MAT224 - LEC5101 - Lecture 2
Linear combinations, span, and independence

Dylan Butson

University of Toronto

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Subspaces review

Definition: A *subspace* of a vector space V is a subset W of V that is itself a vector space with the same operations of vector addition and scalar multiplication as in V.

Note that for $\mathbf{w}, \mathbf{x} \in W$, we only know $\mathbf{w} + \mathbf{x}, c\mathbf{w} \in V$ in general. Thus, the definition of subspace implicitly requires that W satisfies:

- *closed under addition*: for any $\mathbf{w}, \mathbf{x} \in W$, $\mathbf{w} + \mathbf{x} \in W$.
- *closed under scalar mult.*: for any $\mathbf{w} \in W$, $c \in \mathbb{R}$, $c\mathbf{w} \in W$.

Equivalently, we can combine these into a single criterion:

- *closed under both*: for each $\mathbf{x}, \mathbf{w} \in W$ and $c \in \mathbb{R}$, $c\mathbf{w} + \mathbf{x} \in W$.

Proposition: A subset W of V is a subspace if and only if it is non-empty and closed under both addition and scalar mult.

Example:

- $W_1 = \{(x_1, x_2) \in \mathbb{R}^2 | x_2 = 2x_1\} \subset \mathbb{R}^2 = V$ is a subspace.
- $W_2 = \{(x_1, x_2) \in \mathbb{R}^2 | x_2 = x_1^2\} \subset \mathbb{R}^2 = V$ is not.

Discussion: Properties and (non) examples of subspaces

Let $A \in M_{m\times n}(\mathbb{R})$. Show that

(1) the null space $\mathrm{null}(A) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$ is a subspace of $\mathbb{R}^n$; and

(2) the column space $\mathrm{col}(A) = \{A\mathbf{x} \in \mathbb{R}^m \mid \mathbf{x} \in \mathbb{R}^n\}$ - is a subspace of $\mathbb{R}^m$.

Which of the following subsets W of $M_{n\times n}(\mathbb{R})$ are subspaces of $M_{n\times n}(\mathbb{R})$?

(3) $W = \{A \in M_{n\times n}(\mathbb{R}) \mid A \text{ is invertible}\}$

(4) $W = \{A \in M_{n\times n}(\mathbb{R}) \mid \text{ the last column of } A \text{ is zero}\}$

(5) $W = \{A \in M_{n\times n}(\mathbb{R}) \mid A^2 = \mathbf{0}\}$

(6) Give examples of subsets $W \subset \mathbb{R}^2$ that are closed under addition but not scalar multiplication, and vice versa.

Linear Combination and Span

Definition: Let $S = \{\mathbf{x}_1, ..., \mathbf{x}_n\}$ be a (finite) subset of V.

- A vector $\mathbf{v} \in V$ is a *linear combination* of vectors in S if
  $$\mathbf{v} = a_1\mathbf{x}_1 + ... + a_n\mathbf{x}_n \text{ for some } a_1, ..., a_n \in \mathbb{R} .$$
- Span(S) is the set of all linear combinations of vectors in S:
  $$\text{Span}(S) = \{\mathbf{v} = a_1\mathbf{x}_1 + ... + a_n\mathbf{x}_n \in V | a_1, ..., a_n \in \mathbb{R}\} \subset V$$
- If Span$(S) = W$ we say S *spans* the subspace W.

Examples:

Let $\mathbf{x} = (1, 0) \in \mathbb{R}^2$. Then
$\text{Span}(\mathbf{x}) = \{\mathbf{v} = (v_1, 0) \in \mathbb{R}^2\} = \{\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2 | v_2 = 0\}$
Let $\mathbf{x}_1 = (1, 2) \in \mathbb{R}^2$. Then
$\text{Span}(\mathbf{x}_1) = \{\mathbf{v} = (v_1, 2v_1) \in \mathbb{R}^2\} = \{\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2 | v_2 = 2v_1\}$
Let $\mathbf{x}_2 = (2, 4) \in \mathbb{R}^2$. Then
$\text{Span}(\mathbf{x}_2) = \text{Span}(\mathbf{x}_1) = \text{Span}(\{\mathbf{x}_1, \mathbf{x}_2\})$.

Proposition:

(a) The set Span$(S) \subset V$ defines a subspace of V.

(b) If $W \subset V$ is a subspace of V and $S \subset W$, then Span$(S) \subset W$.

Discussion: Examples of Spans and the Sum Operation

Let $\mathbf{v} = (1,0,0), \mathbf{w} = (0,1,2), \mathbf{x} = (1,1,2) \in \mathbb{R}^3$.

(1) Let $S = \{\mathbf{v}, \mathbf{w}\}$, describe $\text{Span}(S)$ as a subset of $\mathbb{R}^3$ as above.

(2) Let $S = \{\mathbf{v}, \mathbf{x}\}$, and do the same.

(3) How do these compare with $\text{Span}(\{\mathbf{v}, \mathbf{w}, \mathbf{x}\})$?

Let $S = \{1, x - 2x^2, 3x^2, 1 + 4x^2\} \subset P_2(\mathbb{R})$

(4) Show $\{1, x, x^2\} \subseteq \text{Span}(S) \subseteq P_2(\mathbb{R})$.

(5) Deduce that $P_2(\mathbb{R}) \subseteq \text{Span}(S)$ and thus $\text{Span}(S) = P_2(\mathbb{R})$.

Bonus: Let W and X be subspaces of V and define

$$W + X = \{\mathbf{v} = \mathbf{w} + \mathbf{x} \in V | \mathbf{w} \in W \text{ and } \mathbf{x} \in X\} \subset V$$

(a) Show that $W + X$ is a subspace of V.

(b) If $W = \text{Span}(\{\mathbf{w}_1, ..., \mathbf{w}_n\})$ and $X = \text{Span}(\{\mathbf{x}_1, ..., \mathbf{x}_m\})$ then

$$W + X = \text{Span}(\{\mathbf{w}_1, ..., \mathbf{w}_n, \mathbf{x}_1, ..., \mathbf{x}_m\})$$

Linear (In)Dependence

Definition: Let $S = \{\mathbf{x}_1, ..., \mathbf{x}_n\}$ be a subset of V.

- A *linear dependence* among the vectors of S is an equation:
$$a_1\mathbf{x}_1 + ... + a_n\mathbf{x}_n = \mathbf{0} \quad \text{for some} \quad a_1, ..., a_n \in \mathbb{R}$$
It is called *trivial* if $a_1 = ... = a_n = 0$ and *non trivial* otherwise.
- S is called *linearly dependent* if it has a non trivial linear dependence.

Example: Let $\mathbf{v} = (1, 0, 0), \mathbf{w} = (0, 1, 2), \mathbf{x} = (1, 1, 2) \in \mathbb{R}^3$.

- The set $S = \{\mathbf{v}, \mathbf{w}, \mathbf{x}\}$ has a non trivial linear dependence:
$$1\mathbf{v} + 1\mathbf{w} + (-1)\mathbf{x} = \mathbf{0}$$

Definition: S is called *linearly independent* if it has no non trivial linear dependence.

Equivalently, S called is linearly independent if: for any equation
$$a_1\mathbf{x}_1 + ... + a_n\mathbf{x}_n = \mathbf{0} \quad \text{for some} \quad a_1, ..., a_n \in \mathbb{R}$$
we must have $a_1 = ... = a_n = 0$.

Example: The set $S_1 = \{\mathbf{v}, \mathbf{w}\}$ is linearly independent.

Discussion: Examples of Linear (In)dependence

(1) Let $S = \{1 + x, x + x^2, 1 - x^2\} \subset P_2(\mathbb{R})$.
Prove that S is linearly dependent.

(2) Let $S = \{\mathbf{x}_1, ..., \mathbf{x}_n\} \subset V$ and $\mathbf{v} \in \text{Span}(S)$.
Prove that $S' = \{\mathbf{x}_1, ..., \mathbf{x}_n, \mathbf{v}\}$ is linearly dependent.

(3) Let $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1) \in \mathbb{R}^3$.
Prove that $S = \{e_1, e_2, e_3\}$ is linearly independent.

(4) Let $S = \{1 + x, x + x^2, 1 + x^2\} \subset P_2(\mathbb{R})$.
Prove that S is linearly independent.

Bonus: Let $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ and $S_1 = \{\mathbf{x}_2, \mathbf{x}_3\}$,$S_2 = \{\mathbf{x}_1, \mathbf{x}_3\}$, and $S_3 = \{\mathbf{x}_1, \mathbf{x}_2\}$.
If S_1, S_2, S_3 are each linearly independent, does this imply S is linearly independent? Prove or give counterexample.