MAT224 - LEC5101 - Lecture 1 // Introduction, vector spaces, and subspaces

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Welcome!

- Hey, I'm Dylan!
- I'm a senior PhD student here; no need to call me 'Doctor' or 'Professor' or whatever - just Dylan is really fine!
- I do a mix of very pure math (algebraic geometry and algebraic topology) and very theoretical physics (quantum field theory and string theory).
- I'm generally very interdisciplinary and happy to talk about lots of areas of math and its applications!
- I'll have 'student-instructor hours' on Tuesdays 3-5, where you can come to ask me about the course, or whatever else you're interested in.
- I'm also a normal person, just like you.

Mat224: boring technical points

- This is mat224, a *second course* in linear algebra. You're expected to be familiar with the main ideas from mat223, including:
- solving systems of linear equations, subspaces of $\mathbb{R}^n$, span, linear independence and dependence, basis, dimension, rank, column space, null space, projections, and diagonalization.
- We will revisit almost all of these ideas this semester, but from a **more conceptual perspective, assuming you understand the underlying computational techniques. In particular, you will be required to write clear proofs.**
- The technical details of the course are in the syllabus, you're responsible for reading it. Highlights:
- Evaluation: two term tests worth .50, one final exam worth .40, a bunch of small 'writing assignments' worth .10.
- Writing assignments: marked for 'careful completion' only, but with feedback. More important to do so you can learn from feedback, than for their actual grade value!

Why learn linear algebra?

- Linear algebra is a key foundational subject in math
- Also, it is the basic language used in many important 'real life' applications:
- Physics: special relativity, quantum mechanics, ...
- Computer science: parallel computing, machine learning (backprop), data science (PCA), ...
- Probability and statistics: linear regression, Markov processes (MCMC, search engine algorithms), ...
- Economics and finance: linear supply-demand models, applications of the ideas in comp. sci. and statistics above, ...
- **You should be excited about learning this new language, and all the opportunities that come with it!**
  Be able to read anything you want on wikipedia!

How (not) to learn linear algebra?

- I really want you to succeed. I'm going to explain my plan:
- **Math is a language!** Linear algebra will be most useful to you in 'real life' if you really learn it as such.
- Also, it will make solving problems on tests much easier:
- Imagine you have to write a reading comprension test in a new language. Memorizing phrases to use in response to some particular questions isn't a good strategy for this, and even less useful in an actual conversation!
- Developing an 'intuitive understanding' gives much more robust outcomes, on tests and in 'real life'.
- In class, time spent just passively copying words or symbols that you don't understand won't be useful. There are lots of great textbooks, summary notes, videos about linear algebra available for free online. Also, I'll post my slides.
- **You're not a robot, don't try to learn like one!**
- So how **should** you learn?

Active Learning

- This class will be run in an **active learning** style: you will be given time to engage with new concepts in real time, discuss with your peers, ask questions, and get feedback!
- Usual story: you spend 3 hours every week sitting here taking notes, read over resulting endless notes the night before a homework deadline or test, realize you never really understood some key idea from a few weeks ago, but now there's no one around to ask about it. ☹
- Our goal: you think hard about concepts while in class, realize ASAP what you don't understand, get clarification right away, and build on that knowledge as lecture continues. ☺
- This will feel like harder work in class if you really commit to it, but it will save you time in the long run if you do!
- Large scale study of undergraduate classes suggests active learning results in **12 percent decrease in failure rate!** (Citation: Janet Rankin, MIT opencourseware lecture)

Typical Class Outline

- I'll begin by explaining a new concept in traditional lecture style.
- I'll put up a slide with a few basic questions about the new idea, and then I'll stop talking for about 5 minutes and walk around the classroom helping out, as you go through 3 steps:
- Think: You think about the questions on your own
- Pair: You discuss them with one or two of your neighbours
- Share: We refocus discussion to the front of the class, I'll ask you to share the answers you think are correct, or any questions you have, and I'll present solutions to the problems.
- Then I'll start discussing another new concept, and we'll repeat.
- Now, lets try it!
- But first, some quick introductions...

Discussion: The well-cultured gameshow

You're on a game show! But the rules are slightly unconventional...

- There are **four** doors, and behind each door is a new **car**!
- Each car has a **single** colour, and is either an **electric car** ☺, or a **gas SUV** ☹.
- **Your goal**: the game show host tells you the following statement, and you must determine if she is telling the truth:

  "There is **no gas SUV** behind any door, unless it is pink!"
- Moreover, the host tells you that:
  1. The car behind the first door is an **electric car** ☺
  2. The car behind the second door is a **gas SUV** ☹
  3. The car behind the third door is pink
  4. The car behind the fourth door is green

What is the **least** number of doors you need to open in order to determine if the statement is true?

(Real) Vector spaces

Definition A *(real) vector space* is a set V together with:

(A) an operation called *vector addition*, which for each pair of vectors $\mathbf{x}, \mathbf{y} \in V$ produces another vector $\mathbf{x} + \mathbf{y}$ in V; and

(B) an operation called *multiplication by a scalar* (a real number), which for each vector $\mathbf{x} \in V$, and each scalar $c \in \mathbb{R}$ produces another vector in V denoted $c\mathbf{x}$; such that

1. For all vectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$
2. For all vectors $\mathbf{x}, \mathbf{y} \in V$, $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
3. There exists a vector $\mathbf{0} \in V$ with the property that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for all vectors $\mathbf{x} \in V$
4. For each vector $\mathbf{x} \in V$, there exists a vector $-\mathbf{x} \in \mathbf{V}$ with the property that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$
5. For all vectors $\mathbf{x}, \mathbf{y} \in V$, and scalars $c \in \mathbb{R}$, $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$
6. For all vectors $\mathbf{x} \in V$, and scalars $c, d \in \mathbb{R}$, $(c + d)\mathbf{x} = c\mathbf{x} + d\mathbf{x}$
7. For all vectors $\mathbf{x} \in V$, and scalars $c, d \in \mathbb{R}$, $(cd)\mathbf{x} = c(d\mathbf{x})$
8. For all vectors $\mathbf{x} \in V$, $1\mathbf{x} = \mathbf{x}$

Discussion: Properties and (non) examples of vector spaces

Recall that $\mathbb{R}^n, M_{m\times n}(\mathbb{R}), P_n(\mathbb{R}), \{0\}$ are vector spaces.
Prove the following properties of vector spaces:

(a) Let V be a vector space and $\mathbf{x} \in V$. Then $0\mathbf{x} = \mathbf{0}$.
(Hint: 0+0=0)

(b) Let V be a vector space and $\mathbf{x} \in V$. Then $(-1)\mathbf{x} = -\mathbf{x}$.

Which of the following sets with operations are vector spaces?

(c) $W = \{(x, y) \in \mathbb{R}^2 | y = 2x\}$ with addition and scalar mult. of $\mathbb{R}^2$.

(d) $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} | x \geq 0\}$ with usual addition and $c\tilde{\cdot}x = |c|x$.

(e) $\mathbb{R}_{>0} = \{x \in \mathbb{R} | x > 0\}$ with $x\tilde{+}y = xy$ and $c\tilde{\cdot}x = x^c$

You can skip the proof for (c), we'll come back to it later.

Problem solving tips:

1. Draw a picture
2. Write what you need to show
3. Use the hypotheses, their definitions, and related theorems.

Subspaces

Definition: A *subspace* of a vector space V is a subset W of V that is itself a vector space with the same operations of vector addition and scalar multiplication as in V.

Example: Consider $W_1 = \{(x, y) \in \mathbb{R}^2 | y = 2x\} \subset \mathbb{R}^2 = V$. Then W is a subspace, as we agreed, and will prove below.

Alternatively, $W_2 = \{(x, y) \in \mathbb{R}^2 | y = x^2\} \subset \mathbb{R}^2 = V$ is not.

Note that for $\mathbf{w}, \mathbf{x} \in W$, we only know $\mathbf{w} + \mathbf{x}, c\mathbf{w} \in V$ in general. Thus, the definition of subspace implicitly requires that W satisfies:

- *closed under addition*: for any $\mathbf{w}, \mathbf{x} \in W$, the element $\mathbf{w} + \mathbf{x} \in W$.
- *closed under scalar multiplication*: for any $\mathbf{w} \in W$, $c \in \mathbb{R}$, the element $c\mathbf{x} \in W$.

Proposition: A subset W of V is a subspace if and only if it is non-empty, closed under addition, and under scalar multiplication.

Corollary: W_1 above is a subspace, while W_2 is not.

Discussion: Properties and (non) examples of subspaces

Let $A \in M_{m\times n}(\mathbb{R})$. Show that

(1) the null space $\mathrm{null}(A) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$ is a subspace of $\mathbb{R}^n$; and

(2) the column space $\mathrm{col}(A) = \{A\mathbf{x} \in \mathbb{R}^m \mid \mathbf{x} \in \mathbb{R}^n\}$ - is a subspace of $\mathbb{R}^m$.

Which of the following subsets W of $M_{n\times n}(\mathbb{R})$ are subspaces of $M_{n\times n}(\mathbb{R})$?

(3) $W = \{A \in M_{n\times n}(\mathbb{R}) \mid A \text{ is invertible}\}$

(4) $W = \{A \in M_{n\times n}(\mathbb{R}) \mid \text{ the last column of } A \text{ is zero}\}$

(5) $W = \{A \in M_{n\times n}(\mathbb{R}) \mid A^2 = \mathbf{0}\}$

(6) Give examples of subsets $W \subset \mathbb{R}^2$ that are closed under addition but not scalar multiplication, and vice versa.

Bonus: Prove that for any two subspaces $W_1, W_2 \subset V$, the intersection $W_1 \cap W_2 = \{\mathbf{x} \in V | \mathbf{x} \in W_1 \text{ and } \mathbf{x} \in W_2\}$ is a subspace.