MAT224 - LEC5101 - Lecture 5
Conceptual overview and the dimension theorem (rank-nullity)

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Review of last week 1: Matrices of linear maps

Let V, W be vector spaces and $\alpha = \{\mathbf{v}_1, ..., \mathbf{v}_n\}, \beta = \{\mathbf{w}_1, ..., \mathbf{w}_m\}$ be bases. Then the following are equivalent data (in bijection):

- a linear map $T : V \to W$
- a list of vectors $\mathbf{y}_j = T(\mathbf{v}_j) \in W$ for $j = 1, ..., n$.
- a matrix of numbers $a_{ij} \in \mathbb{R}$ defined by
  $$T(\mathbf{v}_j) = a_{1j}\mathbf{w}_1 + ... + a_{mj}\mathbf{w}_m$$
  for $i = 1, ..., m$ and $j = 1, ..., n$.

Graphically, we write

$$[T]_\alpha^\beta = \begin{bmatrix} T(\mathbf{v}_1) | & \cdots & | T(\mathbf{v}_n) \end{bmatrix}^\beta = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \in M_{m \times n}(\mathbb{R})$$

The j^{th} column of $[T]_\alpha^\beta$ describes $T(\mathbf{v}_j)$, the image under T of the j^{th} vector $\mathbf{v}_j$ in the basis α, in terms of coordinates defined by β.

Review of last week 2: Kernel and image

Let $T : V \to W$. We made the following definitions:

$$\text{im}(T) = T(V) = \{T(\mathbf{v}) \in W | \mathbf{v} \in V\}$$
$$\ker(T) = T^{-1}(\{\mathbf{0}\}) = \{\mathbf{v} \in V | T(\mathbf{v}) = \mathbf{0}\}$$

Moreover, we learned:

$\text{im}(T) = \text{col}([T]_\alpha^\beta) = \text{Span}(\{T(\mathbf{v}_1), ..., T(\mathbf{v}_n)\})$

$\ker(T) = \text{null}([T]_\alpha^\beta) = \{\mathbf{x} \in V | x_1 T(\mathbf{v}_1) + ... + x_n T(\mathbf{v}_n) = \mathbf{0}\}$

where $[\mathbf{x}]_\alpha = [x_i]$. i.e. 'the set of linear dependences in $\{T(\mathbf{v}_i)\}$'

Slightly more generally, we can ask about $T^{-1}(\{\mathbf{b}\})$. We have:

Proposition: Let $\mathbf{b} \in \text{im}(T)$ so that $\mathbf{b} = T(\mathbf{x}_0)$ for $\mathbf{x}_0 \in V$. Then

$$T^{-1}(\{\mathbf{b}\}) := \{\mathbf{x} \in V | T(\mathbf{x}) = \mathbf{b}\} = \{\mathbf{x}_0 + \mathbf{v} | \mathbf{v} \in \ker(T)\}$$

Corollary: There is a bijection between $\ker(T)$ and $T^{-1}(\{\mathbf{b}\})$.

What does it mean to solve linear equations?

Fix $a_{ij} \in \mathbb{R}$ for $i = 1, ..., m$ and $j = 1, ..., n$.

Question: For each $\mathbf{b} = (b_1, ..., b_m)$, does there exist $\mathbf{x} = (x_1, ..., x_n)$ such that

$$a_{i1}x_1 + ... + a_{in}x_n = b_i \qquad \text{for each } i = 1, ..., m?$$

- If no, then for which $\mathbf{b}$ does there exist such an $\mathbf{x}$?
- If yes, how many 'different' $\mathbf{x}$ are there for fixed $\mathbf{b}$?

Answer: Let $A = [a_{ij}] \in M_{m\times n}(\mathbb{R})$, which defines $T : \mathbb{R}^n \to \mathbb{R}^m$. Then

$$a_{i1}x_1 + ... + a_{in}x_n = b_i \qquad \text{for each } i = 1, ..., m$$

if and only if

$$A\mathbf{x} = \mathbf{b}, \text{ or equivalently } T(\mathbf{x}) = \mathbf{b}.$$

Thus, we have the following answer:

- There exists $\mathbf{x}$ solving the equation if and only if $\mathbf{b} \in \operatorname{im}(T)$.
- For each fixed $\mathbf{b} \in \operatorname{im}(T)$, the set of solutions is $T^{-1}(\{\mathbf{b}\})$, which we showed is in bijection with $\ker(T)$.

Injectivity and surjectivity revisited

Let V, W be vector spaces and $\alpha = \{\mathbf{v}_1, ..., \mathbf{v}_n\}, \beta = \{\mathbf{w}_1, ..., \mathbf{w}_m\}$ be bases, and fix any of the following equivalent pieces of data:

(1) $T : V \to W$
(2) $\{T(\mathbf{v}_1), ..., T(\mathbf{v}_n)\}$ a list of vectors in W
(3) $[T]_\alpha^\beta$ a matrix of numbers $a_{ij} \in \mathbb{R}$
(4) a system of equations $a_{i1}x_1 + ... + a_{in}x_n =?$ for $i = 1, ..., m$

T is surjective if for any $\mathbf{b} \in W$, there is $\mathbf{x} \in V$ with $T(\mathbf{x}) = \mathbf{b}$.

In each of the above pictures, we have an equivalent condition:

(1) $\text{im}(T) = W$
(2) $\text{Span}(\{T(\mathbf{v}_i)\} = W$
(3) $\text{col}([T]_\alpha^\beta) = W$
(4) for any $\mathbf{b} \in W$, there *exists* $\mathbf{x}$ solving $a_{i1}x_1 + ... + a_{in}x_n = b_i$

Similarly: T is injective if $T(\mathbf{x}) = T(\mathbf{y})$ imples $\mathbf{x} = \mathbf{y}$. Equivalently,

(1) $\ker(T) = \{\mathbf{0}\}$
(2) $\{T(\mathbf{v}_i)\}$ is linearly independent.
(3) $\text{null}([T]_\alpha^\beta) = \{\mathbf{0}\}$
(4) For $\mathbf{b} \in \text{im}(T)$, solution to $a_{i1}x_1 + ... + a_{in}x_n = b_i$ is *unique*

Towards The Dimension Theorem

We have reduced the question of **existence** and **uniqueness** of solutions to linear equations to understanding the **image** and **kernel** of a linear map $T : V \to W$.

What can we say about $\text{im}(T)$ and $\ker(T)$ in general? Let's look at some examples:

(1) $T : \mathbb{R}^2 \to \mathbb{R}^3 \quad (x_1, x_2) \mapsto (x_1, x_2, 0)$
- $\dim \ker(T) = 0$, $\dim \text{im}(T) = 2$.

(2) $T : \mathbb{R}^2 \to \mathbb{R}^3 \quad (x_1, x_2) \mapsto (x_1, 0, 0)$
- $\dim \ker(T) = 1$, $\dim \text{im}(T) = 1$.

(3) $T : \mathbb{R}^2 \to \mathbb{R}^3 \quad (x_1, x_2) \mapsto (0, 0, 0)$
- $\dim \ker(T) = 2$, $\dim \text{im}(T) = 0$.

(4) $T : \mathbb{R}^n \to \mathbb{R}^m \quad (x_1, ..., x_n) \mapsto (x_1, ..., x_{n-k}, 0, ..., 0)$
- $\dim \ker(T) = k$, $\dim \text{im}(T) = n - k$.

Claim: Every linear $T : V \to W$ 'looks like this' wrt some bases.

Corollary: Let $T : V \to W$ linear, with $\dim V = n$. Then

$$\dim V = n = k + (n - k) = \dim \ker(T) + \dim \text{im}(T)$$

Discussion: The Dimension Theorem

Theorem: (The Dimension Theorem) Let $T : V \to W$ be a linear map, with V finite dimensional. Then

$$\dim V = \dim \ker(T) + \dim \operatorname{im}(T)$$

Let's prove the dimension theorem using the following steps:

To fix notation, let's say $\dim V = n$.

(1) Since $\ker(T) \subset V$, we know $k := \dim \ker(T) \leq \dim V = n$.

(2) Choose a basis $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ for $\ker(T)$, and extend to a basis $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ for V.

(3) Show that $\{T(\mathbf{v}_{k+1}), ..., T(\mathbf{v}_n)\}$ define a basis for $\operatorname{im}(T)$. (If it were linearly dependent, find a 'new' element of $\ker(T)$)

(4) Conclude that $\dim \operatorname{im}(T) = n - k$.

(5) Use that $n = k + (n - k)$ to prove the theorem.

Suppose $\dim W = m$ is finite, and extend $\{T(\mathbf{v}_{k+1}), ..., T(\mathbf{v}_n)\}$ to a basis for W.

What is $[T]$ with respect to these bases?

Discussion: Applications of the dimension theorem

Let $T : V \to W$ and V and W finite dimensional. Recall:

T is injective if and only if $\ker(T) = \{\mathbf{0}\}$

T is surjective if and only if $\operatorname{im}(T) = W$.

Theorem: $\dim V = \dim \ker(T) + \dim \operatorname{im}(T)$

(a) If $\dim(\ker T) = 0$ can you determine if T is injective? What about surjective? (doesn't require the theorem)

(b) If $\dim(\operatorname{im} T) = \dim W$ can you determine if T is injective? What about surjective? (doesn't require the theorem)

(c) If $\dim(\operatorname{im} T) = \dim V$ can you determine if T is injective? What about surjective?

(d) If $\dim V = \dim W$ and T is injective, can you determine if T is surjective? What about vice versa?

(e) If $\dim V < \dim W$, can T be injective? Can it be surjective?

(f) If $\dim V > \dim W$, can T be injective? Can it be surjective?

Discussion: Further applications, putting it all together!

Recall we had 4 different pictures of a linear map:

(1) $T : V \to W$

(2) $\{ T(\mathbf{v}_1), ..., T(\mathbf{v}_n) \}$ a list of vectors in W

(3) $[T]_\alpha^\beta$ a matrix of numbers $a_{ij} \in \mathbb{R}$

(4) a system of equations $a_{i1}x_1 + ... + a_{in}x_n = ?$ for $i = 1, ..., m$

and for each of these, an interpretation of injective and surjective.

Combine these with the dimension theorem to show:

(1) Let $W = \{\mathbf{x} \in \mathbb{R}^n | a_1 x_1 + ... + a_n x_n = 0\}$. Show $\dim W = n - 1$ unless the a_i are all zero.

(2) Let V be n dimensional and $S = \{\mathbf{v}_1, ..., \mathbf{v}_n\} \subset V$. Show that:

(a) If S is linearly independent, then S must be a basis.

(b) If $\text{Span}(S) = V$ then S must be a basis.

(3) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ a linear map with $[T] \in M_{n \times n}(\mathbb{R})$. Show that if $[T]$ has linearly dependent columns, then T is not surjective.