- M_0 affine, normal, Poisson
- M a res^{n} of sing., $\pi^*\{,\}$ sympl.

Let $M \xrightarrow{\pi} M_0$ a <u>conical</u> sympl. resolution.

$\mathbb{C}^\times_\hbar \curvearrowright M, M_0$, π equivariant, $M_0^{\mathbb{C}^\times} = pt$, $\{,\}$ wt 1.

Def^{n}: A quantization Q of M is
- A sheaf of associative $\mathbb{C}[[\hbar]]$ algebras Q over M.
- an iso^{m} $Q/\hbar Q \xrightarrow{\simeq} \mathcal{O}_M$. intertwining Poisson strs.

Th^{m}: $\{\text{Quantizations of } M\} \xrightarrow{\simeq} \{[\Omega]\} \oplus \hbar H^2(M;\mathbb{C})[[\hbar]]$.

Def^{n}: A quantization is $\mathbb{C}^\times_\hbar$-equivariant if σ is $\mathbb{C}^\times$-equivariant, where we let $\mathbb{C}^\times_\hbar \curvearrowright H^2(M)$ wt 1.

$= \{\sigma : D_\hbar \xleftarrow{\longleftarrow} D_\hbar \times H^2(M)\}$, $\sigma(t) = pt \times \Omega$

Cor^{y}: $\{\mathbb{C}^\times_\hbar\text{-equivariant quantizations}\} \xrightarrow{\simeq} H^2(M;\mathbb{C})$.

Fix a quantization $Q = Q_\lambda$, $\lambda \in H^2(M)$.

Let $D = Q[\hbar^{-1}] \supset D(m) = \hbar^{-m} Q$ (in gen. $n \in \mathbb{N}$)

$A = \Gamma(M, D)^{\mathbb{C}^\times_\hbar} \supset A(m) = \Gamma(M, D(m))^{\mathbb{C}^\times}$

$$\boxed{\begin{array}{l} D(0) = Q \\ D(-1) = \hbar Q \\ D(0)/D(-1) = \mathcal{O}_M \end{array}}$$

$A_\bullet$: a filtered algebra with $\mathrm{gr}_\bullet A = \mathcal{O}_{M_0}$

(For M affine, this is just the Rees construction from last time)

<u>Warning</u>: BLPW notationally preferences - / $\mathbb{C}[[\hbar]]$ picture on M
- Filtered picture on M_0.

I'll follow this convention.

<u>Example</u>: $M = T^*G/B \xrightarrow{\pi} \mathcal{N} = M_0$. $\lambda \in H^2(M) = \mathfrak{h}^\vee$ a wt.

$Q_\lambda = D^\lambda_M$ = microlocalization of D^λ_X for $X = G/B$.

$A_\bullet = U_\lambda \mathfrak{g}$ as a filtered algebra.

Let $D\text{-Mod} = \{N$ a sheaf of $\mathbb{C}^\times_\hbar$-equivariant D-mod's $/M$ | $\exists\, N(0) \subset N$ a $\mathbb{C}^\times_\hbar$-eq., coherent $D(0)$ lattice$\}$

("good filtration")

Def^{n}: Let $N(m) = \hbar^{-m} N(0)$. $N(0)/N(-1) \in \mathcal{O}_M\text{-Mod}$ is called the classical limit of N; analoge of "sing. support".

N is called holonomic if $\mathrm{supp}(N(0)/N(-1))$ is Lagrangian.

Let $\Gamma_\hbar = \Gamma(M, -)^{\mathbb{C}^\times_\hbar} : D\text{-Mod} \longrightarrow A\text{-Mod}$ (finitely generated)

and $\mathrm{Loc}_M : A\text{-Mod} \longrightarrow D\text{-Mod}$ its left adjoint.

Th^{m}: For generic $\lambda \in H^2(M)$, these are mutually inverse derived equivalences. For "positive" λ, they are exact.

If $\lambda - \mu$ is integral, $D\text{-Mod}_\lambda \xrightarrow{\simeq} D\text{-Mod}_\mu$

$\downarrow \qquad\qquad\qquad \downarrow\simeq$

$D\text{-Mod}_\mu \xrightarrow{\simeq} A\text{-Mod}_\mu$

Let
$$\begin{array}{ccc} L & \longrightarrow\!\!\!\!\to & L_0 \\ \cap & & \cap \\ M & \longrightarrow\!\!\!\!\to & M_0 \end{array}$$

$A\text{-Mod}_{L_0}$ = full subcategory s.t. $\mathrm{supp}(\mathrm{gr}_\bullet N) \subset L_0$.

$D\text{-Mod}_L$ = full subcategory s.t. $\mathrm{supp}(N(0)/N(-1)) \subset L$.

Then for λ as above

$\Gamma_\hbar : D\text{-Mod}_L \underset{}{\overset{\simeq}{\rightleftarrows}} A\text{-Mod}_{L_0} : \mathrm{Loc}$.

$\mathbb{R}\Gamma_\hbar : D^b_L(D\text{-Mod}) \overset{\simeq}{\rightleftarrows} D^b_{L_0}(A\text{-Mod}) : \mathbb{L}\,\mathrm{Loc}$.

Let T = maximal torus in group of Hamiltonian automorphisms commuting with $\mathbb{C}^\times_\hbar$

<u>Assume</u> M^T is finite.

Fix $\xi \in X_\bullet(T)$ a generic cocharacter, so $M^\xi = M^T$.

Let $M^\xi_{(0)}$ = union of attracting sets of fixed points $\subset M_{(0)}$.

$$\mathcal{O}^\xi_\lambda := D_\lambda\text{-Mod}_{M^\xi} \xrightarrow{\simeq} A_\lambda\text{-Mod}_{M^\xi_0}.$$

<u>Example:</u> Let $M = T^*X$ a symplectic resolution.

(Warning: this is very restrictive, implies X is D-affine).

Then D_λ = microlocalization of λ-twisted differential operators.

$D_\lambda\text{-Mod}$ = category of λ-twisted D-modules.

$D_\lambda\text{-Mod}_{M^\xi} = D_\lambda\text{-Mod}$ with singular support $\subset M^\xi$

$\simeq \mathrm{Perv}_{\mathcal{S}_\xi}(X)$ where $\mathcal{S}_\xi$ is stratification by attracting sets.

<u>E.g.:</u> For $X = G/B$,

$\mathcal{O}^\xi_\lambda = \lambda$ block of usual category $\mathcal{O}$, with Borel determined by ξ.

<u>General Story:</u> Let $M \twoheadrightarrow M_0$ a conical sympl. res. $T \subset M$ with finitely many fixed points.

"<u>Thm</u>": $\mathcal{O}^\xi_\lambda$ is a <u>highest weight category</u>. (today)

Further, it admits a <u>graded lift</u>, and is <u>Koszul</u>.

For $M, M^!$ "symplectic dual" varieties, $H^2(M) = T_{M^!}$ and vice versa.

$\mathcal{O}^\xi_\lambda(M)$ and $\mathcal{O}^\lambda_\xi(M^!)$ are "Koszul dual".

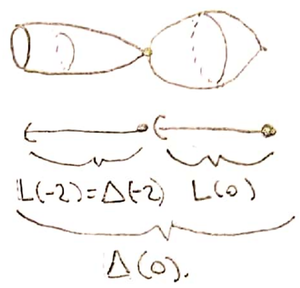
In particular, $D^b\mathcal{O}^\xi_\lambda(M) \xrightarrow{\simeq} D^b\mathcal{O}^\lambda_\xi(M^!)$

and

$$\mathrm{End}_{\mathcal{O}}(\oplus P) \simeq \mathrm{Ext}^\bullet_{\mathcal{O}^!}(\oplus L).$$

<u>Example:</u> $X = \mathbb{P}^1 = G/B$ for $G = SL_2$. $\lambda = 0$.

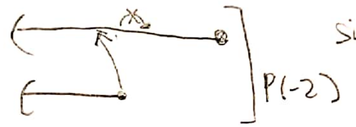
$M_\xi = \mathbb{P}^1 \cup T^\vee_\infty \mathbb{P}^1$.



$\Delta(-2) \hookrightarrow \Delta(0) \twoheadrightarrow L(0)$

$P(0) \xrightarrow{\simeq} \Delta(0)$.

$\Delta(0) \hookrightarrow P(-2) \twoheadrightarrow \Delta(-2)$



Since $(P(-2) : \Delta(0)) = [\Delta(0) : L(-2)]$ by BGG reciprocity.

$$\mathrm{End}(P(0) \oplus P(-2)) = \begin{bmatrix} \mathbb{C} & \mathbb{C} \\ \mathbb{C} & \mathbb{C}[\varepsilon]/\varepsilon^2 \end{bmatrix}, \quad \mathrm{Ext}^\bullet\begin{pmatrix} L(0) \\ \oplus \\ L(-2) \end{pmatrix} = \begin{bmatrix} \mathbb{C} & \mathbb{C}[-1] \\ \mathbb{C}[-1] & \mathbb{C}[u]/u^2 \end{bmatrix}_{|u|=2}$$

Let $\mathcal{A}$ be an abelian category, with finitely many (say n) iso classes of simple objects, and such that any object has a finite composition series.

<u>Def^n:</u> $\mathcal{A}$ is called <u>highest weight</u> if there are exact sequences

$$U_i \hookrightarrow P_i \twoheadrightarrow \Delta_i \qquad 1 \le i \le n$$

s.t.

1. $\mathrm{Hom}_{\mathcal{A}}(\Delta_i, \Delta_j) = 0$ for $i > j$.
2. $U_i \in \mathrm{Filt}(\Delta_{i+1}, \dots, \Delta_n)$ for all i.
3. $\bigoplus_i P_i$ is a projective generator
4. $\mathrm{End}(\Delta_i)$ is a division ring for all i (no idemp.)

<u>Equivalently:</u> Let $\mathcal{A}_i = \mathrm{Serre}(\Delta_1, \dots, \Delta_i)$.

Then $0 = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \dots \subseteq \mathcal{A}_n = \mathcal{A}$ a chain of full subcat's s.t. each $\mathcal{A}_i \hookrightarrow \mathcal{A}_{i+1}$ is a homological recollement.

$[f_{\alpha+\beta}, f_\beta] = 0$

Parabolic Category $\mathcal{O}$ for SL_3:

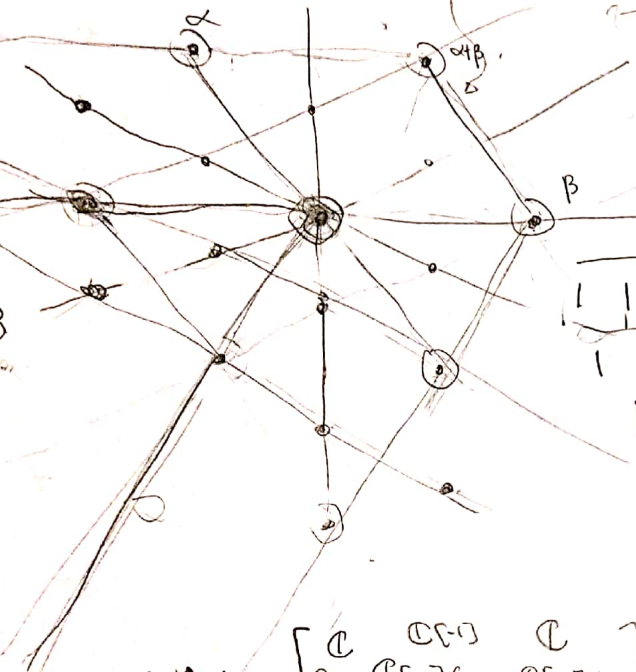
finite dim^l simple for levi

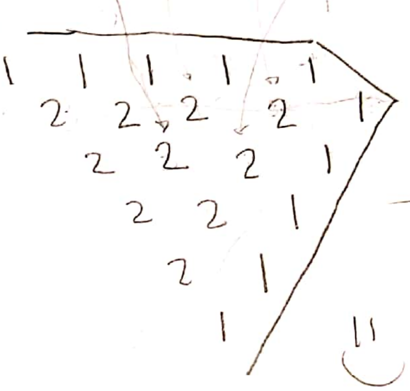
$f_\beta(\ \vdots\ )$

$f_{\alpha+\beta} f_\beta f_{\alpha+\beta} v \quad f_\beta f_{\alpha+\beta} v \quad f_{\alpha+\beta} v \quad f^2_{\alpha+\beta} v$

$f_{\alpha+\beta} f^2_\beta (f_\alpha v) \quad f^2_\beta f_\alpha v \quad f_\beta (f_\alpha v) \quad f_{\alpha+\beta} f_\beta f_\alpha v$

weight multiplicities?!





$$\mathrm{Ext}^\bullet(L) = \begin{bmatrix} \mathbb{C} & \mathbb{C}[-1] & \mathbb{C} \\ \mathbb{C}[-1] & \mathbb{C}[u]/u^2 & \mathbb{C}[u]/u^2 \\ \mathbb{C} & \mathbb{C}[u]/u^2 & \mathbb{C}[u]/u^3 \end{bmatrix}$$

$(P(\lambda) : L(\mu)) = [M(\mu) : L(\lambda)]$

in gen. $\dim \mathrm{Hom}(P(\lambda), M) = [M : L(\lambda)]$

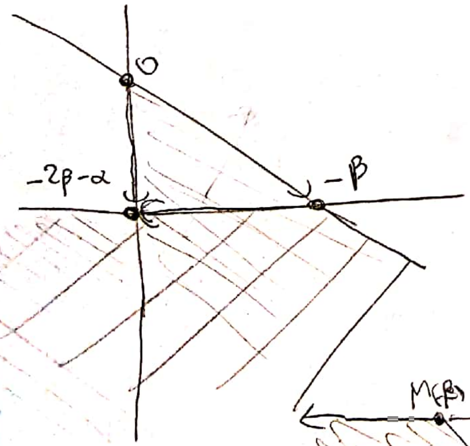
No finite $\Rightarrow$

$M(-2\beta-\alpha) \to M(-\beta) \to M(0) \twoheadrightarrow L(0)$

$M(-2\beta-\alpha) \to M(-\beta) \twoheadrightarrow L(-\beta)$

$$\mathrm{End}(P) = \begin{bmatrix} \mathbb{C} & \mathbb{C} & 0 \\ \mathbb{C} & \mathbb{C}[\varepsilon]/\varepsilon^2 & \mathbb{C} \\ 0 & \mathbb{C} & \mathbb{C}[\varepsilon]/\varepsilon^2 \end{bmatrix}$$

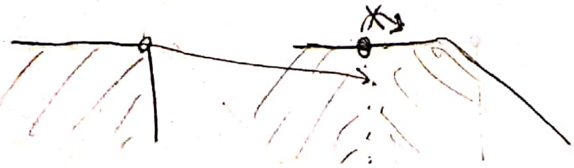
$M(-2\beta-\alpha) = L(-2\beta-\alpha)$



$P(0) \xrightarrow{\cong} M(0)$

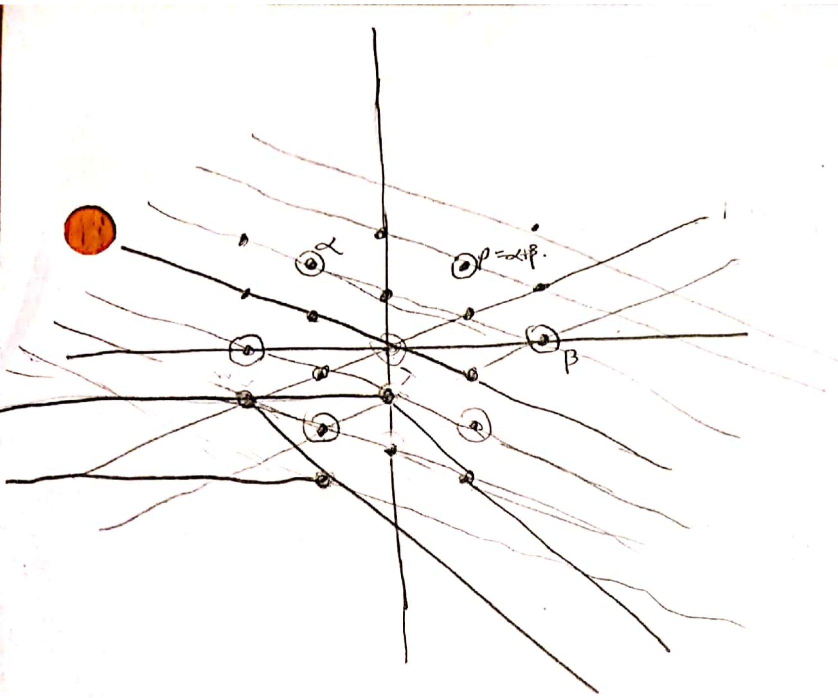
$M(0) \to P(-\beta) \twoheadrightarrow M(-\beta)$

$M(-\beta) \to P(-2\beta-\alpha) \twoheadrightarrow M(-2\beta-\alpha)$



$\mathrm{End}(P(-\beta)) = \mathbb{C}[\varepsilon]/\varepsilon^2$

$\mathrm{End}(P(-2\beta-\alpha)) = \mathbb{C}[\varepsilon]/\varepsilon^2.$



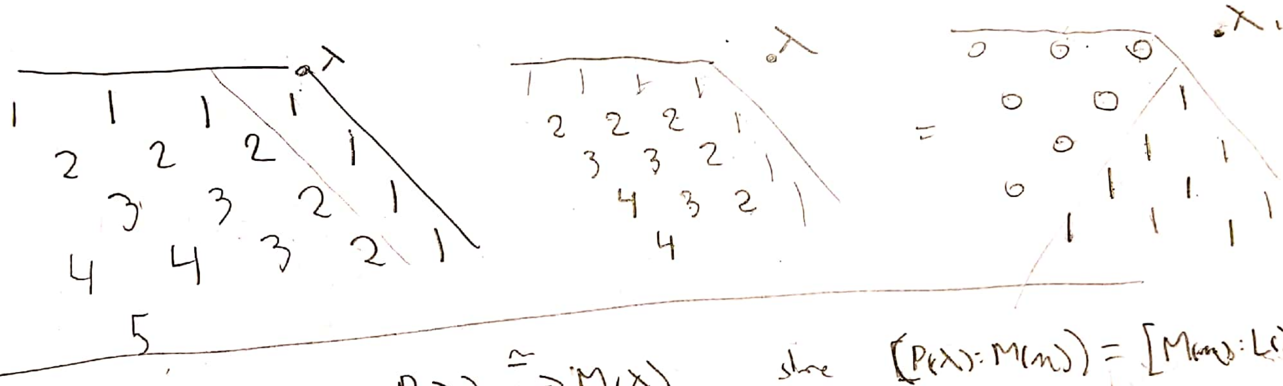
$$M(s_\beta\cdot\lambda) \longrightarrow M(\lambda) \longrightarrow L(\lambda)$$

$$M(s_\alpha s_\beta\cdot\lambda) \longrightarrow M(s_\beta\cdot\lambda) \longrightarrow L(s_\beta\cdot\lambda)$$

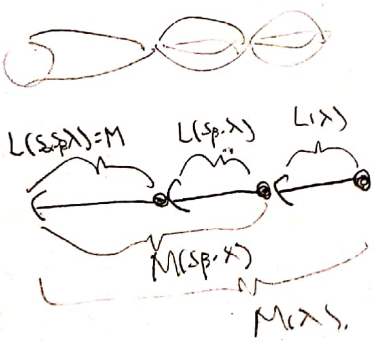
$$M(s_\alpha s_\beta\cdot\lambda) = L(s_\alpha s_\beta\cdot\lambda)$$

$$\Rightarrow \mathrm{Ext}^\bullet(L) = \begin{bmatrix} \mathbb{C} & \mathbb{C} & 0 \\ \mathbb{C} & \mathbb{C}[u]/u^2 & \mathbb{C} \\ 0 & \mathbb{C} & \mathbb{C}[u]/u^2 \end{bmatrix}$$

<u>NB:</u> [illegible] nilp. for f_β

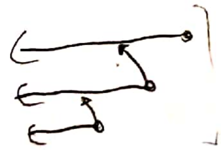


$P(\lambda) \xrightarrow{\simeq} M(\lambda)$ since $[P(\lambda):M(\mu)] = [M(\mu):L(\lambda)]$ $= 0$ for $\mu < \lambda$.



$$M(\lambda) \longrightarrow P(s_\beta\lambda) \longrightarrow M(s_\beta\lambda)$$

$$M(\lambda) \longrightarrow M(s_\beta\cdot\lambda) \longrightarrow P(s_\alpha s_\beta\cdot\lambda) \twoheadrightarrow M(s_\alpha s_\beta\cdot\lambda)$$



$$\mathrm{End}(P) = \begin{bmatrix} \mathbb{C} & \mathbb{C} & \mathbb{C} \\ \mathbb{C} & \mathbb{C}[\varepsilon]/\varepsilon^2 & \mathbb{C}[\varepsilon]/\varepsilon^2 \\ \mathbb{C} & \mathbb{C}[\varepsilon]/\varepsilon^2 & \mathbb{C}[\varepsilon]/\varepsilon^3 \end{bmatrix}$$

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