MAT224 - LEC5101 - Lecture 10
Vector spaces over fields and triangular form

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Review: Diagonalization

Let V a finite dimensional vector space and $T : V \to V$ linear.

Fix a basis $\alpha = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ for V. By definition, TFAE:

- $[T]_\alpha^\alpha$ is diagonal with $a_{ii} = \lambda_i$ for each $i = 1, ..., n$
- $T(\mathbf{v}_i) = \lambda_i \mathbf{v}_i$ for each $i = 1, ..., n$
- $\mathbf{v}_i$ is an eigenvector with eigenvalue λ_i for each $i = 1, ..., n$

Definition: T is diagonalizeable if there exists such a basis α.

In sum: **Diagonalizable iff there is a basis of eigenvectors.**

Let $E_\lambda = \{\mathbf{v} \in V | T(\mathbf{v}) = \lambda \mathbf{v}\} = \ker(T - \lambda I)$ the λ-eigenspace.

We showed these spaces are mutually linearly independent, i.e.:

Proposition: For $\lambda_i \in \mathbb{R}$ distinct, we have $E_{\lambda_1} \oplus ... \oplus E_{\lambda_k} \subset V$.

That is, $E_{\lambda_i} \cap (\sum_{j \neq i} E_{\lambda_j}) = \{\mathbf{0}\}$ for each $j = 1, .., k$.

Corollary: We can adjoin the bases of all eigenspaces without introducing any linear dependences.

This gives a basis for V of eigenvectors iff these actually span V.

Corollary: T is diagonalizeable iff $V = E_{\lambda_1} \oplus ... \oplus E_{\lambda_k}$.

Review: Diagonalizability and the characteristic polynomial

When does $V = E_{\lambda_1} \oplus ... \oplus E_{\lambda_k}$ for $\lambda_1, ..., \lambda_k$ the eigenvalues for T?

Recall $p_T(\lambda) = \det(T - \lambda I)$. $E_{\lambda_i} \neq \{\mathbf{0}\}$ iff λ_i a root of $p_T(\lambda)$.

Now, suppose $p_T(\lambda) = (\lambda - \lambda_1)^{m_1} ... (\lambda - \lambda_k)^{m_k} \cdot \tilde{p}(\lambda)$

where $\tilde{p}(\lambda)$ has no roots. e.g. $\tilde{p}(\lambda) = \lambda^2 + 1$.

Note that $m_1 + ... + m_k \leq n = \dim V$, with equality iff $\tilde{p} = 1$.

Proposition: For each $i = 1, ..., k$, we have $1 \leq \dim E_{\lambda_i} \leq m_i$.

Theorem: $T : V \to V$ is diagonalizable if and only if

(1) $m_1 + ... + m_k = n$ (Equivalently, $p_T(\lambda)$ has n roots)

(2) $\dim E_{\lambda_i} = m_i$ for each $i = 1, ..., k$.

How can these fail?

(1) $p_T(\lambda)$ does not have n roots. e.g. $[T] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

(2) $p_T(\lambda)$ has repeated roots, $\dim E_{\lambda_i} < m_i$. e.g. $[T] = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$

Corollary: If $p_T(\lambda)$ has n distinct roots, then T is diagonalizable.

Complex Vector Spaces and Diagonalization

We'll now try to resolve the issue (1) above. This will motivate us to study vector spaces over the 'field' $\mathbb{C}$ instead of $\mathbb{R}$.

Let $[T] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, recall $p_T(\lambda) = \lambda^2 + 1$, has no roots $\lambda_1 \in \mathbb{R}$.

However, we know $p_T(\lambda) = (\lambda + i)(\lambda - i)$, where $i^2 = -1$.

Thus, if we allow $\mathbb{C}$ numbers, then we find $\lambda_1 = i, \lambda_2 = -i$.

Exercise: Find 'eigenvectors' $\mathbf{v} \in \ker(T - \lambda_i I)$ for λ_1, λ_2 above.

Warning: you need to use $\mathbf{v} = (a, b)$ for $a, b \in \mathbb{C}$ instead of just $\mathbb{R}$.

Solution: $\mathbf{v} = (1, -i) \in \ker(T - iI)$, $\mathbf{v} = (1, i) \in \ker(T - (-i)I)$.

Thus, using $\mathbb{C}$ eigenvalues and eigenvectors, $[T]$ is diagonalizable.

This will turn out to always solve the issue (1) above, since:

Theorem: Let $p(\lambda) = a_n\lambda^n + ... + a_1\lambda + a_0$, with $a_i \in \mathbb{C}$, $a_n \neq 0$.

Then $\quad p(\lambda) = (\lambda - \lambda_1)^{m_1}...(\lambda - \lambda_k)^{m_k} \quad$ for some $\lambda_i \in \mathbb{C}$.

Note $m_1 + ... + m_k = n$ always. Thus (1) is indeed resolved.

Fields

Instead of just $\mathbb{C}$, we will generalize from $\mathbb{R}$ to any 'field':

Definition: A *field* is a set F together with two operations, called:

(A) Addition, which takes $a, b \in F$ and produce $a + b \in F$.

(B) Mult., which takes $a, b \in F$ and produces $a \odot b \in F$, such that

1. For all $a, b, c \in F$, $(a + b) + c = a + (b + c)$
2. For all $a, b \in F$, $a + b = b + a$.
3. There exists a unique element $0 \in F$ with the property that $a + 0 = a$ for all $a \in F$
4. For each $a \in F$, there exists a unique $-a \in F$ with the property that $a + (-a) = 0$
5. For all $a, b, c \in F$, $a \odot (b \odot c) = (a \odot b) \odot c$
6. For all $a, b \in F$, $a \odot b = b \odot a$
7. For all $a, b, c \in F$, $a \odot (b + c) = a \odot b + a \odot c$
8. There exists a unique element $1 \in F$ with the property that $1 \odot a = a$ for all $a \in F$
9. For all non-zero $a \in F$, there exists a unique $a^{-1} \in F$ with the property that $a \odot (a^{-1}) = 1$

Example: The sets $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are fields. **Exercise:** $\mathbb{Z}$ is not.

Vector Spaces over Fields

Definition A *vector space* over a field F is a set V together with:

(A) an operation called *vector addition*, which for each pair of vectors $\mathbf{x}, \mathbf{y} \in V$ produces another vector $\mathbf{x} + \mathbf{y}$ in V; and

(B) an operation called *multiplication by a scalar*, which for each vector $\mathbf{x} \in V$, and each scalar $c \in F$ produces another vector in V denoted $c\mathbf{x}$; such that

1. For all vectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$
2. For all vectors $\mathbf{x}, \mathbf{y} \in V$, $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
3. There exists a vector $\mathbf{0} \in V$ with the property that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for all vectors $\mathbf{x} \in V$
4. For each vector $\mathbf{x} \in V$, there exists a vector $-\mathbf{x} \in \mathbf{V}$ with the property that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$
5. For all vectors $\mathbf{x}, \mathbf{y} \in V$, and scalars $c \in F$, $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$
6. For all vectors $\mathbf{x} \in V$, and scalars $c, d \in F$, $(c + d)\mathbf{x} = c\mathbf{x} + d\mathbf{x}$
7. For all vectors $\mathbf{x} \in V$, and scalars $c, d \in F$, $(cd)\mathbf{x} = c(d\mathbf{x})$
8. For all vectors $\mathbf{x} \in V$, $1\mathbf{x} = \mathbf{x}$

Linear algebra over arbitrary fields F

In sum, we just replace $\mathbb{R}$ by F everywhere in the definition.

Examples: $F^n = \{(x_1, ..., x_n) | x_i \in F\}$ is a F vector space

$P_n(F) = \{a_n x^n + ... + a_1 x + a_0 | a_i \in F\}$ is a F vector space

$M_{m \times n}(F)$, the set of matrices with entries in F, is a F vector space

Fact: We can now literally repeat every definition and theorem from the course so far, replacing $\mathbb{R}$ by F, and they remain true.

Thus, you now know how to do linear algebra over any field F.

Now, let's go back to our motivation: simpler diagonalization.

Note $\mathbb{R} \subset \mathbb{C}$ so $M_{n \times n}(\mathbb{R}) \subset M_{n \times n}(\mathbb{C})$. Moreover, we have

Corollary: Let V a $\mathbb{C}$ vector space of dimension n and $T : V \to V$.

Then $\quad p(\lambda) = (\lambda - \lambda_1)^{m_1} ... (\lambda - \lambda_k)^{m_k} \quad$ for some $\lambda_i \in \mathbb{C}$.

Thus, T is diagonalizable iff $\dim E_{\lambda_i} = m_i$ for each $i = 1, ..., k$.

It remains to understand exactly when this works and how it fails.

Triangularization

Definition $[A] \in M_{n\times n}(F)$ is called triangular if $a_{ij} = 0$ for $i > j$.

Lemma: Let $A \in M_{k\times k}(F), B \in M_{(n-k)\times(n-k)}(F)$, and $C \in M_{k\times(n-k)}(F)$. Then $\det \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = \det(A)\det(B)$.

Corollary: If A is triangular, then $\det(A) = a_{11}...a_{nn}$.

Throughout, let V an F vector space and $T : V \to V$ linear.

Definition: A subspace $W \subset V$ is T-invariant if $T(W) \subset W$.

Proposition: Let $\alpha = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ a basis for V. Then $[T]_\alpha^\alpha$ is triangular if and only Span$(\mathbf{v}_1, ...\mathbf{v}_k)$ is invariant for $k = 1, ..., n$.

Now, suppose $p_T(\lambda)$ has n roots. (e.g. always true over $F = \mathbb{C}$.)

Lemma: For any invariant subspace $W \subset V$, there exists $W \subset \tilde{W} \subset V$ with $\dim \tilde{W} = \dim W + 1$ and $\tilde{W}$ invariant.

Theorem: Any $T : V \to V$ such that $p_T(\lambda)$ has n roots is triangularizable.