MAT224 - LEC5101 - Lecture 3 // Basis and dimension

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Intuitive overview of last week

Definition: Let $S = \{\mathbf{x}_1, ..., \mathbf{x}_n\}$ be a (finite) subset of V.

- A vector $\mathbf{v} \in V$ is a *linear combination* of vectors in S if

$$\mathbf{v} = a_1\mathbf{x}_1 + ... + a_n\mathbf{x}_n \qquad \text{for some} \qquad a_1, ..., a_n \in \mathbb{R}$$

- Span(S) is the set of all linear combinations of vectors in S:
  $\text{Span}(S) = \{\mathbf{v} = a_1\mathbf{x}_1 + ... + a_n\mathbf{x}_n \in V | a_1, ..., a_n \in \mathbb{R}\} \subset V$

Natural Question: For each set $S = \{\mathbf{x}_1, ..., \mathbf{x}_n\}$ of n vectors in V, how can we predict the 'size' of the resulting subspace Span(S)?

Example: Let $V = \mathbb{R}^3$ and $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in V$ vectors in V.

- Let $S = \{\mathbf{x}_1\}$. Then Span(S) is is a 'line', same 'size' as $\mathbb{R}$.
  Unless: $\mathbf{x}_1 = \mathbf{0}$. (including $\mathbf{0}$ in S never changes Span(S))
- Let $S = \{\mathbf{x}_1, \mathbf{x}_2\}$. Then Span($S$) is a plane, same 'size' as $\mathbb{R}^2$.
  Unless: $\mathbf{x}_1$ or $\mathbf{x}_2$ is $\mathbf{0}$, or $\mathbf{x}_2 = a\mathbf{x}_1$ for $a \in \mathbb{R}$.
- Let $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$. Then Span($S$) is all of $V = \mathbb{R}^3$.
  Unless: Some $\mathbf{x}_i$ is 0, or $\mathbf{x}_2 = a\mathbf{x}_1$, or $\mathbf{x}_3 = b\mathbf{x}_1$, or $\mathbf{x}_3 = c\mathbf{x}_2$,
  OR $\mathbf{x}_3 = a_1\mathbf{x}_1 + a_2\mathbf{x}_2$, etc. (which generalize the above).

Goal today: Properly state and answer the question

Partial Answer: For each set $S = \{\mathbf{x}_1, ..., \mathbf{x}_n\}$ of n vectors in V, the resulting subspace Span(S) is the 'size' of $\mathbb{R}^n$.
Unless: the vectors of S 'satisfy some linear relation'.

Definition: (Recall from last week)

- A *linear dependence* among the vectors of S is an equation:

  $$a_1\mathbf{x}_1 + ... + a_n\mathbf{x}_n = \mathbf{0} \qquad \text{for some} \qquad a_1, ..., a_n \in \mathbb{R}$$

- S is called *linearly dependent* if it has a linear dependence.
- S is called *linearly independent* if it does not have one.

Slightly Better Answer: Span(S) is the 'size' of $\mathbb{R}^n$ if and only if S is linearly independent.
Span(S) gets 'smaller' for each 'different' linear dependence in S.

But what does 'size' really mean? We want to define *dimension*.

Tentative Definition: A subspace $W \subset V$ has *dimension* n if $W = \text{Span}(S)$ for $S = \{\mathbf{x}_1, ..., \mathbf{x}_n\}$ linearly independent.

Problem: Need to check this is 'well-defined', independent of S.

Discussion: The notion of Basis

Let V be a vector space and $\mathbf{x}_1, ..., \mathbf{x}_n \in V$.

Definition: A set $S = \{\mathbf{x}_1, ..., \mathbf{x}_n\}$ is called a *basis* for V if

(a) $V = \text{Span}(S)$

(a) S is linearly independent.

Examples: The list $1, x, x^2, \ldots, x^n$ is a basis for $P_n(\mathbb{R})$.
The list $\mathbf{e}_1 = (1, 0, ..., 0), \mathbf{e}_2 = (0, 1, 0, ..., 0), ..., \mathbf{e}_n = (0, ..., 0, 1)$ is a basis for $\mathbb{R}^n$.
The set $\{[e_{ij}]\}_{i=1,\ldots,n}^{j=1,\ldots,m}$ is a basis for $M_{n\times m}(\mathbb{R})$.

Which of the following define a basis for $\mathbb{R}^3$? (With proof)

(1) $\mathbf{x}_1 = (1, 1, 0), \mathbf{x}_2 = (0, 0, 1)$

(2) $\mathbf{x}_1 = (1, 1, 0), \mathbf{x}_2 = (0, 0, 1), \mathbf{x}_3 = (1, -1, 0)$

(3) $\mathbf{x}_1 = (1, 1, 0), \mathbf{x}_2 = (0, 0, 1), \mathbf{x}_3 = (1, -1, 0), \mathbf{x}_4 = (0, 1, 0)$

Bonus: Let $V, \mathbf{x}_i$ as above, and $S = \{\mathbf{x}_1, ..., \mathbf{x}_n\}$.
Show that S is a basis if and only if the following holds:
For every vector $\mathbf{v} \in V$, there **exists** a **unique** choice of scalars

$$a_1, ..., a_n \in \mathbb{R} \quad \text{such that} \quad \mathbf{v} = a_1\mathbf{x}_1 + ... + a_n\mathbf{x}_n$$

Constructing a Basis

Question: Does there always exist a basis? **Answer:** Yes, almost...

Counterexample: The vector space $P(\mathbb{R})$ of all polynomials.

Thus, we need to assume the following, and we do for this slide:

Assumption: There exists a finite set S such that $V = \text{Span}(S)$.

Lemma: Let $S = \{\mathbf{x}_1, ..., \mathbf{x}_n\}$ be linearly independent, and $\mathbf{v} \in V$.

$S' = \{\mathbf{x}_1, ..., \mathbf{x}_n, \mathbf{v}\}$ is linearly dependent if and only if $\mathbf{v} \in \text{Span}(S)$.

Equivalently, S' is linearly independent if and only if $\mathbf{v} \notin \text{Span}(S)$.

Example: Recall our discussion about subspaces of $\mathbb{R}^3$.

Theorem: Then there exists a basis for V. Moreover,

For any $S = \{\mathbf{x}_1, ..., \mathbf{x}_k\}$ linearly independent, there exists
$S' = \{\mathbf{x}_1, ..., \mathbf{x}_k, \mathbf{x}_{k+1}, ..., \mathbf{x}_n\}$ such that S' is a basis for V.

For any $S = \{\mathbf{x}_1, ..., \mathbf{x}_m\}$ such that $\text{Span}(S) = V$, there exists
$S' = \{\mathbf{x}_{i_1}, ..., \mathbf{x}_{i_n}\} \subset S$ such that S' is a basis for V.

Warning: This used the above assumption.

Discussion: Constructing a basis

Recall $P_2(\mathbb{R}) = \{a + bx + cx^2 | a, b, c \in \mathbb{R}\}$.

True or False: (with proof!)

1. If $p(x), q(x), r(x) \in P_2(\mathbb{R})$ such that $p(x) + q(x) = q(x) + r(x)$, then the list $p(x), q(x), r(x)$ cannot be a basis for $P_2(\mathbb{R})$.
2. The list $1 + 2x - x^2, 1 + x + x^2$ is a basis for the subspace $W = \{a + bx + cx^2 \in P_2(\mathbb{R}) \mid -3a + 2b + c = 0\}$.

Let $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ a list of vectors in V, such that $V = \text{Span}(S)$, and the list $\{\mathbf{x}_1, \mathbf{x}_2\}$ is a basis for V.

True or False: (with proof!)

1. The list $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ must be linearly dependent.
2. Suppose the list $\mathbf{x}_1, \mathbf{x}_3$ is also basis for V. Then it follows that the list $\mathbf{x}_2, \mathbf{x}_3$ is a basis for V.

Dimension

Lemma: Let S be a spanning set for V with m elements.
Then any linearly independent set S' in V has at most m elements.

Example: We know $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ spans $\mathbb{R}^3$. Any linearly independent set of vectors in $\mathbb{R}^3$ has at most 3 elements.

Key Corollary: Let $S_1 = \{\mathbf{x}_1, ..., \mathbf{x}_n\}$ and $S_2 = \{\mathbf{y}_1, ..., \mathbf{y}_m\}$ both be a basis of V. Then $n = m$, i.e. the lists are the same length.

Definition: The *dimension* of V is the number of elements in some (or equivalently any) basis for V.

For V satisfying our assumption, this number is finite by previous theorem. Any V not satisfying this is called *infinite dimensional*.

Examples: $\dim \mathbb{R}^n = n$, $\dim P_n(\mathbb{R}) = n + 1$, $\dim M_{m\times n}(\mathbb{R}) = mn$.

Theorem: Let U be a subspace of a finite dimensional V. Then $\dim U \leq \dim V$, with equality if and only if $U = V$.

Discussion: Dimension

What dimension are the following subspaces of $\mathbb{R}^3$: (with proof)

(1) $W = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 | \ x_3 = 0\}$

(2) $W = \text{Span}(\{(1, 0, 0), (1, 1, 0), (1, -1, 0)\})$

Let $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}, \mathbf{x_4} \in V$ and $U = \text{Span}\{\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}, \mathbf{x_4}\}$.

Suppose that $\mathbf{x_3} = \mathbf{x_1} - \mathbf{x_2}$ and $\mathbf{x_4} = 2\mathbf{x_1} + 3\mathbf{x_2} - \mathbf{x_3}$.

(3) What are the possible dimensions of U?

(4) Suppose $\dim U = 2$. Must $\{\mathbf{x_3}, \mathbf{x_4}\}$ be linearly independent?

Bonus: Let W_1, W_2 be subspaces of V, and recall

$$W_1 + W_2 = \{\mathbf{w} + \mathbf{x} \mid \mathbf{w} \in W_1 \text{ and } \mathbf{x} \in W_2\}$$

Show that $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$.