MAT224 - LEC5101 - Lecture 8
The Determinant

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Review: Matrices of linear maps, yet again.

Let V, W be vector spaces and $\alpha = \{\mathbf{v}_1, ..., \mathbf{v}_n\}, \beta = \{\mathbf{w}_1, ..., \mathbf{w}_m\}$ be bases. Then the following are equivalent data (in bijection):

- a linear map $T : V \to W$
- a list of vectors $\mathbf{y}_j = T(\mathbf{v}_j) \in W$ for $j = 1, ..., n$.
- a matrix of numbers $a_{ij} \in \mathbb{R}$ defined by

$$T(\mathbf{v}_j) = a_{1j}\mathbf{w}_1 + ... + a_{mj}\mathbf{w}_m \qquad\qquad [T(\mathbf{v}_j)]^\beta = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

  for $i = 1, ..., m$ and $j = 1, ..., n$.

Graphically, we write

$$[T]_\alpha^\beta = \big[\, T(\mathbf{v}_1) | \quad \cdots \quad | T(\mathbf{v}_n) \big]^\beta = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \in M_{m\times n}(\mathbb{R})$$

The j^{th} column of $[T]_\alpha^\beta$ describes $T(\mathbf{v}_j)$, the image under T of the j^{th} vector $\mathbf{v}_j$ in the basis α, in terms of coordinates defined by β.

Towards Determinants

Let V be a vector space with $\dim V = n$ and $T : V \to V$ linear.

Then T defines a system of n linear equations in n unknowns:

Let $\alpha = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ a basis for V and $[T]_\alpha^\alpha = [a_{ij}] \in M_{n\times n}(\mathbb{R})$.

Then $T(\mathbf{x}) = 0$ if and only if

$$a_{i1}x_1 + ... + a_{in}x_n = 0 \qquad \text{for each } i = 1, ..., m.$$

We've learned that the following are equivalent:

(1) $\ker(T) \neq \{\mathbf{0}\}$

(2) $\{T(\mathbf{v}_i)\}$ is linearly dependent.

(3) $\mathrm{null}([T]_\alpha^\alpha) \neq \{\mathbf{0}\}$

(4) there exists a non-trivial solution to $a_{i1}x_1 + ... + a_{in}x_n = 0$

The determinant $\det([T])$ is a quantity we calculate to determine when any of these equivalent properties hold:

$\det([T]) = 0$ **if and only if any of the above conditions hold.**

2×2 determinants

Suppose $\dim V = 2$, $\alpha = \{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for V, so $V \cong \mathbb{R}^2$.

For $T : V \to V$, the columns of $[T]$ are just the vectors $T(\mathbf{v}_i)$:

$$[T]_\alpha^\alpha = \begin{bmatrix} T(\mathbf{v}_1) & T(\mathbf{v}_2) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad T(\mathbf{v}_1) = \begin{bmatrix} a \\ c \end{bmatrix} \; T(\mathbf{v}_2) = \begin{bmatrix} b \\ d \end{bmatrix}$$

Main idea: The vectors $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$ are linearly dependent if and only if the parallelogram they form has area 0.

Proposition: The signed area of the parallelogram with sides (a, c) and (b, d) is $ad - bc$.

Definition: For $A \in M_{2\times 2}(\mathbb{R})$, we define $\det(A) = ad - bc$.

Exercise: Calculate the following determinants:

$$\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \qquad \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad \begin{bmatrix} a & 2a \\ b & 2b \end{bmatrix} \qquad \begin{bmatrix} a & 2a \\ b & 3b \end{bmatrix}$$

Exercise: Show that $\det A = 0$ if and only if A is not invertible. (Hint: Consider the above examples)

General Determinants

Let $\dim V = n$, $\alpha = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ a basis for V, so $V \cong \mathbb{R}^n$.

Let $T : V \to V$, given by $A = [T]_\alpha^\alpha \in M_{n \times n}(\mathbb{R})$.

We want an analogous criterion for T to be invertible.

Main idea: The columns $T(\mathbf{v}_i)$ of $[T]$ form an n-dimensional parallelogram. The n-dimensional volume of this parallelogram will equal 0 if and only if the vectors $T(\mathbf{v}_i)$ are linearly dependent , i.e. if and only if T is not invertible.

Tentative Definition: $\det([T])$ is the (signed) volume of the n-dimensional parallelogram with sides $T(\mathbf{v}_i)$.

Giving an explicit formula for this volume in arbitrary dimension is quite complicated. We'll use an abstract approach to characterize it uniquely, and determine its useful computational properties.

We want function $\det : M_{n \times n}(\mathbb{R}) \to \mathbb{R}$.
Natural question: Is it a linear function?

Multilinear Functions

Note that $M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2} = \mathbb{R}^n \oplus \mathbb{R}^n \oplus ... \oplus \mathbb{R}^n = (\mathbb{R}^n)^{\oplus n}$ where

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \quad \mapsto \quad \left(\begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix}, \dots, \begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix}, \dots, \begin{bmatrix} a_{1n} \\ \vdots \\ a_{nj} \end{bmatrix} \right)$$

Definition: A function $f : (\mathbb{R}^n)^{\oplus n} \to \mathbb{R}$ is called *multilinear* if
$f(\mathbf{x}_1, ..., \mathbf{x}_j + c\mathbf{y}_j, ..., \mathbf{x}_n) = f(\mathbf{x}_1, ..., \mathbf{x}_j, ..., \mathbf{x}_n) + cf(\mathbf{x}_1, ..., \mathbf{y}_j, ..., \mathbf{x}_n)$
or equivalently, $f(\mathbf{x}_1, ..., -, ..., \mathbf{x}_n) : \mathbb{R}^n \to \mathbb{R}$ is linear, for each i.

We have argued that $\det : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ should be multilinear. This is one of its two key properties.

Exercise: Prove that $\det : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ defined by $\det(A) = ad - bc$ is multilinear.

Natural Question: What other property does the function det satisfy, which guarentees it vanishes on linearly dependent sets?

Alternating Functions and the Definition of Determinant

Definition: A function $f : (\mathbb{R}^n)^{\oplus n} \to \mathbb{R}$ is called alternating if
$f(\mathbf{x}_1, ..., \mathbf{x}_i, ..., \mathbf{x}_j, ..., \mathbf{x}_n) = -f(\mathbf{x}_1, ..., \mathbf{x}_j, ..., \mathbf{x}_i, ..., \mathbf{x}_n)$
for each $i, j = 1, ..., n$, with $i \neq j$.

Exercise: Prove that $\det : \mathbb{R}^2 \oplus \mathbb{R}^2 \to \mathbb{R}$ defined by
$\det((a, c), (b, d)) = ad - bc$ is alternating.

Exercise: Let $f : (\mathbb{R}^n)^{\oplus n} \to \mathbb{R}$ be alternating. Show that
$f(\mathbf{x}_1, ..., \mathbf{x}_n) = 0$ whenever $\mathbf{x}_i = \mathbf{x}_j$ for some $i \neq j$.

Proposition: Let $f : (\mathbb{R}^n)^{\oplus n} \to \mathbb{R}$ be alternating and multilinear.
Then $f(\mathbf{x}_1, ..., \mathbf{x}_n) = 0$ whenever $\mathbf{x}_1, ..., \mathbf{x}_n$ is linearly dependent.

Theorem: There exists a unique function $f : (\mathbb{R}^n)^{\oplus n} \to \mathbb{R}$
satisfying the following properties:

- f is multilinear
- f is alternating
- $f(I) = 1$

The Determinant

Recall that $M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2} = \mathbb{R}^n \oplus \mathbb{R}^n \oplus ... \oplus \mathbb{R}^n = (\mathbb{R}^n)^{\oplus n}$.

Definition: The determinant $\det : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ is the unique function satisfying:

- det is multilinear
- det is alternating
- $\det(I) = 1$, where I is the identity matrix.

Exercise: Show that if A is not invertible then $\det(A) = 0$.

Definition: Let $A \in M_{n\times n}(\mathbb{R})$ and fix $1 \leq i, j \leq n$, then $A_{ij} \in M_{(n-1)\times(n-1)}(\mathbb{R})$ is the matrix obtained by deleting the i^{th} row and the j^{th} column from A; A_{ij} is called the $(i,j)^{th}$ *minor*.

Proposition: Let $A \in M_{n\times n}(\mathbb{R})$ and fix $1 \leq i \leq n$. Then

$$\det(A) = \sum_{j=1}^{n} (-1)^{i-j} a_{ij} \det(A_{ij})$$

Exercise: Compute the following determinants:

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \qquad \begin{bmatrix} a & b & 0 \\ d & e & 0 \\ 0 & 0 & i \end{bmatrix} \qquad \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Properties of Determinants

Proposition: Let $A \in M_{n\times n}(\mathbb{R})$ be a diagonal matrix with entries $\lambda_1, ..., \lambda_n$. Then $\det(A) = \lambda_1 \cdot ... \cdot \lambda_n$.

Proposition: Let $c \in \mathbb{R}$ and $A \in M_{n\times n}(\mathbb{R})$, and $\tilde{A} \in M_{n\times n}(\mathbb{R})$ differ by a column operation from A. Then $\det(A) = \det(\tilde{A})$.

Proposition: Let $A \in M_{n\times n}(\mathbb{R})$. A is invertible iff $\det(A) \neq 0$.

Proposition: Let $A, B \in M_{n\times n}(\mathbb{R})$. Then

$$\det(AB) = \det(A)\det(B).$$

Corollary: If A is invertible then $\det(A^{-1}) = \det(A)^{-1}$.

Exercise: Let $A, S \in M_{n\times n}(\mathbb{R})$ and suppose S is invertible. Show

$$\det(S^{-1}AS) = \det(A)$$

Exercise: Let $T : V \to V$ a linear map and α, β be two bases.Then

$$\det([T]_\alpha^\alpha) = \det([T]_\beta^\beta)$$