowright R = \Bbbk[\alpha_s]$, $\deg \alpha_s = 2$

$s(\alpha_s) = -\alpha_s \qquad R^s = \Bbbk[\alpha_s^2]$

$B_{id} = R$ regular bimodule, $\qquad B_s = R \otimes_{R^s} R\,(1)$.

$B_s \otimes_R B_s \simeq B_s(1) \oplus B_s(-1)$ categorifies quadratic relation in Hecke alg

$$\mathrm{Parity}_B(\mathbb{P}^1, \Bbbk) \xrightarrow{\sim} \mathrm{SBim}(\mathfrak{h}, S_2)$$
$$\mathcal{E}_{id} = \underline{\Bbbk}_{pt} \longmapsto R$$
$$\mathcal{E}_s = \underline{\Bbbk}_{\mathbb{P}^1}\{1\} \longmapsto B_s$$

$\mathrm{End}^\bullet(R) \simeq R$, $\quad \mathrm{Hom}^\bullet(B_s, R) = R\cdot\mathrm{mult}$, $\quad \mathrm{Hom}^\bullet(R, B_s) = R\cdot\mathrm{comult}$.

$\mathrm{End}^\bullet(B_s) = R\cdot id \oplus R\cdot(\mathrm{comult} \circ \mathrm{mult}) \simeq R \oplus R(-2) \simeq B$-equiv cohomology of $\mathbb{P}^1$.

<u>Rmk</u> SBim is not well-behaved (does not categorify the Hecke alg) in general, e.g. for affine flag var and positive char coeff. Must use Parity instead.

§ <u>Ext-enhanced Soergel bimodules</u> [Hogancamp-M]

$\mathrm{SBim} \subset R\text{-gmod-}R \subset D^b(R\text{-gmod-}R)$, $\quad \lceil 1 \rfloor :=$ cohomological shift.
additive $\qquad$ abelian

<u>Def</u> Category $\mathrm{SBim}^{\mathrm{Ext}}$ of <u>Ext-enhanced Soergel bimodules</u>: obj are $\oplus$ of $B\lceil n \rfloor$, $B \in \mathrm{SBim}$, $n \in \mathbb{Z}$. Hom spaces are $\mathbb{Z}\times\mathbb{Z}$-graded, determined by

$$\mathrm{Hom}^{i,j}_{\mathrm{SBim}^{\mathrm{Ext}}}(B\lceil n \rfloor, B'\lceil m \rfloor) := \mathrm{Ext}^{i+m-n}_{R\text{-gmod-}R}(B, B'(j)).$$

(Contains SBim as (nonfull) subcategory, $i = 0$.)

<u>Ex</u> $\mathrm{Hom}^{0,\bullet}_{\mathrm{SBim}^{\mathrm{Ext}}}(B, B') = \mathrm{Hom}^\bullet_{\mathrm{SBim}}(B, B')$ if $B, B' \in \mathrm{SBim}$.

$\mathrm{Hom}^{i,j}_{\mathrm{SBim}^{\mathrm{Ext}}}(R, B) = HH^i(B(j)) = i^{\text{th}}$ Hochschild, $B \in \mathrm{SBim}$

$$\mathrm{End}^{\bullet,\bullet}_{SBim^{Ext}}(R) = \check{\Lambda} \otimes_{\Bbbk} R$$
$\check{\Lambda}$ ← exterior alg in $\mathfrak{h}$, $\deg \mathfrak{h} = (1,-2)$; R ← symm alg in $\mathfrak{h}^*$, $\deg \mathfrak{h}^* = (0,2)$

$\mathrm{End}^{1,-4}(B_s) \simeq \Bbbk$, (one-dim'l self Ext of B_s)

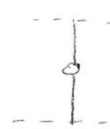
The same homological procedure
$$(\mathrm{Parity}(B\backslash G/B, \Bbbk), \underset{B}{*}) \rightsquigarrow (\mathrm{Tilt}(U\backslash\!\backslash G /\!/ U, \Bbbk), \hat{*})$$
can be repeated starting with $SBim^{Ext}$ to produce
$$(SBim^{Ext}(\mathfrak{h}, W), \underset{R}{\otimes}) \rightsquigarrow (\mathrm{Tilt}^{Ext}(U\backslash\!\backslash G /\!/ U, \Bbbk), \hat{*})$$

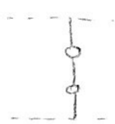
<u>Rmk</u> Expect such defn also for "$\mathrm{Parity}^{Ext}(B\backslash G/B, \Bbbk)$," but $\mathrm{Parity}(B\backslash G/B, \Bbbk)$ does not sit in natural abelian category ...

<u>Conj</u> (Hogancamp – M)

$\exists$ monoidal additive equiv.
$$(\mathrm{Parity}^{Ext}(B\backslash G/B, \Bbbk), \underset{B}{*}) \xrightarrow{\sim} (\mathrm{Tilt}^{Ext}(U^\vee \backslash\!\backslash G^\vee /\!/ U^\vee, \Bbbk), \hat{*})$$
for any symmetrizable Kac-Moody $G_{/\mathbb{C}}$, $\Bbbk$ field of char $\Bbbk \neq 2$. Contains [AMRW] monoidal KD as subcategories.

In [HM], we define Parity^{Ext} for GL_2 using $SBim^{Ext}$, then define such monoidal functor by generator and relations. For this, we give diagrammatic presentation of $SBim^{Ext}$ for GL_2 extending the Elias – Khovanov diagrammatics.

new generators : $\boxed{r}$ $r \in \check{\Lambda}$,  : $B_s \longrightarrow B_s[1](-4)$

new rels :  $= 0$,  $= \boxed{\alpha_s^\vee}$, $\alpha_s^\vee \in \check{\Lambda}$, etc.

Checking relations for GL_2 already involves complicated computations that seem to work out miraculously ...

§ <u>Motivation : symmetry in HOMFLY-PT link homology</u>

HOMFLY-PT homology $HHH^{\bullet\bullet\bullet}(L)$ = $(\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z})$-graded v.s.

via Hecke cat [Khovanov] [Rozansky-Khovanov]

decat (Poincaré polyn $\mathcal{P}_L(a,q,t)$, $\mathcal{P}_L|_{t=-1} = P_L$ up to normalization.)

link L $\xrightarrow[\text{via Hecke alg }(GL_n)\text{ [Jones]}]{}$ HOMFLY-PT polynomial $P_L(a,q)$

Symmetry $P_L(a,q) = P_L(a,q^{-1})$ $\longleftrightarrow$ involution ι of Hecke alg

Preparal

$\boxed{\text{no such symmetry for HOMFLY-PT homology} \xleftrightarrow{\text{related}} \text{no Koszul self-duality of (Ext-enhanced) Hecke category } (GL_n)}$

Indeed,

$L =$ closure of braid $\beta \rightsquigarrow$ Rouquier cx $F_\beta \in K^b SBim \hookrightarrow K^b SBim^{Ext}$

$$HHH^{\cdots}(L) = \bigoplus_{i,j,k} \mathrm{Hom}_{K^b SBim^{Ext}}(R, F_\beta[i](j)\ulcorner k \lrcorner).$$

So Koszul self-duality of $D^{mix, Ext}(B\backslash GL_n/B) := K^b(SBim^{Ext}(GL_n))$ fixing R, F_β would immediately imply symmetry.

Instead, expect Ext-enhanced Koszul duality involving (Langlands-dual) free-monodromic cx.

<u>Rmk</u> Gorsky–Hogancamp defined <u>y-ified Soergel bimodules</u> and conjectured its Koszul self-duality, which would imply symmetry of <u>~~y-Ified~~ homology</u> deforming HHH.

Work in progress with Hogancamp: relation between y-ified and free-monodromic, Koszul duality for y-ified conjectured for arbitrary G.