Mixed geometry, gradings, and Koszulity

3/18/20

§ Varieties over $\mathbb{F}_q$

$X_0/\mathbb{F}_q$, $X = X_0 \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$, $\Gamma = \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) = \langle \sigma \rangle$

$\sigma : a \mapsto a^q$

$X(\overline{\mathbb{F}}_q) = \{ \mathrm{Spec}\, \overline{\mathbb{F}}_q \to X = X_0 \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$

$\Gamma \circlearrowright$ on $X(\overline{\mathbb{F}}_q)$; $\Gamma \leftarrow$ induce same action $\rightarrow \Gamma$

$F_X := \mathrm{id}_{X_0} \otimes \sigma^{-1}$

$\sigma(x_0 : x_1 : \dots : x_n) = (\sigma(x_0) : \dots : \sigma(x_n)) \in \mathbb{P}^n(\overline{\mathbb{F}}_q)$

$\ell \neq p$. A Weil sheaf on X_0 is $\tilde{\mathcal{G}} = (\mathcal{G}, \varepsilon_{\mathcal{G}})$, with $\mathcal{G}$ a $\overline{\mathbb{Q}}_\ell$-sheaf on X, $F_X^* \mathcal{G} \xrightarrow{\varepsilon_{\mathcal{G}}} \mathcal{G}$.

$\tilde{\mathcal{G}}(-m) := (\mathcal{G}, q^m \varepsilon_{\tilde{\mathcal{G}}})$

Eg:

$\tilde{\mathcal{G}} \mapsto (\Gamma(\mathcal{G}), \varepsilon_{\tilde{\mathcal{G}}} \circ F_X^*)$

$\pi_1(\mathrm{Spec}\,\mathbb{F}_q) \cong \hat{\mathbb{Z}}$ $\quad \{\text{Weil shvs}/\mathrm{Spec}\,\mathbb{F}_q\} \xrightarrow{\sim} \{(V, A) \mid V \in \mathrm{Vect}_{\overline{\mathbb{Q}}_\ell}, A \in \mathrm{Aut}(V)\}$

$\updownarrow \wr$

$\pi_1(S^1) \cong \mathbb{Z}$ $\quad \{\overline{\mathbb{Q}}_\ell \text{ local systems on } S^1\}$

In general, $\tilde{\mathcal{G}} \rightsquigarrow H^i(X, \mathcal{G}) \circlearrowleft F^*$.

Eg: $X_0 = \mathbb{A}^1 \setminus \{0\}$ $\qquad 1 \to \mu_\ell \to \mathbb{G}_m \xrightarrow{\ell} \mathbb{G}_m \to 1$

$$0 \to H^0(X, \mathbb{G}_m)/\ell \to H^1(X, \mu_\ell) \to \underbrace{H^1(X, \mathbb{G}_m)[\ell]}_{\mathrm{Pic}\, X = 0} \to 0$$

$H^0(X,\mathbb{G}_m)/\ell \cong \mathcal{O}(X)^\times/\ell \xrightarrow[\mathrm{ord}_0]{\sim} \mathbb{Z}/\ell$

$H^1(X, \mathbb{F}_\ell) \cong H^1(X, \mu_\ell) \otimes_{\mathbb{Z}/\ell} \mu_\ell^* \cong \mu_\ell^* \cong \mathbb{Z}/\ell(-1)$ $\qquad \sigma \overset{q}{\circlearrowright} \mu_\ell \Rightarrow \sigma^{-1} \overset{q}{\circlearrowright} \mu_\ell^*$

$H^1(X, \overline{\mathbb{Q}}_\ell) = \overline{\mathbb{Q}}_\ell(-1)$

Eg: Cycle classes

Excision & Künneth $\Rightarrow H_c^{2n}(\mathbb{A}^n, \overline{\mathbb{Q}}_\ell) \cong \overline{\mathbb{Q}}_\ell(-n)$

X smooth of dim n $\Rightarrow H_c^{2n}(X, \mathbb{Q}_\ell) \cong \overline{\mathbb{Q}}_\ell(-n)$

§ Perverse sheaves and gradings $X = \bigsqcup X_\alpha$ Whitney   ②

$\tilde{\mathcal{L}}_\alpha = i_{\alpha!*} \overline{\mathbb{Q}}_{\ell, X_\alpha}[n_\alpha]$

Thm (Deligne, BBDG): If $\overline{X}_\alpha$ has a resolution, λ is an ev of $F^* \circlearrowright H^j(X, \tilde{\mathcal{L}}_\alpha)$, then $|\tau(\lambda)|^2 = q^{j+n_\alpha}$ $\forall$ ~~...~~ $\tau : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$.

Pf: Decomposition thm & Weil I.

Serre subcat of {perv. Weil shvs} gen by $\tilde{\mathcal{L}}_\alpha(-m)$ = "mixed $\overline{\mathbb{Q}}_\ell$ sheaves"

$w(\tilde{\mathcal{L}}_\alpha(-m)) := -(n_\alpha + 2m)$. $\tilde{\mathcal{F}} \in \widetilde{\mathrm{Perv}}(X)$ is pure of wt m if $\tilde{\mathcal{L}} \in JH(\tilde{\mathcal{F}}) \Rightarrow w(\tilde{\mathcal{L}}) = m$.

Warning: $\widetilde{\mathrm{Perv}}(X) \to \mathrm{Perv}(X)$ is not a grading.

$X_0 = \mathrm{Spec}\, \mathbb{F}_q$ $\mathrm{Ext}^1(\underset{\text{wt } 0}{\overline{\mathbb{Q}}_\ell}, \underset{}{\overline{\mathbb{Q}}_\ell}) \cong \overline{\mathbb{Q}}_\ell$ $F^* = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$

Thm (BBDG): For $\tilde{\mathcal{F}} \in \widetilde{\mathrm{Perv}}(X)$, $\exists!$ increasing filtration $W_\bullet \tilde{\mathcal{F}}$ st $gr^W_m \tilde{\mathcal{F}}$ is pure of weight m.

$\tilde{\mathcal{P}} = \{\tilde{\mathcal{F}} \in \widetilde{\mathrm{Perv}}(X) \mid gr^W_m \tilde{\mathcal{F}} \text{ is semisimple } \forall m\} \overset{\text{ff.}}{\subset} \widetilde{\mathrm{Perv}}(X)$

$\mathcal{P} = \mathrm{Perv}(X)$

$\boxed{\text{Thm}}$: Let $X_0 = G/P$ (argument more general).

Then 1) $\tilde{\mathcal{P}} \to \mathcal{P}$ is a grading

2) $\tilde{\mathcal{P}}$ is Koszul.

Eg: $i : \{0\} \hookrightarrow \mathbb{P}^1$

$\mathcal{O}_0$	$\widetilde{\mathrm{Perv}}(\mathbb{P}^1)$	weight
$L(-2)$	$i_* \overline{\mathbb{Q}}_\ell$	0
$L(0)$	$\overline{\mathbb{Q}}_{\ell, \mathbb{P}^1}[1]$	-1
$L(-2)$	$i_* \overline{\mathbb{Q}}_\ell(-1)$	-2

§ Proof, Part I: Geometric preliminaries

Lemma 1: $\exists$ resolutions $Y_\alpha \xrightarrow{\pi_\alpha} \overline{X}_\alpha$ $\forall \alpha$ st

1) $\pi_\alpha^{-1}(X_\beta) \to X_\beta$ is smooth (fibration) $\forall \beta$

2) $H^*(Y_\alpha, \overline{\mathbb{Q}}_\ell)$ is spanned by algebraic cycles.

Pf: Demazure resolution.

Lemma 2: With conclusion in Lemma 1,

~~$i_\beta^* \tilde{\mathcal{L}}_\alpha$ is constant on X_β, and vanishes if j is odd.~~

$F^* \circlearrowright H^j(X, \mathcal{L}_\alpha)$ acts by $q^{(j+n_\alpha)/2}$.

Pf: Use the resolution $Y_\alpha \to X_\alpha$ and decomposition theorem.

Lemma 3: $\forall j, \alpha, \beta$, $H^j i_\beta^* \tilde{\mathcal{L}}_\alpha \cong \begin{cases} \overline{\mathbb{Q}}_{\ell, X_\beta}\left(\frac{-j-n_\alpha}{2}\right)^{\oplus ?} & \text{if } 2 \mid j+n_\alpha \\ 0 & \text{else} \end{cases}$

Pf: X_α are rad(P)-orbits $\Rightarrow$ $i_\beta^* \mathcal{L}_\alpha$ is constant.

KLparity vanishing

X_β is affine space $\Rightarrow$ $H^j i_\beta^* \mathcal{L}_\alpha = 0$ if $2 \nmid j+n_\alpha$.

$E_2^{p,q} = \bigoplus_{n_\beta = -p} H_c^{p+q}(X_\beta, i_\beta^* \mathcal{L}_\alpha) \Rightarrow H_c^{p+q}(X, \mathcal{L}_\alpha)$

"checkerboard vanishing" $\Rightarrow$ degeneration. Done by Lemma 2. $\square$

§ Proof, Part II: Key computations

Lemma 4: $\forall$ Weil sheaves $\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2$, $\exists$ SES

$$0 \to \mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2[i-1])_{F^\circ} \to \mathrm{Ext}^i(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2) \to \mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2[i])^{F^\circ} \to 0$$

Pf: By $\otimes$-Hom adjunction, shifting, can assume $\tilde{\mathcal{F}}_1 = \tilde{\overline{\mathbb{Q}}}_\ell$ and $i=1$.

$$E_0^{p,q} = \begin{cases} \mathcal{F}_2^p & q=0,1 \\ 0 & \text{else} \end{cases} \Rightarrow R\mathrm{Hom}(\tilde{\overline{\mathbb{Q}}}_\ell, \tilde{\mathcal{F}}_2) \qquad \left(\mathrm{Hom}(\tilde{\overline{\mathbb{Q}}}_\ell, \tilde{\mathcal{F}}_2) = \mathrm{Hom}(\overline{\mathbb{Q}}_\ell, \mathcal{F}_2)^{F^\circ} \quad H^*(\underset{\langle\sigma\rangle}{\overset{\shortparallel}{\mathbb{Z}}}, M) = (M \xrightarrow{1-\sigma} M) \right)$$

$$E_2^{p,q} = \begin{cases} H^p(\mathcal{F}_2)^{F^\circ} & q=0 \\ H^p(\mathcal{F}_2)_{F^\circ} & q=1 \\ 0 & \text{else} \end{cases} \qquad \underline{\text{degenerates}}$$

$\square$

Lemma 5: If $n_\alpha - n_\beta$ is even, then $\mathrm{Ext}^1(\mathcal{L}_\beta, \mathcal{L}_\alpha) = 0$

If $n_\alpha - n_\beta$ is odd, then $F^\circ \curvearrowright \mathrm{Ext}^1(\mathcal{L}_\beta, \mathcal{L}_\alpha)$ acts by $q^{\frac{n_\alpha - n_\beta + 1}{2}}$

Pf: By duality, can assume $X_\alpha \not\subset \overline{X}_\beta \setminus X_\beta$.

SES $0 \to \mathcal{K}_\beta \to \mathcal{M}_\beta \to \mathcal{L}_\beta \to 0$ gives

$$\mathrm{Hom}(\mathcal{K}_\beta, \mathcal{L}_\alpha) \to \mathrm{Ext}^1(\mathcal{L}_\beta, \mathcal{L}_\alpha) \to \mathrm{Ext}^1(\mathcal{M}_\beta, \mathcal{L}_\alpha)$$

=0 since K filtered by Vermas of lower weight

$$\cong H^1(X_\beta, i_\beta^* \mathcal{L}_\alpha[-n_\beta])$$

$$\underset{X_\beta \text{ is an affine space}}{=} H^0(X_\beta, \mathcal{H}^{1-n_\beta} i_\beta^* \mathcal{L}_\alpha)$$

$$\text{112 Lemma 3} \quad \overline{\mathbb{Q}}_{\ell, X_\beta}\left(\frac{n_\beta - n_\alpha - 1}{2}\right)^{\oplus ?}$$

$$\cong \overline{\mathbb{Q}}_\ell\left(\frac{n_\beta - n_\alpha - 1}{2}\right)^{\oplus ?}$$

$\square$

§ Proof, Part III: Final steps

Lemma 6: 1) If $P \in \mathcal{P}$ is indecomp proj, $\exists\, \tilde{P} \in \tilde{\mathcal{P}}$ st $\nu(\tilde{P}) \cong P$.

2) Any such $\tilde{P}$ is proj in $\boxed{\tilde{\mathcal{P}}}$

Pf: 1) $P^i := \mathrm{rad}^i P$. Lift P/P^i inductively.

P/P^1 simple $\rightsquigarrow$ $\widetilde{P/P^1}$ weight w.

$$P^i/P^{i+1} \overset{\text{general}}{\cong} \bigoplus_\alpha \mathrm{Ext}^1(P/P^i, \mathcal{L}_\alpha)^* \otimes \mathcal{L}_\alpha$$

$$\mathrm{Ext}^1(P/P^i, \mathcal{L}_\alpha) \overset{\text{general}}{\hookrightarrow} \mathrm{Ext}^1(P^{i-1}/P^i, \mathcal{L}_\alpha) \circlearrowleft F^* \text{ acts by } q^{\frac{n_\alpha - w + i}{2}}$$

(Lemma 5)

pure wt $w-i+1$ — wt $-n_\alpha$ (usual $\tilde{\mathcal{L}}_\alpha$)

$$\Rightarrow \widetilde{P^i/P^{i+1}} := \bigoplus_\alpha \mathrm{Ext}^1(P/P^i, \mathcal{L}_\alpha)^* \otimes \tilde{\mathcal{L}}_\alpha$$

Class in $\mathrm{Ext}^1(P/P^i, P^i/P^{i+1})$ is F^*-inv ✓ (Lemma 4)

2) $0 \to \mathrm{Hom}(P, \mathcal{L})_{F^*} \to \mathrm{Ext}^1(\tilde{P}, \tilde{\mathcal{L}}) \to \mathrm{Ext}^1(P, \mathcal{L})^{F^*} \to 0$

$\mathrm{Hom}(P,\mathcal{L})_{F^*} = \mathrm{Hom}(P/P^1, \mathcal{L})_{F^*}$ ← nonzero iff $\widetilde{P/P^1} \cong \tilde{\mathcal{L}}$; $\mathrm{Ext}^1(P,\mathcal{L})^{F^*} = 0$

Corresponding extension is not gr^W-semisimple (cf. Warning)

$\Rightarrow \mathrm{Ext}^1_{\tilde{\mathcal{P}}}(\tilde{P}, \tilde{\mathcal{L}}) = 0$. □

Pf of $\boxed{\text{Thm}}$: $\tilde{\mathcal{L}}_1, \tilde{\mathcal{L}}_2 \in \mathrm{Irr}\, \tilde{\mathcal{P}}$ of wts w_1, w_2

WTS $\mathrm{Ext}^i(\tilde{\mathcal{L}}_1, \tilde{\mathcal{L}}_2) = 0$ unless $i = w_1 - w_2$.

$$\begin{tikzcd}
\mathrm{Ext}^i_{\tilde{\mathcal{P}}}(\tilde{\mathcal{L}}_1, \tilde{\mathcal{L}}_2) \arrow[r, hook, "\text{Lemma 6}"] \arrow[d] & \mathrm{Ext}^i(\mathcal{L}_1, \mathcal{L}_2) \\
\mathrm{Ext}^i(\tilde{\mathcal{L}}_1, \tilde{\mathcal{L}}_2) \arrow[r, two heads, "\text{Lemma 4}"] & \mathrm{Ext}^i(\mathcal{L}_1, \mathcal{L}_2)^{F^*} \arrow[u, hook]
\end{tikzcd}$$

$\Rightarrow$ Suffices to show $\mathrm{Ext}^i(\mathcal{L}_1, \mathcal{L}_2)^{F^*} = 0$ unless $i = w_1 - w_2$.

$$\mathrm{Ext}^i(\mathcal{L}_1, \mathcal{L}_2)^{F^*} \overset{\text{parity vanish}}{\cong} \bigoplus_j \mathrm{Hom}_{\bar{\mathbb{Q}}_\ell}(H^j(\mathcal{L}_1), H^{j+i}(\mathcal{L}_2))^{F^*}$$

$\circlearrowleft\ \lambda \in F^*\text{-ev}$ $\quad$ $\circlearrowleft\ \mu \in F^*\text{-ev}$

Thm $\Rightarrow$ $|\tau(\lambda)|^2 = q^{j+w_1}$ $\qquad$ $|\tau(\mu)|^2 = q^{j+i+w_2}$ □