1 Addendum

1.1 Question from 'Discussion: Constructing a Basis'

I got a bit confused in class solving a system of linear equations at the end of the following problem. Here, I repeat the main part of the proof as in class, and then solve the relevant system as required.

Problem: Recall $P_2(\mathbb{R}) = \{a + bx + cx^2 | a, b, c \in \mathbb{R}\}$.

True or false: (with proof)

The list $1 + 2x - x^2, 1 + x + x^2$ is a basis for the subspace

$$W = \{a + bx + cx^2 \in P_2(\mathbb{R}) \mid -3a + 2b + c = 0\}$$

Solution: The statement is true.

Let $S = \{1 + 2x - x^2, 1 + x + x^2\}$ be the list above. To begin, note that the elements of the list are easily checked to be linearly independent. Thus, it remains to check that $\text{Span}(S) = W$.

Now, one can check that both polynomials satisfy the condition that $-3a + 2b + c = 0$, so that S is a subset of W. Then, since W is a subspace, we can apply the proposition from lecture 2 to conclude that $\text{Span}(S) \subset W$.

It remains to show $W \subset \text{Span}(S)$. To do this, we (as always) take an arbitrary element $\mathbf{w} \in W$ and we need to show that $\mathbf{w} \in \text{Span}(S)$.

By definition, we have $\mathbf{w} = a + bx + cx^2$ for some $a, b, c \in \mathbb{R}$, such that $-3a + 2b + c = 0$. On the other hand, basically by definition we have

$$\text{Span}(S) = \{(s + t) + (2s + t)x + (-s + t)x^2 |\ s, t \in \mathbb{R}\}\ .$$

Thus, we must check that for any fixed $a, b, c \in \mathbb{R}$ as above, we can find $s, t \in \mathbb{R}$ so that

$$a + bx + cx^2 = (s + t) + (2s + t)x + (-s + t)x^2$$

Equivalently, we need to solve the system of linear equations

$$s + t = a \qquad 2s + t = b \qquad -s + t = c$$

These seem overdetermined, were it not for the condition on a, b, c above. As we show below, after accounting for it, there does exist a solution.

Adding, or subtracting, the first and third equations, we arrive at the solutions

$$t = (a + c)/2 \qquad s = (a - c)/2\ ,$$

respectively. These succesfully solve the first and third equation, so we just need to check they solve the second. Plugging in, we find:

$$2s + t = (3a - c)/2 = b\ ,$$

as desired, where we used the constraint $-3a+2b+c = 0$ in the second equality. Thus, the system admits a solution.

In summary, we've shown that $\mathbf{w} \in \text{Span}(S)$, so that indeed $W \subset \text{Span}(S)$, as desired. This completes the proof that $W = \text{Span}(S)$, and that S is a basis for W.

1.2 An alternative conceptual solution

A more conceptual way to preform the final step, checking that $W = \text{Span}(S)$, is the following: (Unfortunately, we couldn't use this method at the time, because we hadn't learned about dimension until later in class.)

The previously established facts show that $\text{Span}(S)$ is two dimensional, and that $\text{Span}(S) \subset W$. Further, one can independently check that W is also two dimensional, as the solution set to a single linear equation in a three dimensional space (for example, this will follow immidiately from the dimension theorem, which we'll prove next class). Then, a proposition from the section on dimension in this lecture says that $\dim(\text{Span}(S)) \leq \dim(W)$, with equality if and only if $\text{Span}(S) = W$. Since we have observed that we have this equality of dimensions, it follows that $\text{Span}(S) = W$.

1.3 Solutions to questions from 'Discussion: Dimension'

Note: These aren't full solutions, just answers without proof for your reference. If you want a full explanation, come by my office hours, Tuesday 3-5 in the Huron lounge!

(1) It has dimension 2.

(2) It has dimension 2.

(3) It has possible dimensions 0,1, or 2.

(4) Yes, they must be linearly independent.