Let $\mathcal{O}_X$ be commutative Poisson algebra. (sheaf of Poisson algebras on X)

Def^{n}: A <u>formal quantization</u> of $\mathcal{O}_X$ is:

An associative alg. $A_\hbar / \mathbb{C}[[\hbar]]$, $\alpha : A_\hbar / \hbar A_\hbar \xrightarrow{\simeq} \mathcal{O}_X$ as (sheaves of) assoc. alg's

such that ① $A_\hbar$ is flat and complete over $\mathbb{C}[[\hbar]]$

② α intertwines $\frac{1}{\hbar}[\,,\,]$ with $\{\,,\,\}_X$. (TBE)

Suppose $\mathcal{O}_X$ is $\mathbb{Z}$-graded, with $\{\,,\,\}$ of deg -1.

A <u>graded</u> quantization of $\mathcal{O}_X$ is:

A $\mathbb{Z}$-graded associative algebra $A / \mathbb{C}[\hbar]$ (deg $+1$), $\alpha : A/\hbar A \to \mathcal{O}_X$

s.t. ① A is free $/ \mathbb{C}[\hbar]$

② α intertwines gradings and $\frac{1}{\hbar}[\,,\,]$ with $\{\,,\,\}_X$.

switch

"Algebraization":

Prop^{n}: $\left\{ \begin{matrix} A \text{ a formal quant}^{n} \\ \mathbb{C}^\times \circlearrowright A \text{ inducing} \\ \text{correct } A_0 \text{ grading} \end{matrix} \right\} \underset{\longleftarrow}{\xrightarrow{\simeq}} \left\{ \begin{matrix} \text{graded quant}^{n}\text{'s} \\ A \text{ of } \mathcal{O}_X \end{matrix} \right\}$

Def^{n}: A filtered quantization of $\mathcal{O}_X$ is:

A filtered assoc. alg. $A_\bullet$, $\alpha : \mathrm{gr}_\bullet A \xrightarrow{\simeq} \mathcal{O}_X$ as ass. alg.

s.t. ② α intertwines the gradings and $\{\,,\,\}$'s.

NB: $[\bar a, \bar b] \in A_{i+j-1} < A_{i+j}$, $\{a,b\} = \mathrm{gr}_{i+j-1}([\bar a, \bar b])$.

Prop^{n}: $\left\{ \begin{matrix} \text{graded quantization} \\ A \text{ of } \mathcal{O}_X \end{matrix} \right\} \underset{\longleftarrow}{\xrightarrow{\simeq}} \left\{ \begin{matrix} \text{filtered quant.} \\ A_\bullet \text{ of } \mathcal{O}_X \end{matrix} \right\}$

The Reese Construction:

$$\left\{ \begin{matrix} \mathbb{G}_m\text{-equivariant} \\ \text{VB's on } \mathbb{A}^1_\hbar \end{matrix} \right\} \underset{\longleftarrow}{\longrightarrow} \left\{ \begin{matrix} \text{Filtered VS's} \\ V_\bullet \end{matrix} \right\}$$

$$( )|_{\hbar=1} \to \mathrm{Vect} \xleftarrow{\text{ob}}, \quad ( )|_{\hbar=0} \to \mathrm{Vect}_{\mathbb{Z}} \xleftarrow{\mathrm{gr.}}$$

$$\simeq \mathbb{A}^\hbar|_{\hbar=1} = \Gamma(\mathbb{A}^1 \setminus \{0\}, \mathbb{A}^\hbar)^{\mathbb{G}_m}$$

$\mathbb{C}[\hbar] \circlearrowright A^\hbar \hookrightarrow A^\hbar[\hbar^{-1}] \simeq V[\hbar^{\pm 1}] \longleftrightarrow V = (A^\hbar[\hbar^{-1}])^0$, with $\mathbb{C}[\hbar^{\pm 1}] \circlearrowright A^\hbar[\hbar^{-1}]$

$$\bigcup_{m \in \mathbb{Z}} A^\hbar_m \hookrightarrow \bigcup_{m \in \mathbb{N}} \hbar^{-m} A^\hbar = \bigcup_{m \in \mathbb{N}} \hbar^{-m} V[\hbar] = \bigcup_{m \in \mathbb{N}} V_m$$

$$A^\hbar \cap \hbar^{-m} V[\hbar] \quad \hbar V_1 \quad \hbar^{-1} V_0 \quad \hbar^{-1} V_1 \qquad V_m = \hbar^{-m} A^\hbar \cap V.$$

$$= \sum_{m \in \mathbb{Z}} \hbar^m V_m[\hbar] \quad V_{-1} \quad V_0 \quad V_1 \quad V \quad = \{\sigma \in V \mid \hbar^m \sigma \in A^\hbar\}$$

$$\simeq \bigoplus \hbar^m V_m \subset V[\hbar^{\pm 1}] \quad \hbar V_{-1} \quad \hbar V_0 \quad \hbar V_1 \quad A^\hbar$$

Example: Let $\mathcal{O}_X = S(\mathfrak{g}) = \mathcal{O}(\mathfrak{g}^\vee)$, $\{\,,\,\} = [\,,\,] : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$ deg -1 wrt. VS grading.

Let $A_\bullet = U(\mathfrak{g})$ with PBW filtration:

For $\bar x, \bar y \in \mathfrak{g} \underset{\text{(choose)}}{\leftrightarrows} U\mathfrak{g}_1$, $[\bar x, \bar y] \in U\mathfrak{g}_1 \subset U\mathfrak{g}_2$.

$\Rightarrow \{x,y\} = \mathrm{gr}_1([\bar x, \bar y]) = [x,y]$ ✓.

$$A^\hbar \cong A^\hbar_0 + A^\hbar_1 + A^\hbar_2 + \dots \subset U\mathfrak{g}[\hbar^{\pm 1}]$$

$$= U\mathfrak{g}_0[\hbar] + \hbar U\mathfrak{g}_1[\hbar] + \hbar^2 U\mathfrak{g}_2[\hbar] + \cdots$$

$$= U\mathfrak{g}_0 \oplus \hbar U\mathfrak{g}_0 \oplus \hbar^2 U\mathfrak{g}_0 \ \ 0 \ \ 0 \quad \hbar U\mathfrak{g}_1 \oplus \hbar^2 U\mathfrak{g}_1 \quad \hbar^2 U\mathfrak{g}_2 \ \subset \ U\mathfrak{g} \oplus \hbar U\mathfrak{g} \oplus \hbar^2 U\mathfrak{g} \oplus \cdots$$

$$\simeq \mathbb{C}[\hbar]\langle x \in \mathfrak{g} \rangle / (x \otimes y - y \otimes x = \hbar [x,y]) =: U_\hbar \mathfrak{g}$$

Let $X \xrightarrow{\rho} S$ a Poisson scheme.

A formal quantization of X/S is:

- A sheaf of associative $\rho^{-1}(\mathcal{O}_S)[[\hbar]]$-algebras A
- An iso$^{\underline{\sim}}$ $\alpha : A/\hbar A \xrightarrow{\simeq} \mathcal{O}_X$ of $\rho^{-1}(\mathcal{O}_S)$-alg's

s.t. ①, ② as before. Similarly for graded, filtered.

E.g.: $D_{\underline{X},\hbar} = \mathbb{C}[\hbar]\langle \mathcal{O}_X, \Theta_{X/S} \,|\, f * g = fg \quad f * v = fv. \quad v * f = fv + \hbar v(f) \quad v*w - w*v = \hbar[v,w] \rangle$

this naturally "microlocalizes" to $D_{T^\vee \underline{X},\hbar}$ a quantization of $X = T^\vee X$

Let X, G as above, $D_{\hbar,X}$ a quantization of X/S

$G \circlearrowright D_{\hbar,X}$ by alg. automorphisms, inducing $\mathfrak{g} \xrightarrow{\phi} \mathrm{Der}(D_{\hbar,X})$
$\quad\curvearrowright G_a$

Suppose $\exists$ $\mu^\vee_\hbar : \mathfrak{g} \to \Gamma(X, D_\hbar)$ s.t. $\phi(\xi) = \frac{1}{\hbar}[\mu^\vee(\xi), -]$

and the following commutes $\mu^\vee \searrow \; \downarrow \hbar = 0 \quad \Gamma(X, \mathcal{O}_X)$.

Then the preceding sheafifies to define a (graded) quantization

$$R(D_{X,\hbar}, \mathfrak{g}, \lambda) \text{ of } \mu^{-1}(\mathcal{O}_\lambda)/G.$$

$$R(D_{X,\hbar}, \mathfrak{g}) \text{ of } X/G \twoheadrightarrow \mathfrak{g}^\vee/\!/G$$

Let X/S symplectic, $G \circlearrowright X \overset{\pi}{\twoheadrightarrow} X/G$

is a free action with smooth quotient, π affine.

Assume $\exists$ $\mu : X \to \mathfrak{g}^\vee$ s.t. $\{\underbrace{\langle d\mu, \xi\rangle}_{\mu^\vee(\xi)}, -\} = X_\xi$, $\xi \in \mathfrak{g}$

$\Leftrightarrow \mu^\vee : \mathcal{O}(\mathfrak{g}^\vee) \to \mathcal{O}(X)$

Then

$$X/G \xrightarrow{\bar{\mu}} \mathfrak{g}^\vee/\!/G \text{ is fibrewise symplectic.}$$

i.e.

$$X/\!/\!/_\lambda G = \mu^{-1}(\mathcal{O}_\lambda)/G \text{ symplectic } \lambda \in \mathfrak{g}^\vee/G.$$

(connected) $\mathfrak{g} = G$

Note: $\mathbb{C}[X/\!/\!/_\lambda G] = [\mathcal{O}_X/\mathcal{O}_X \cdot \{\mu(\xi) - \langle \lambda, \xi\rangle \,|\, \xi \in \mathfrak{g}\}]^{\mathfrak{g}} = \mathcal{O}_X^{\mathfrak{g}}/\mathcal{O}_X(\text{" "})^{\mathfrak{g}}$

For A assoc., $U\mathfrak{g} \xrightarrow{\phi} \mathrm{Der}\, A$
$\mu^\vee : U\mathfrak{g} \to A$ s.t. $[\mu^\vee(\xi), -] = \phi(\xi)$

$$R(A, \mathfrak{g}, \lambda) = [A/A\{\mu^\vee(\xi) - \langle \lambda, \xi \rangle\}]^{\mathfrak{g}}.$$

Example: Let $\underline{X}$ smooth, $\mathcal{L}$ a line bundle on $\underline{X}$.

Then $\mathbb{C}^\times \circlearrowright \underline{X}' = |\mathcal{L}| \setminus \underline{X} \xrightarrow{\pi} X$.

Then

$$D^{\alpha\mathcal{L}}_{T^\vee \underline{X},\hbar} = R(D_{T^\vee \underline{X}', \hbar}, \mathfrak{g}, \alpha)$$

$$= (\pi_* D_{\hbar, T^\vee \underline{X}'})^{\mathbb{C}^\times} / (\pi_* D_{\hbar, T^\vee \underline{X}})^{\mathbb{C}^\times}(eu - \hbar\alpha)$$

Example: "Hyperbolic Enveloping Algebra"

For $(\mathbb{C}^\times)^k \hookrightarrow (\mathbb{C}^\times)^n \twoheadrightarrow (\mathbb{C}^\times)^{n-k}$, $\quad D_{\hbar, T^\vee\mathbb{C}^n} = \mathbb{C}[\hbar]\langle x_i, \partial_j \rangle / [x_i, \partial_j] = \hbar$

$\quad\; \| \qquad\qquad\;\; \circlearrowright \qquad\qquad \|$
$\quad\; T \qquad\quad\;\; T^\vee\mathbb{C}^n \qquad\quad T_F$

$$U^{T,n}_\lambda = R(D_{T^\vee\mathbb{C}^n,\hbar}, \mathfrak{t}, \lambda) = D^T_{T^\vee\mathbb{C}^n,\hbar} / D^T_{T^\vee\mathbb{C}^n,\hbar} \cdot \langle \ker \lambda \rangle$$

(Gelfand-Kazhdan)

Constructing + Classifying quantizations: (Fedosov, Kontsevich)

Idea: The formal nbhd of any smooth point is (non-canonically!) iso$^{\cong}$ to $\mathbb{D}$. The formal Darboux th^{m} says this holds symplectically. We know how to quantize $\mathcal{O}_{\mathbb{D}}$ to formal Weyl algebra $D_{\hbar,\mathbb{D}}$.

Construction: Do this in families. Classification: obstruction to trivializing the family classifies quantization.

Let $(G,\mathfrak{h})$ a HC pair.

Def^{n}: A $(G,\mathfrak{h})$ torsor is $G \curvearrowright M \twoheadrightarrow X$ a G-torsor, $\omega \in \Omega^1_M(\underline{\mathfrak{h}}_M)^G$ an $\mathfrak{h}$-valued connection.

Let $H^1(X,(G,\mathfrak{h})) = \{iso^{\cong}$ classes of $(G,\mathfrak{h})$-torsors on $X\}$.

Given M, $\mathrm{Loc}_M : H^\bullet((G,\mathfrak{h});V) \to H^\bullet_{dR}(X, V \overset{G}{\times} M)$

Example: $\mathbb{C} \hookrightarrow \mathcal{O}_{\mathbb{D}} \twoheadrightarrow \mathbb{H}^{Ham}_{\mathbb{D}}$ induces

$$(\mathbb{C}, \mathrm{Lie}\,\mathbb{C}) \hookrightarrow (\hat{G},\hat{\mathfrak{h}}) \twoheadrightarrow (\mathrm{Symp}(\mathbb{D}), \mathbb{H}^{Ham}_{\mathbb{D}})$$

$c = [\omega_{\mathbb{D}}] \in H^2((\mathrm{Symp}, \mathbb{H}^{Ham}_{\mathbb{D}}), \mathbb{K})$ is the universal class.

$\mathrm{Loc}(M^s, c) = [\omega_X] \in H^2_{dR}(X;\mathbb{C})$ obstructs lifting!

Thus, we classify the reduction of str (equiv. the sympl. form) by the obstruction to "trivializing" (in certain sense) the torsor.

Prop^{n}: $Q(X,\Omega) \xrightarrow{\cong} H^1_{M^{symp}}(X, \langle \mathrm{Aut}(D_{\hbar,\mathbb{D}}), \mathrm{Der}(D_{\hbar,\mathbb{D}})\rangle)$ defines a bijection. Given a class, construct quantization by descent. Given a quantization, we get a class by Formal Darboux Th^{m}.

Def^{n}: $M^{coor}_X : \mathrm{Sch}^{op} \to \mathrm{Sets}$.

$$T \mapsto \{h : T \to X,\ \hat{\mathcal{O}}_{X,h} \xrightarrow{\cong} \mathcal{O}_T \hat{\otimes} \mathcal{O}_{\mathbb{D}}\}$$

Prop^{n}: M^{coor}_X is rep$_s$ $M^{coord}_X \to X$ an $(\mathrm{Aut}(\mathbb{D}), \mathbb{H}_{\mathbb{D}})$-tors.

Prop^{n}: $\left\{\begin{matrix}\text{symplectic} \\ \text{strs on } X\end{matrix}\right\} \rightleftarrows \left\{\begin{matrix}\text{reductions of str. of} \\ M^{coor}_X \text{ to } \langle \mathrm{Symp}(\mathbb{D}), \mathbb{H}^{Ham}_{\mathbb{D}}\rangle\end{matrix}\right\}$

Construction: Let $V \in (G,\mathfrak{h})$-Mod viewed as additive grp. and fix extension $(V, \mathrm{Lie}\,V) \hookrightarrow (\hat{G},\hat{\mathfrak{h}}) \twoheadrightarrow (G,\mathfrak{h})$. There exists universal class $c \in H^2((G,\mathfrak{h}),V)$ s.t.

① $H^1_M(X,(\hat{G},\hat{\mathfrak{h}})) \neq \emptyset$ iff $\mathrm{Loc}(M,c) = 0$.

② In such case, $H^1_M(X,(\hat{G},\hat{\mathfrak{h}}))$ affine for $H^1_{dR}(X, \mathrm{Loc}(M,V))$.

Now, $\mathbb{C}[\![\hbar]\!] \hookrightarrow \hbar^{-1} \cdot D_{\hbar,\mathbb{D}} \twoheadrightarrow (\mathrm{Aut}(D_{\hbar,\mathbb{D}}), \mathrm{Der}_{\hbar,\mathbb{D}})$ yields a universal class $c \in H^2(\ \ ;\mathbb{C}[\![\hbar]\!])$.

$\mathrm{Per} : H^1_{M^{symp}}(X, \langle \mathrm{Aut}(D_{\hbar,\mathbb{D}}), \mathrm{Der}(D_{\hbar,\mathbb{D}})\rangle) \to H^2_{dR}(X,\mathbb{C})[\![\hbar]\!]$

$\mathrm{Per}(M) \in \mathrm{Loc}(M,c)$ $\qquad \{[\Omega]\} \oplus \hbar H^2_{dR}(X)[\![\hbar]\!]$

Th^{m}: If $H^i(X,\mathcal{O}_X) = 0$, $i=1,2,3$, then "" is an iso^{m}.

Def^{n}: A quantization is called canonical if $\mathrm{Per}(M) = [\Omega]$.

The th^{m} says there exists a unique canonical quantization of each such X.

Deformations + Quantizations.

Now, suppose X_0 a symplectic resolution

X/S its universal Poisson deformation $/ S = H_2(X)$

$D_{X/S,\hbar}$ the canonical quantization of X/S.

For any section $\sigma : \mathrm{Spf}\,\mathbb{C}[[\hbar]] \xrightarrow{\leftarrow} \mathrm{Spf}\,\mathbb{C}[[\hbar]] \times S$ with $\sigma(pt) = \{0\}$, $\sigma^* Q$ a quantization of X_0 for any Q a quant. of X/S.

Moreover,

$$\mathrm{Per}(\sigma^* D_{X/S,\hbar}) - [\Omega_0] = \sigma \in \hbar H^2(X_0)[[\hbar]]$$

(where we identify sections σ w. such)

Now, suppose "$\mathbb{C}^\times_t \curvearrowright X/S$", lifting $\mathbb{C}^\times \curvearrowright H^2(X)$ linearly, and induced $\mathbb{C}^\times_t \curvearrowright X_0$ satisfies $t^*\Omega = t^n \Omega$ ($\{-,-\}$ wt $-n$)

We also let $\mathbb{C}^\times_t \curvearrowright \mathbb{C}[[\hbar]]$, w. $t.\hbar = n\hbar$

Then we have

$D_\hbar \in Q(X/S)$ lifts to a graded quantization

iff $\mathrm{Per}(D_\hbar) \in \hbar H^2(X)$

$$\boxed{\text{Upshot: } H^2(X_0) \text{ also parametrizes graded quantizations of}}$$

$\hbar H^2(X)[[\hbar]]$