MAT224 - LEC5101 - Lecture 7
Composition of linear maps, isomorphisms, and change of basis

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Review: Matrices of linear maps, yet again.

Let V, W be vector spaces and $\alpha = \{\mathbf{v}_1, ..., \mathbf{v}_n\}, \beta = \{\mathbf{w}_1, ..., \mathbf{w}_m\}$ be bases. Then the following are equivalent data (in bijection):

- a linear map $T : V \to W$
- a list of vectors $\mathbf{y}_j = T(\mathbf{v}_j) \in W$ for $j = 1, ..., n$.
- a matrix of numbers $a_{ij} \in \mathbb{R}$ defined by

$$T(\mathbf{v}_j) = a_{1j}\mathbf{w}_1 + ... + a_{mj}\mathbf{w}_m \qquad [T(\mathbf{v}_j)]^\beta = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

  for $i = 1, ..., m$ and $j = 1, ..., n$.

Graphically, we write

$$[T]_\alpha^\beta = \begin{bmatrix} T(\mathbf{v}_1) | & \cdots & | T(\mathbf{v}_n) \end{bmatrix}^\beta = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \in M_{m\times n}(\mathbb{R})$$

The j^{th} column of $[T]_\alpha^\beta$ describes $T(\mathbf{v}_j)$, the image under T of the j^{th} vector $\mathbf{v}_j$ in the basis α, in terms of coordinates defined by β.

Review: Matrix multiplication works

Given the matrix $[T]_\alpha^\beta$, we can recover T as:

$$T(\mathbf{x}) = T(\sum_{j=1}^{n} x_j \mathbf{v}_j) = \sum_{j=1}^{n} x_j T(\mathbf{v}_j) = \sum_{j=1}^{n} x_j \left(\sum_{i=1}^{m} a_{ij} \mathbf{w}_i \right)$$
$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) \mathbf{w}_i$$

Thus, given $[\mathbf{x}]^\alpha = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, we find $[T(\mathbf{x})]^\beta = [T]_\alpha^\beta \cdot [\mathbf{x}]^\alpha$

This is the proof that **matrix multiplication algorithm works**.

This is the conceptual reason why matrices are used to solve linear equations. (previously, this was a bonus exercise)

Discussion: Composition of linear maps

Let U, V, W be vector spaces, $S : U \to V$ and $T : V \to W$ linear.

We define the composition $TS : U \to W$ by $TS(\mathbf{y}) = T(S(\mathbf{y}))$

Exercise: Show that $TS : U \to W$ is linear.

Example: Let
$S : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $S(y_1, y_2) = (y_1, y_2, 2y_2)$
$T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x_1, x_2, x_3) = (x_2, x_3)$.
Then $TS : \mathbb{R}^2 \to \mathbb{R}^2$ is calculated directly by

$$TS(y_1, y_2) = T(y_1, y_2, 2y_2) = (y_2, 2y_2)$$

Exercise: Let
$S : P_2(\mathbb{R}) \to P_1(\mathbb{R})$ defined by $S(p(x)) = \frac{d}{dx} p(x)$
$T : P_1(\mathbb{R}) \to P_3(\mathbb{R})$ defined by $T(p(x)) = x^2 p(x)$

Calculate $[S]_\gamma^\alpha$, $[T]_\alpha^\beta$, and $[TS]_\gamma^\beta$ in terms of the standard bases $\gamma = \{1, x, x^2\}$, $\alpha = \{1, x\}$ and $\beta = \{1, x, x^2, x^3\}$.

Composition and matrix multiplication

Let U, V, W be vector spaces, $S : U \to V$ and $T : V \to W$ linear.

Fx bases $\gamma = \{\mathbf{u}_1, ..., \mathbf{u}_p\}$, $\alpha = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ and $\beta = \{\mathbf{w}_1, ..., \mathbf{w}_m\}$, for U, V and W, respectively. Then we have matrices:

$$[T]_\alpha^\beta = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad [S]_\gamma^\alpha = \begin{bmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & b_{jk} & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix}$$

Natural Question: How do we find $[TS]_\gamma^\beta$?

$$TS(\mathbf{u}_k) = T\left(\sum_{j=1}^n b_{jk}\mathbf{v}_j\right) = \sum_{j=1}^n b_{jk}\left(\sum_{i=1}^m a_{ij}\mathbf{w}_i\right) = \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij}b_{jk}\right)\mathbf{w}_i$$

Thus $[TS(\mathbf{u}_k)]^\beta = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} b_{1k} \\ \vdots \\ b_{nk} \end{bmatrix} = [T]_\alpha^\beta [S(\mathbf{u}_k)]^\alpha$

and so $[TS]_\gamma^\beta = \left[TS(\mathbf{u}_1) \middle| \quad \cdots \quad \middle| TS(\mathbf{u}_p)\right]^\beta = [T]_\alpha^\beta \cdot [S]_\gamma^\alpha$

Exercise: Check this works in the example we just calculated.

Review: Injective, Surjective, Bijective

Let $S, \tilde{S}$ be sets and $f : S \to \tilde{S}$. Recall the following:

Definition: f is *injective* if knowing $f(s) = f(t)$ implies $s = t$.

f is *surjective* if for each $\tilde{s} \in \tilde{S}$ there exists $s \in S$ with $f(s) = \tilde{s}$.

f is *bijective* if it is injective and surjective.

Moreover, we showed:

(1) f is surjective if and only if $f(S) = \tilde{S}$, if and only if:
For each $\tilde{s} \in \tilde{S}$, $f^{-1}(\{\tilde{s}\})$ is non-empty, i.e. $f^{-1}(\{\tilde{s}\}) \neq \emptyset$.

(2) f is injective if and only if:
For each $\tilde{s} \in \tilde{S}$, $f^{-1}(\{\tilde{s}\})$ is either a single point $\{s\}$ or empty $\emptyset$.

(3) f is bijective if and only:
For each $\tilde{s} \in \tilde{S}$, there **exists** a **unique** $s \in S$ such that $f(s) = \tilde{s}$.

Exercise: Let $f : S \to \tilde{S}$ be bijective. Then there exists a unique function $g : \tilde{S} \to S$ such that $g(f(s)) = s$ and $f(g(\tilde{s})) = \tilde{s}$ for each $s \in S$ and $\tilde{s} \in \tilde{S}$.
We call g the inverse of f, and write $g(s) = f^{-1}(s)$.
Warning: This is not to be confused with $f^{-1}(\{s\})$.

Discussion: Isomorphisms and Inverses

Let V, W be finite dimensional vector spaces and $T : V \to W$ a linear transformation. Recall:

T is injective if and only if $\ker(T) = \{\mathbf{0}\}$

T is surjective if and only if $\operatorname{im}(T) = W$.

Exercise: Suppose $T : V \to W$ is injective and surjective, and that V, W are finite dimensional. Show that $\dim V = \dim W$.

Exercise: Suppose $T : V \to W$ and $\dim V = \dim W$. Show that T injective if and only if T surjective, if and only if T bijective.

Definition: We say $T : V \to W$ is an *isomorphism* if it is bijective. We also say V and W are *isomorphic* vector spaces.

In this case, there exists a unique function $T^{-1} : W \to V$ such that $T^{-1}T = \mathbb{1}_V$ and $TT^{-1} = \mathbb{1}_W$.

Proposition: Choose bases α, β. Then $[T^{-1}]_\beta^\alpha = ([T]_\alpha^\beta)^{-1}$.

For $A \in M_{n\times n}(\mathbb{R})$, you know how to calculate A^{-1} from mat223.

Discussion: Towards Change of Basis

Given V, W with bases α, β, we can encode $T : V \to W$ in $[T]_\alpha^\beta$.

Natural Question: How does $[\mathbf{x}]^\alpha, [T]_\alpha^\beta$ depend on choice α, β?

Let $T = \mathbb{1} : V \to V$. Then for any basis α, $[\mathbb{1}]_\alpha^\alpha = [1]$.

Given another $\alpha' = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$, we have $[\mathbb{1}]_{\alpha'}^\alpha = \begin{bmatrix} \mathbf{v}_1 | & \cdots & | \mathbf{v}_n \end{bmatrix}^\alpha$

Note the special propery: $[\mathbb{1}]_{\alpha'}^\alpha [\mathbf{x}]^{\alpha'} = [\mathbb{1}(\mathbf{x})]^\alpha = [\mathbf{x}]^\alpha$

Definition: We call $[\mathbb{1}]_{\alpha'}^\alpha$ the *change of basis matrix* from α' to α.

Example: Let $\alpha = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis for $\mathbb{R}^2$, let $\alpha' = \{(1,1),(1,-1)\}$ and let $\mathbf{x} = (3,1) \in \mathbb{R}^2$.Then

$$[\mathbf{x}]^\alpha = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad [\mathbf{x}]^{\alpha'} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad [\mathbb{1}]_{\alpha'}^\alpha = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and } [\mathbf{x}]^\alpha = [\mathbb{1}]_{\alpha'}^\alpha [\mathbf{x}]^{\alpha'}$$

Exercise: $V = P_1(\mathbb{R})$, $\alpha = \{1 + x, -x\}$, $\alpha' = \{1 + 2x, 1 + 3x\}$, and $\mathbf{x} = 1 - x$. Compute:

$$[\mathbf{x}]^\alpha \qquad [\mathbf{x}]^{\alpha'} \qquad [\mathbb{1}]_{\alpha'}^\alpha \qquad [\mathbb{1}]_{\alpha'}^\alpha \cdot [\mathbf{x}]^{\alpha'}$$

Discussion: Change of Basis Continued

We can use a similar trick to describe $[T]_\alpha^\beta$ under change of basis:

$$[T]_\alpha^\beta[\mathbb{1}]_{\alpha'}^\alpha = [T\mathbb{1}]_{\alpha'}^\beta = [T]_{\alpha'}^\beta \quad \text{and} \quad [\mathbb{1}]_\beta^{\beta'}[T]_\alpha^\beta = [\mathbb{1}T]_\alpha^{\beta'} = [T]_\alpha^{\beta'}$$

Exercise: Let $V = W = \mathbb{R}^2$, α the standard basis, $\alpha' = \{(2,0),(1,-1)\}$, and $T : V \to V$ be $T(x_1, x_2) = (x_1 + x_2, x_2)$
Compute $\quad [T]_\alpha^\alpha \quad [\mathbb{1}]_\alpha^{\alpha'} \quad [\mathbb{1}]_{\alpha'}^\alpha \quad [T]_{\alpha'}^\alpha \quad [T]_\alpha^{\alpha'}$

We can summarize this by: $\quad [\mathbb{1}]_\beta^{\beta'}[T]_\alpha^\beta[\mathbb{1}]_{\alpha'}^\alpha = [T]_{\alpha'}^{\beta'}$

In the case $V = W$, $\alpha = \beta$, $\alpha' = \beta'$: $\quad [\mathbb{1}]_\alpha^{\alpha'}[T]_\alpha^\alpha[\mathbb{1}]_{\alpha'}^\alpha = [T]_{\alpha'}^{\alpha'}$
In particular, taking $T = \mathbb{1}$, we find that $[\mathbb{1}]_\alpha^{\alpha'} = ([\mathbb{1}]_{\alpha'}^\alpha)^{-1}$.

Definition: $A, B \in M_{n\times n}(\mathbb{R})$ are called *similar* if $A = S^{-1}BS$.

Proposition: $A, B \in M_{n\times n}(\mathbb{R})$ are similar if and only if there exists $T : V \to V$ and bases α, β for V s.t. $A = [T]_\alpha^\alpha$ and $B = [T]_\beta^\beta$.