MAT224 - LEC5101 - Lecture 11
Nilpotent and Jordan Canonical Form

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Overview: Diagonalization to Jordan Form

Let V a finite dimensional F vector space and $T : V \to V$ linear.

Fix a basis $\alpha = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ for V. By definition, TFAE:

- $[T]_\alpha^\alpha$ is diagonal with $a_{ii} = \lambda_i$ for each $i = 1, ..., n$
- $T(\mathbf{v}_i) = \lambda_i \mathbf{v}_i$ for each $i = 1, ..., n$
- $\mathbf{v}_i$ is an eigenvector with eigenvalue λ_i for each $i = 1, ..., n$

Definition: T is diagonalizeable if there exists such a basis α.

In sum: **Diagonalizable iff there is a basis of eigenvectors.**

Let $E_\lambda = \{\mathbf{v} \in V | T(\mathbf{v}) = \lambda \mathbf{v}\} = \ker(T - \lambda I)$ the λ-eigenspace.

Proposition: For $\lambda_i \in \mathbb{R}$ distinct, we have $E_{\lambda_1} \oplus ... \oplus E_{\lambda_k} \subset V$.

Corollary: T is diagonalizeable iff $V = E_{\lambda_1} \oplus ... \oplus E_{\lambda_k}$.

Let $p_T(\lambda) = \det(T - \lambda I) = (\lambda - \lambda_1)^{m_1} ... (\lambda - \lambda_k)^{m_k} \cdot \tilde{p}(\lambda)$.

Note that $m_1 + ... + m_k \leq n = \dim V$, with equality iff $\tilde{p} = 1$.

Proposition: For each $i = 1, ..., k$, we have $1 \leq \dim E_{\lambda_i} \leq m_i$.

Theorem: $T : V \to V$ is diagonalizable if and only if

(1) $m_1 + ... + m_k = n$ (Equivalently, $p_T(\lambda)$ has n roots)

(2) $\dim E_{\lambda_i} = m_i$ for each $i = 1, ..., k$.

Assumption: From now on we assume F is such that all polynomials factor. (E.g. $F = \mathbb{C}$, but not $F = \mathbb{R}$).

Then, $p_T(\lambda) = \det(T - \lambda I) = (\lambda - \lambda_1)^{m_1} ... (\lambda - \lambda_k)^{m_k}$

with $m_1 + ... + m_k = n$, and T is diagonalizeable iff $\dim E_{\lambda_i} = m_i$.

This can still fail, i.e. $\dim E_{\lambda_i} < m_i$. e.g. $[T] = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$. But:

Theorem: Let $T : V \to V$. Then there exists a basis α for V s.t. $[T]_\alpha^\alpha$ is a 'block diagonal' matrix, with each block given by

$$\begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & 0 & \lambda_i & 1 \\ 0 & \dots & 0 & 0 & \lambda_i \end{bmatrix} . \quad \text{e.g.} \quad \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}$$

Equivalently, $V = \oplus_i K_{\lambda_i}$ for $E_{\lambda_i} \subset K_{\lambda_i} \subset V$, with $\dim K_{\lambda_i} = m_i$.

E.g. $V = F^5$, $K_{\lambda_1} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, $K_{\lambda_2} = \text{Span}\{\mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$, $E_{\lambda_1} = \text{Span}\{\mathbf{v}_1\}$ and $E_{\lambda_2} = \text{Span}\{\mathbf{v}_3, \mathbf{v}_5\}$. What are $\mathbf{v}_2, \mathbf{v}_4$?

Definition: Matrices of this type are called λ_i *Jordan blocks*.

Blocks of a matrix and triangularization review

Definition: A subspace $W \subset V$ is T-invariant if $T(W) \subset W$.

Definition: A *block* of size m in a matrix $A \in M_{n\times n}(F)$ is a list of m successive columns of A which are 0 except in those m rows.

Proposition: A subspace $W \subset V$ is T-invariant if and only if for any basis α' of W extended to a basis α of V, $[T]$ has a block of size dim W in the columns/rows corresponding to α'.

Eg: For $[T] = \begin{bmatrix} 1 & 3 & 0 & 0 & 6 \\ 3 & 4 & 0 & 0 & 3 \\ 0 & 2 & 4 & 2 & 0 \\ 5 & 4 & 3 & 4 & 1 \\ 4 & 6 & 0 & 0 & 8 \end{bmatrix}$ $W = \text{Span}\{\mathbf{v}_3, \mathbf{v}_4\}$ is invariant.

Definition $[A] \in M_{n\times n}(F)$ is called triangular if $a_{ij} = 0$ for $i > j$.

Corollary: Let $\alpha = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ a basis for V. Then $[T]_\alpha^\alpha$ is triangular if and only Span$(\mathbf{v}_1, ...\mathbf{v}_k)$ is invariant for $k = 1, ..., n$.

Theorem: Any $T : V \to V$ such that $p_T(\lambda)$ has n roots is triangularizable. (By our assumption on F, this applies for any T.)

Summary: For any $T : V \to V$, $[T]$ can be made triangular , but it can not necessarily be made diagonal. How close can we get?

Proposition: If A is triangular, then $\det(A) = a_{11}...a_{nn}$.

Corollary: Triangular and diagonal matrices with same diagonal entries λ_i have the same char. polynomial $(\lambda - \lambda_1)^{m_1}...(\lambda - \lambda_k)^{m_k}$

We need to learn how to distinguish between linear maps T s.t.

$$[T] = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}, \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix} \text{ and } \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

Exercise: Chech that all these matrices satisfy $p_T(\lambda) = (\lambda - \lambda_1)^3$.

Exercise: Compute $\dim E_{\lambda_1}$ for each of the above matrices. 1, 2, 3

Note that $1 \leq \dim E_{\lambda_1} \leq 3 = m_1$, with equality iff diagonalizable.

Exercise: What are the dimensions of the blocks, or equivalently invariant subspaces, in each of the above cases? 3, (1,2), (1,1,1)

Now, we need to learn how to find the block sizes of T , or equivalently, how to choose a basis to so that $[T]$ of this form.

Towards nilpotent normal form

Let's begin with the simplest case: $T : V \to V$ with $p_T(\lambda) = \lambda^n$.

Then the only possible eigenvalue is $\lambda_1 = 0$, and $E_0 = \ker(T)$.

Corollary: T is diagonalizable iff $E_0 = V$ iff T is the zero map.

Thus, we see that being diagonalizable is 'rare' among all such T.

Definition: A map $N : V \to V$ is *nilpotent* if $N^k = 0$ for some k.

Exercise: Compute N^2, N^3 for $[N] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Theorem: $T : V \to V$ is nilpotent if and only if $p_T(\lambda) = \lambda^n$.

Definition: Let $K_0 = \{\mathbf{v} \in V | T^k\mathbf{v} = 0\}$ for some k.

Note $E_0 \subset K_0$. For T nilpotent $K_0 = V$, T diagonal iff $E_0 = V$.

Exercise: For each $[N]$ above, what is the subspace $E_0 \subset K_0 = V$?

Question: For T nilpotent, how do we choose a basis so that $[T]$ only has off-diagonal 1's, such as in the above examples?

Nilpotent normal form

Throughout let $T : V \to V$ nilpotent, so $p_T(\lambda) = \lambda^n$ and $V = K_0$.

Then for each $\mathbf{v} \in V$, there is some minimal k such that $T^k \mathbf{v} = 0$.

Definition: The set $\{T^{k-1}\mathbf{v}, T^{k-2}\mathbf{v}, ..., \mathbf{v}\}$ is called the cycle of $\mathbf{v}$.

$C(\mathbf{v}) = \mathrm{Span}\{T^{k-1}\mathbf{v}, T^{k-2}\mathbf{v}, ..., \mathbf{v}\} \subset V$ the cyclic subspace.

The integer $k = \dim C(\mathbf{v})$ is called the length of the cycle of $\mathbf{v}$.

A vector $\mathbf{v} \in V$ is called maximal if $\mathbf{v} \notin \mathrm{im}(T)$.

Exercise: Let $[T_1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $[T_2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Compute:

the maximal vectors $\mathbf{v} \in V$, and their $C(\mathbf{v})$. What if $\mathbf{v}$ is not max.?

Prop.: Let $\mathbf{v}_1, ..., \mathbf{v}_j \in V$ maximal and linearly independent.

Then $C(\mathbf{v}_i)$ are mutually lin. indep.: $C(\mathbf{v}_1) \oplus ... \oplus C(\mathbf{v}_j) \subset V$.

Theorem: Let $j = \dim E_0$. Then there exist $\mathbf{v}_1, ..., \mathbf{v}_j$ maximal, such that $C(\mathbf{v}_1) \oplus ... \oplus C(\mathbf{v}_j) = V$.

Thus, $[T]$ has j Jordan blocks of size $k(j) = \dim C(\mathbf{v}_j)$ with $\lambda = 0$.

Towards Jordan Form

Now, let $T : V \to V$ any linear map (not necessarily nilpotent).

We can still define K_0, and we have $E_0 = \ker(T) \subset K_0 \subset V$.

Definition: The subspace $K_\lambda = \{\mathbf{v} \in V | (T - \lambda)^k \mathbf{v} = \mathbf{0}\}$ is the *generalized eigenspace* of T corresponding to eigenvalue λ.

Exercise: Compute E_{λ_1} and K_{λ_1} for each of the following:

$$[T] = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}, \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix} \text{ and } \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

Proposition: For each eigenvalue λ_i: K_{λ_i} is an invariant subspace.

We have $\dim K_{\lambda_i} = m_i$ the multiplicity of λ_i in $p_T(\lambda)$.

The subspaces K_{λ_i} are mutually linearly independent.

Corollary: $V = K_{\lambda_1} \oplus ... \oplus K_{\lambda_k}$, as $m_1 + ... + m_k = n = \dim V$.

Thus, if we choose a basis for each K_{λ_i}, all together they form a basis of V, and $[T]$ is block diagonal with respect to this basis.

Jordan Form

Now, let's focus on $E_{\lambda_i} \subset K_{\lambda_i}$ for a single eigenvalue λ_i.

Since K_{λ_i} is invariant under T, we can consider $T|_{K_{\lambda_i}} : K_{\lambda_i} \to K_{\lambda_i}$.

Note that by definition, $T|_{K_{\lambda_i}} - \lambda I$ is nilpotent.

Thus, we can apply the nilpotent normal form theorem from earlier:

Let $j_i = \dim \ker(T|_{K_{\lambda_i}} - \lambda I) = \dim E_{\lambda_i}$. Then there exist $\mathbf{v}_{l_1}, ..., \mathbf{v}_{l_{j_i}}$ maximal vectors for $T|_{K_{\lambda_i}} - \lambda I$, s.t. $K_{\lambda_i} = C(\mathbf{v}_{l_1}) \oplus ... \oplus C(\mathbf{v}_{l_{j_i}})$.

Thus, $T|_{K_{\lambda_i}}$ decomposes into j_i Jordan blocks of size $\dim C(\mathbf{v}_l)$.

Theorem: Let $T : V \to V$ with $p_T(\lambda) = (\lambda - \lambda_1)^{m_1} ... (\lambda - \lambda_k)^{m_k}$.

There exists a basis α for V s.t.

- T decomposes into k blocks of dimension m_i (the K_{λ_i}'s)
- Each of these decomposes into j_i smaller blocks (the $C(\mathbf{v}_l)$'s)
- Each smaller block is of Jordan type with diagonal λ_i:

Example: The matrix from beginning of class has three blocks: a 2d λ_1 block, a 2d λ_2 block, and another 1d λ_2 block.