Koszul duality
GRT learning seminar

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1 Category $\mathcal{O}$ for $\mathfrak{sl}_2$

1.1 Indecomposable objects in $\mathcal{O}$ for $\mathfrak{sl}_2$

Consider the block $\mathcal{O}_0$ for $\mathfrak{g} = \mathfrak{sl}_2$. Objects here are $U(\mathfrak{g})$-modules that are finitely generated, where $\mathfrak{n}$ acts locally finitely, $\mathfrak{h}$ acts diagonally, and $Z(U(\mathfrak{g})) = \mathbb{C}(h^2 + 2h + 4yx)$ acts via the same central character as it does on the trivial representation.

More concretely, let's describe some interesting classes of objects in $\mathcal{O}_0$. The simple objects in $\mathcal{O}_0$ are the trivial representation $L(0)$, and the Verma module $L(-2) = M(-2) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}(-2)$. The Verma module $M(0) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}(0)$ fits into the exact sequence

$$0 \to M(-2) \to M(0) \to L(0) \to 0.$$

We will use the head filtration (take the largest semisimple quotient, repeat) to denote $M(0)$ as

$$M(0) \cong \begin{matrix} L(0) \\ L(-2) \end{matrix}$$

Since 0 is a dominant weight, $M(0)$ is a projective module ([Hum08], Proposition 3.8), so we'll say $P(0) = M(0)$. Category $\mathcal{O}$ also has enough projectives ([Hum08], Theorem 3.8), in particular, $L(-2)$ has a projective cover denoted $P(-2)$. To find out the structure of $P(-2)$, we can use BGG reciprocity

$$(P(\lambda) : M(\mu)) = [M(\mu) : L(\lambda)],$$

where $(P(\lambda) : M(\mu))$ denotes the multiplicity of $M(\mu)$ in $P(\lambda)$'s standard filtration, and $[M(\mu) : L(\lambda)]$ denotes the multiplicity of $L(\lambda)$ in $M(\mu)$'s simple filtration (equivalently, these are the coefficients of the classes in $K(\mathcal{O})$ of $[P(\lambda)]$, $M(\mu)$, respectively, in the bases $\{M(\nu)\}$, $\{L(\nu)\}$, respectively).

From what we had above, we find that $(P(-2) : M(-2)) = (P(-2 : M(0))) = 1$. Since $P(-2)$ has a surjective map to $M(-2) = L(-2)$, it must fit into an exact sequence

$$0 \to M(0) \to P(-2) \to M(-2) \to 0,$$

so, using the head filtration, we can represent $P(-2)$ as

$$P(-2) \cong \begin{matrix} L(-2) \\ L(0) \\ L(-2) \end{matrix}$$

It turns out that the five objects $L(0), L(-2), M(0), M^\vee(0), P(-2)$ are the only indecomposable objects in $\mathcal{O}_0$.

1.2 Aside: Perverse sheaves on $\mathbb{P}^1$

Via Beilinson-Bernstein localization and the Riemann-Hilbert correspondence, the category $\mathcal{O}_0$ is equivalent to the category of perverse sheaves on $\mathbb{P}^1$ with respect to the stratification $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$. Let

$$\infty \xhookrightarrow{i} \mathbb{P}^1 \xhookleftarrow{j} \mathbb{A}^1$$

denote the inclusion maps. The indecomposable objects in this category can be matched up with the ones from $\mathcal{O}_0$ as follows:

$$\begin{aligned} L(0) &\leftrightarrow IC(\mathbb{A}^1) = \underline{\mathbb{C}}(\mathbb{P}^1) \\ L(-2) &\leftrightarrow IC(\infty) = \underline{\mathbb{C}}_\infty[-1] \\ M(0) &\leftrightarrow j_! \underline{\mathbb{C}}(\mathbb{A}^1) \\ M^\vee(0) &\leftrightarrow j_* \underline{\mathbb{C}}(\mathbb{A}^1) \\ P(-2) &\leftrightarrow \mathcal{P}_\infty \end{aligned}$$

where $\underline{\mathbb{C}}(\mathbb{A}^1)$ denotes the constant sheaf on $\mathbb{A}^1$ and $\mathcal{P}_\infty$ is the "big projective". The exact sequences above can be realized by using exact triangles, for example, if we let $\mathcal{F} = \underline{\mathbb{C}}(\mathbb{P}^1) \in D^+(\mathbb{P}^1)$, then we can use a distinguished triangle

$$j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F}$$

to get a short exact sequence of sheaves

$$0 \to IC(\infty)[-1] \to j_! \underline{\mathbb{C}}(\mathbb{A}^1) \to IC(\mathbb{A}^1) \to 0$$

in $D(X)$. We'll need the perverse sheaf language when we want to talk about mixed sheaves, but for now we'll work with algebras instead.

1.3 Endomorphism ring of a projective generator in $\mathcal{O}_0$

Consider the object $P = P(0) \oplus P(-2)$. This a **projective generator** of $\mathcal{O}_0$, meaning it is a projective object surjecting onto every simple object in $\mathcal{O}_0$. Consider the ring $A = \mathrm{End}_{\mathcal{O}}(P)$. There is a canonical equivalence of categories

$$\mathrm{Hom}_{\mathcal{O}_0}(P, -) : \mathcal{O}_0 \to \mathrm{Mof} - A,$$

where $\mathrm{Mof} - A$ denotes the category of finitely generated right A-modules. To be able to use this equivalence, we should compute A. This involves computing four Hom sets.

1. $\mathrm{Hom}_{\mathcal{O}_0}(P(0), P(0)) = \mathbb{C}a$, where a is the identity map on $P(0) = M(0)$, which is a Verma module, so it does not have any nontrivial endomorphisms.
2. $\mathrm{Hom}_{\mathcal{O}_0}(P(-2), P(-2)) = \mathbb{C}\{b, \varepsilon\}$, where ε is the nilpotent endomorphism sending the $L(-2)$ in the head into the $L(-2)$ in the socle, and everything else to zero, and b is the identity map on $P(-2)$.
3. $\mathrm{Hom}_{\mathcal{O}_0}(P(0), P(-2)) = \mathbb{C}e$, where $e : P(0) \to P(-2)$ is the inclusion map.
4. $\mathrm{Hom}_{\mathcal{O}_0}(P(-2), P(0)) = \mathbb{C}f$, where f is the composition of the maps $P(-2) \twoheadrightarrow L(-2) \hookrightarrow P(0)$.

Notice that $ef = \varepsilon$ and $fe = 0 = \varepsilon^2$. So we may express the algebra $A = \mathrm{End}_{\mathcal{O}}(P)$ as the path algebra of the quiver

$$\begin{tikzcd} \bullet_a \arrow[r, bend left, "e"] & \bullet_b \arrow[l, bend left, "f"] \end{tikzcd}$$

where a and b are the constant paths at the two vertices and we have an extra relation $fe = 0$.

1.4 The Ext-algebra

Let $\mathcal{C}$ be an abelian category with enough projectives and injectives. Recall that $\mathrm{Hom}_{\mathcal{C}}(-,-)$ is a left-exact bifunctor, and we define $\mathrm{Ext}^i_{\mathcal{C}}(-,-) = R^i \mathrm{Hom}_{\mathcal{C}}(-,-)$. To compute $\mathrm{Ext}^i_{\mathcal{C}}(A, B)$ for $A, B \in \mathcal{C}$, consider the projective (resp. injective) resolutions of A (resp. B)

$$\cdots \to P^{-1} \to P^0 \to A \to 0$$

and

$$0 \to B \to I^0 \to I^1 \to \cdots,$$

then

$$H^{-i}(\mathrm{Hom}_{\mathcal{C}}(P^\bullet, B)) = \mathrm{Ext}^i_{\mathcal{C}}(A, B) = H^i(\mathrm{Hom}_{\mathcal{C}}(A, I^\bullet)).$$

Note that $\mathrm{Ext}^0(A,B) = \mathrm{Hom}_{\mathcal{C}}(A,B)$ and that the $\mathrm{Ext}^i_{\mathcal{C}}(A,B)$ are abelian groups under addition.

There is an alternative characterization of $\mathrm{Ext}_{\mathcal{C}}$, which will be convenient for us. Recall that $\mathrm{Ext}^1_{\mathcal{C}}(A,B)$ can be identified with the set of **extensions** of A by B, i.e. exact sequences

$$0 \to B \to E \to A \to 0.$$

up to isomorphism, i.e. if two extensions fit into a commutative diagram of the form

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r] & E' \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

then they are considered equivalent. We then define

Definition 1.1 (see [HS97], p148). *An n-**extension** of A by B is an exact sequence of the form*

$$E : 0 \to B \to E_n \to \ldots \to E_1 \to A \to 0.$$

Definition 1.2. *Two n-extensions E, E' satisfy the relation $E \rightsquigarrow E'$ if there is a commutative diagram*

$$\begin{tikzcd} E: & 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & E_n \arrow[r] \arrow[d] & \cdots \arrow[r] & E_1 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\ E': & 0 \arrow[r] & B \arrow[r] & E'_n \arrow[r] & \cdots \arrow[r] & E'_1 \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

This is no longer a symmetric relation (because of the lack of the five lemma), so to get an equivalence relation, we need to take the symmetric closure of $\rightsquigarrow$, so we define

Definition 1.3. *Two n-extensions E, E' are **equivalent**, denoted $E \sim E'$ if there exists a chain*

$$E \rightsquigarrow F_1 \leftsquigarrow F_2 \rightsquigarrow \cdots \rightsquigarrow F_k \leftsquigarrow E'.$$

Similar to how this is done for $\mathrm{Ext}^1_{\mathcal{C}}$, we can define an abelian group structure on n-extensions of A by B. We have the following theorem

Theorem 1.4 ([ML63], Theorem 6.4). *The abelian group of equivalence classes of n-extenstions of A by B is isomorphic to* $\mathrm{Ext}^n(A,B)$.

Given $E \in \mathrm{Ext}^m_{\mathcal{C}}(B,A)$ and $E' \in \mathrm{Ext}^n_{\mathcal{C}}(C,B)$, we can consider some representative exact sequences

$$E : 0 \to A \to E_1 \to \cdots \to E_m \to B \to 0$$

and

$$E' : 0 \to B \to E'_1 \to \cdots \to E'_m \to C \to 0.$$

We can splice the two sequences together to form a longer exact sequence

$$E \circ E' : 0 \to A \to E_1 \to \cdots \to E_m \to E'_1 \to \cdots \to E'_m \to C \to 0.$$

which then represents a class in $\mathrm{Ext}^{m+n}_{\mathcal{C}}(C,A)$. If you are familiar with derived categories, you might notice that this is basically the composition of morphisms there.

In particular, if $A \in \mathcal{C}$ is an object, then, $\mathrm{Ext}^\bullet_{\mathcal{C}}(A,A)$ is an algebra, and for any other object B, $\mathrm{Ext}^\bullet_{\mathcal{C}}(A,B)$ is a right $\mathrm{Ext}^\bullet_{\mathcal{C}}(A,A)$-module, and $\mathrm{Ext}^\bullet_{\mathcal{C}}(B,A)$ is a left $\mathrm{Ext}^\bullet_{\mathcal{C}}(A,A)$-module.

1.5 The Ext-algebra of a simple generator in $\mathcal{O}_0$

Warning: we are ignoring some Langlands duality issues here. Oh well. Let $L = L(-2) \oplus L(0)$, and consider the algebra $\mathrm{Ext}^\bullet_{\mathcal{O}_0}(L,L)$. Since our category is so nice, this splits into a direct sum, so we have to compute four $\mathrm{Ext}^\bullet$-groups. We'll drop the subscripts from the Ext's, they are all computed in $\mathcal{O}_0$. Since the L's are simple, there are no nonzero Hom's between them.

1. $\mathrm{Ext}^1(L(-2), L(0)) = \mathbb{C}$, corresponding to the exact sequence

$$E : 0 \to L(0) \to M^\vee(0) \to L(-2) \to 0,$$

and there are no higher Ext's.

2. $\mathrm{Ext}^1(L(0), L(-2)) = \mathbb{C}$, corresponding to the exact sequence

$$F : 0 \to L(-2) \to M(0) \to L(0) \to 0,$$

and there are no higher Ext's.

3. $\mathrm{Ext}^0(L(-2), L(-2)) = \mathbb{C}$, and let A denote the identity map on $L(-2)$.

4. $\mathrm{Ext}^0(L(0), L(0)) = \mathrm{Ext}^2(L(0), L(0)) = \mathbb{C}$. Let B denote the identity map on $L(0)$. The nontrivial 2-extension can be represented by the exact sequence

$$\Upsilon : 0 \to L(0) \to M^\vee(0) \to M(0) \to L(0) \to 0.$$

Note that we have $E \circ F = \Upsilon$.

On the other hand, we have the following diagram, where ι stands for the natural inclusion maps,

$$\begin{tikzcd}
F \circ E : & 0 \arrow[r] & L(-2) \arrow[r] & M(0) \arrow[rr] & & M^\vee(0) \arrow[r] & L(-2) \arrow[r] & 0 \\
& 0 \arrow[r] & L(-2) \arrow[r, "\begin{bmatrix} 1 \\ \iota \end{bmatrix}"] \arrow[u, equal] \arrow[d, equal] & L(-2) \oplus P(0) \arrow[rr, "\begin{bmatrix} -\iota & \iota \end{bmatrix}"'] \arrow[u, "\begin{bmatrix} 0 & 1 \end{bmatrix}"'] \arrow[d, "\begin{bmatrix} 1 & 0 \end{bmatrix}"'] & & P(-2) \arrow[r] \arrow[u] \arrow[d] & L(-2) \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \\
0 : & 0 \arrow[r] & L(-2) \arrow[r, "1"'] & L(-2) \arrow[rr, "0"'] & & L(-2) \arrow[r, "1"'] & L(-2) \arrow[r] & 0
\end{tikzcd}$$

showing that $F \circ E \sim 0$ in $\mathrm{Ext}^2(L(-2), L(-2))$.

So the ring $\mathrm{Ext}^\bullet(L, L)$ can be expressed as the path algebra of the quiver

$$\begin{tikzcd}
\bullet_A \arrow[r, bend left, "E"] & \bullet_B \arrow[l, bend left, "F"]
\end{tikzcd}$$

with the extra relation $FE = 0$

1.6 A grading on $\mathcal{O}_0$

We notice that

$$\mathrm{Ext}^\bullet_{\mathcal{O}_0}(L, L) \cong \mathrm{End}_{\mathcal{O}}(P).$$

We also notice something rather strange, the ring $\mathrm{Ext}^\bullet(L, L)$ is naturally graded (by cohomological degree), but there is no a priori grading on $\mathrm{End}_{\mathcal{O}}(P)$. We'll define a grading on $\mathrm{End}_{\mathcal{O}}(P)$ which will subsequently be related to the cohomological grading on $\mathrm{Ext}^\bullet_{\mathcal{O}_0}(L, L)$ by Koszul duality.

Let $\mathbb{C}[\mathfrak{h}^*] = \mathbb{C}[x]$. This carries a W-action given by $s(x) = -x$. Then $\mathbb{C}[x]^W = \mathbb{C}[x^2]$ is the ring of invariants, and let $C = \mathbb{C}[x]/\langle x^2 \rangle$ be the ring of coinvariants. We may notice that

$$\mathrm{End}_{\mathfrak{g}}(P(-2)) \cong \mathbb{C}[x]/x^2.$$

In fact, this generalizes to (we are following the exposition in [Str03] here)

Theorem 1.5 ([Soe90], Endomorphismensatz). *Let λ be an integral dominant weight and let W_λ be its stabilizer under the dot action of W. Let $C = \frac{\mathbb{C}[\mathfrak{h}]}{\mathbb{C}[\mathfrak{h}]^W_+}$ be the algebra of coinvariants, and C^λ be C^{W_λ} the ring of W_λ-invariants in C. Then*

$$\mathrm{End}_{\mathfrak{g}}(P(w_0^\lambda \cdot \lambda)) \cong C^\lambda,$$

where w_0^λ is the longest element of W/W_λ.

Recall that

$$C \cong H^\bullet(G/B, \mathbb{C}),$$

and accordingly, we will consider C (and $C^\lambda \cong H^\bullet(G/P_\lambda, \mathbb{C})$) as an evenly graded algebra. We will transport this grading to $\operatorname{End}_{\mathfrak{g}}(P, P)$ and then to $\mathcal{O}$.

Theorem 1.6 ([Soe90], Struktursatz 9)**.** *Let λ be an integral dominant weight, then the functor*

$$\begin{aligned} \mathbb{V} = \mathbb{V}_\lambda : \mathcal{O}_\lambda &\to C^\lambda - \mathrm{Mof} \\ M &\mapsto \operatorname{Hom}_{\mathfrak{g}}\left(P(w_0^\lambda \cdot \lambda), M\right) \end{aligned}$$

is fully faithful on projective objects. In other words, for $w, \nu \in W/W_\lambda$, there is an isomorphism of vector spaces

$$\operatorname{Hom}_{\mathfrak{g}}\left(P(w \cdot \lambda), P(\nu \cdot \lambda)\right) \cong \operatorname{Hom}_{C^\lambda}\left(\mathbb{V}P(w \cdot \lambda), \mathbb{V}P(\nu \cdot \lambda)\right).$$

Let's do a sanity check for $\mathcal{O}_0$ of $\mathfrak{sl}_2$. We have $C^\lambda = C = \mathbb{C}[x]/x^2$.

1. We have already computed that $\mathbb{V}P(-2) = \mathbb{V}\operatorname{Hom}_{\mathfrak{g}}(P(-2), P(-2)) = \mathbb{C}[x]/x^2$.
2. We have $\mathbb{V}P(0) = \operatorname{Hom}_{\mathfrak{g}}(P(-2), P(0)) = \mathbb{C}$.

We compute the remaining cases:

1. $\operatorname{Hom}_C(\mathbb{V}P(0), \mathbb{V}P(0)) = \operatorname{Hom}_C(\mathbb{C}, \mathbb{C}) = \mathbb{C}$,
2. $\operatorname{Hom}_C(\mathbb{V}P(0), \mathbb{V}P(-2)) = \operatorname{Hom}_C(\mathbb{C}, \mathbb{C}[x]/x^2) = \mathbb{C}$.

1.7 Translation functors

To extend the grading to other objects in $\mathcal{O}$, we'll need the help of the translation functors. Let λ, μ be weights such that $\lambda - \mu$ is integral. Define

$$\begin{aligned} \theta_\lambda^\mu : \mathcal{O}_\lambda &\to \mathcal{O}_\mu, \\ M &\mapsto \mathrm{pr}_\mu\left(M \otimes_{\mathbb{C}} E(\mu - \lambda)\right), \end{aligned}$$

where $E(\mu - \lambda)$ is the finite-dimensional simple $\mathfrak{g}$-module with extremal weight $\lambda - \mu$. This functor is exact, since tensoring with a finite-dimensional module and projection to a block both are. If s is a simple reflection and the stabilizer $W_\mu = \{w \in W | w \cdot \lambda = \lambda\}$ is $\{1, s\}$ (so, μ is as close to regular as it can be while being on the s-wall), then we define the functor (translation through the s-wall)

$$\theta_s = \theta_\mu^\lambda \circ \theta_\lambda^\mu.$$

Let's compute some examples of translation functors. For $\mathfrak{g} = \mathfrak{sl}_2$, let $\lambda = 0, \mu = -1$. Then $E(-1) = E(1) = L(1) = \mathbb{C}^2$, the standard representation.

We can explicitly compute $M(0) \otimes L(1)$. Let v_0 be a highest weight vector for $M(0)$, then we can choose a basis $\{v_0, v_1, v_2, \dots\}$ for $M(0)$ such that

$$\begin{aligned} h \cdot v_i &= (-2i)v_i \\ e \cdot v_i &= (-i+1)v_{i-1} \\ f \cdot v_i &= (i+1)v_{i+1} \end{aligned}$$

Similarly, we can choose a basis $\{l_0, l_1\}$ of $L(1)$ consisting of weight vectors such that $e \cdot l_1 = l_0$ and $f \cdot l_0 = l_1$. Then we have a decomposition of $M(0) \otimes L(1)$ (rightward arrows are the action of e, leftward ones are the action of f):

$$\begin{tikzcd}
\text{weight space} & -5 & -3 & -1 & 1 \\
\cdots \arrow[r, bend left] & \mathbb{C}(l_0 \otimes v_3 + l_1 \otimes v_2) \arrow[l, bend left] \arrow[r, bend left] & \mathbb{C}(l_0 \otimes v_2 + l_1 \otimes v_1) \arrow[l, bend left] & \mathbb{C}(l_0 \otimes v_1 + l_1 \otimes v_0) \arrow[l, bend left] \arrow[r, bend left] & \mathbb{C}(l_0 \otimes v_0) \arrow[l, bend left] \\
\cdots \arrow[r, bend left] & \mathbb{C}(3l_0 \otimes v_3 + 2l_1 \otimes v_2) \arrow[l, bend left] \arrow[r, bend left] & \mathbb{C}(2l_0 \otimes v_2 + l_1 \otimes v_1) \arrow[l, bend left] \arrow[r, bend left] & \mathbb{C}(l_0 \otimes v_1) \arrow[l, bend left] &
\end{tikzcd}$$

so we see that $M(0) \otimes L(1) = M(1) \oplus M(-1)$. Therefore $\theta_0^{-1}(M(0)) = M(-1)$. Similarly we can compute $\theta_0^{-1}(M(-2)) = M(-1)$. Exactness of θ_0^{-1} implies that $\theta_0^{-1} L(0) = 0$.

Next we compute $M(-1) \otimes L(1)$, choose bases as before

$$\begin{tikzcd}
\text{weight space} & -6 & -4 & -2 & 0 \\
\cdots \arrow[r, bend left] & \mathbb{C}(l_0 \otimes v_3 + l_1 \otimes v_2) \arrow[l, bend left] \arrow[r, bend left] & \mathbb{C}(l_0 \otimes v_2 + l_1 \otimes v_1) \arrow[l, bend left] \arrow[r, bend left] & \mathbb{C}(l_0 \otimes v_1 + l_1 \otimes v_0) \arrow[l, bend left] & \mathbb{C}(l_0 \otimes v_0) \arrow[l, bend left] \\
\cdots \arrow[ur, dotted] \arrow[r, dotted, bend left] & \mathbb{C}(3l_0 \otimes v_3 + 2l_1 \otimes v_2) \arrow[l, bend left] \arrow[ur, dotted] \arrow[r, dotted, bend left] & \mathbb{C}(2l_0 \otimes v_2 + l_1 \otimes v_1) \arrow[l, bend left] \arrow[ur, dotted] \arrow[r, dotted, bend left] & \mathbb{C}(l_0 \otimes v_1) \arrow[l, bend left] \arrow[ur] &
\end{tikzcd}$$

(the dotted arrows indicate that e sends a basis vector to a linear combination of basis vectors with nonzero coefficients). We see that

$$M(-1) \otimes L(1) \cong P(-2) \in \mathcal{O}_0,$$

so

$$\theta^0_{-1}(M(-1)) = P(-2),$$

and

$$\theta_s(M(0)) = P(-2).$$

Therefore we obtained $P(-2)$ from the dominant Verma module $M(0)$ using translation functor. Since we also have $P(0) = M(0)$, these are all the projectives in $\mathcal{O}_0$.

This is true in general, if λ is a regular dominant weight and $M(\lambda)$, then the projective module $P(w \cdot \lambda)$ is a direct summand of

$$\theta_{s_1} \cdots \theta_{s_r} M(\lambda).$$

where $w = s_r \cdots s_3 s_2 s_1$ be a reduced expression for $w \in W$. Moreover, it is the unique indecomposable direct summand not isomorphic to some $P(v \cdot \lambda)$ for $v < w$.

We can describe how the translation functors interact with the functor $\mathbb{V}$

Theorem 1.7 ([Soe90], Theorem 10)**.** *Let λ be a regular weight and s be a simple reflection. There is a natural equivalence of functors $\mathcal{O}_\lambda \to C - \mathrm{Mof}$*

$$\mathbb{V}\theta_s \cong C \otimes_{C^s} \mathbb{V}.$$

Corollary 1.8 ([Str03], Corollary 1.10.)**.** *Let $w = s_r \cdots s_3 s_2 s_1$ be a reduced expression for $w \in W$. Then the module $\mathbb{V}P(w \cdot \lambda)$ is isomorphic to the unique direct summand of*

$$C \otimes_{C^{s_1}} C \otimes_{C^{s_2}} C \otimes_{C^{s_3}} \cdots \otimes_{C^{s_r}} \mathbb{C}$$

which is not isomorphic to some $\mathbb{V}P(v \cdot \lambda)$ with $v < w$.

To get a singular version of the above theorem/corollary, we can replace C by C^λ.

We are finally ready to equip the ring $\mathrm{End}_{\mathfrak{g}}(P, P)$ with a grading.

Theorem 1.9 ([Str03], Theorem 2.1.)**.** *Let λ be regular integral dominant, and Q, Q' be projective objects. The functor $\mathbb{V}$ induces a grading on $\mathrm{Hom}_{\mathfrak{g}}(Q, Q')$. In particular, $\mathrm{End}_{\mathfrak{g}}(P)$ can be considered as a grading ring.*

Proof. (sketch) We have already seen that $\mathrm{End}_{\mathfrak{g}}(P(w_0\lambda)) \cong C$ is graded. Any of the subrings $C^s \subseteq C$ are graded, so C is a graded C^s-module. The trivial module $\mathbb{C}$ is also a graded C^s-module. If M and N are graded C^s-modules, one can show that $M \otimes_{C^s} N$ is a graded C^s-module. Any indecomposable projective object in $\mathcal{O}_0$ is obtained as a direct summand of successive tensor products of graded modules and hence can also be equipped with a grading. $\square$

Remark 1.10. *All of the gradings of the indecomposable objects are only determined up to a shift. The standard convention is to consider $\mathbb{V}P(w \cdot \lambda)$ as a graded C-module with highest degree in $l(w)$.*

For $\mathfrak{g} = \mathfrak{sl}_2$, we then have $\mathbb{V}P(-2) = \mathbb{C} + \mathbb{C}x$, where x is in degree $1 = l(s)$, and the constants are in degree -1. For $\mathbb{V}P(0) = \mathbb{C}$, the constants are in degree 0. With these choices, we have

$$\begin{tikzcd}
\text{degree} & \mathbb{V}P(0) & \mathbb{V}P(-2) \\
1 & & \mathbb{C}x \\
0 & \mathbb{C} \arrow[ur] & \\
-1 & & \mathbb{C}
\end{tikzcd}$$

is a morphism of degree 1 and so is the morphism

$$\begin{tikzcd}
\text{degree} & \mathbb{V}P(-2) & \mathbb{V}P(0) \\
1 & \mathbb{C}x & \\
0 & & \mathbb{C} \\
-1 & \mathbb{C} \arrow[ur] &
\end{tikzcd}$$

So our ring is nonnegatively graded, with only the identity maps of the two indecomposable projectives being in degree 0. In the general case, this convention leads to a nonnegatively grading of the ring $\mathrm{End}_{\mathfrak{g}}(P, P)$.

We have defined gradings on the projectives, and using the grading on we can define graded versions of the simple modules. In particular, we want the canonical projections $P(\lambda) \to L(\lambda)$ to be degree 0 maps.

We can also define graded lifts of the functors

Definition 1.11 ([Str03], Definition 3.3)**.** *Let B and D be graded rings. We call a functor $F : \mathrm{Mof} - B \to \mathrm{Mof} - D$ **gradeable** if there exists a functor $\widetilde{F} : \mathrm{grMof} - B \to \mathrm{grMof} - D$ which induces F after forgetting the grading.*

The translation functors θ_s are gradeable and we have $\theta_s P(0) \cong P(s \cdot 0)$ as *graded modules*. We also have the following Theorem which can be used to define graded versions of Verma modules

Theorem 1.12 ([Str03], Theorem 3.6)**.** *For $w \in W$ and s a simple reflection such that $ws > w$, there is an exact sequence of graded modules*

$$0 \to M(w \cdot 0)\langle 1 \rangle \to \theta_s M(w \cdot 0) \to M(ws \cdot 0) \to 0.$$

As an example, this leads to the exact sequence of graded modules (for $w = e$)

$$0 \to M(0)\langle 1 \rangle \to P(-2) \to M(-2) \to 0.$$

Let's list the indecomposable objects in $\mathcal{O}_0$ for $\mathfrak{sl}_2$ with their gradings:

1. The simple module $L(0)$ is concentrated in degree -1.

2. The simple modules $L(-2)$ is concentrated in degree -1.

3. The Verma module $M(0)$ has a quotient isomorphic to $L(0)$ in degree -1 and a submodule isomorphic to $L(-2)\langle 1 \rangle$ in degree 0.

4. The dual Verma module $M^\vee(0)$ has a submodule isomorphic to $L(0)$ in degree -1 and a quotient isomorphic to $L(-2)\langle -1 \rangle$ in degree -2.

5. The projective module $P(-2)$ has a submodule isomorphic to $L(-2)\langle 2 \rangle$ in degree 1, a subquotient $L(0)\langle 1 \rangle$ in degree 0 and a quotient $L(-2)$ in degree -1. Note that the two $L(-2)$'s that appear as subquotients appear in different degrees.

2 Mixed categories

2.1 Mixed categories and graded algebras

What did we learn in the above section? We had a category, $\mathcal{O}_0$, equivalent to the category $\mathrm{Mof} - \mathrm{End}_{\mathfrak{g}}(P, P)$. We defined a grading on the ring $\mathrm{End}_{\mathfrak{g}}(P, P)$, and on modules and morphisms in $\mathrm{Mof} - \mathrm{End}_{\mathfrak{g}}(P, P)$, making it into a category $\mathrm{grMof} - \mathrm{End}_{\mathfrak{g}}(P, P)$ of graded modules for the graded ring $\mathrm{End}_{\mathfrak{g}}(P, P)$. This leads us to the notion of *mixed categories*.

Definition 2.1 ([BGS96], Definition 4.1.1.)**.** *A **mixed category** is an artinian category $\mathcal{M}$ equipped with a map $w : \mathrm{Irr}(\mathcal{M}) \to \mathbb{Z}$ (called **weight**) such that for any two irreducible objects M, N, one has $\mathrm{Ext}^1(M, N) = 0$ if $w(M) \geq w(N)$. (**Caution:** we are switching from the standard convention here so that the notation stays consistent with the grading shift)*

The only irreducible objects in $\mathcal{O}$ are the simple objects. Note that in the graded version of category $\mathcal{O}$, we do not have an exact sequence of *graded* modules

$$0 \to L(-2) \to M(0) \to L(0) \to 0$$

(note that $w(L(-2)) = -1$ and $w(L(0)) = -1$), but we do have an exact sequence of graded modules

$$0 \to L(-2)\langle 1\rangle \to M(0) \to L(0) \to 0.$$

and we do have the exact sequence of graded modules

$$0 \to L(0) \to M^{\vee}(0) \to L(-2)\langle -1\rangle \to 0.$$

It is also worth pointing out that $\mathrm{Ext}^i_{\mathcal{O}}(L(0), L(0)\langle j\rangle) = 0$ unless $i = j = 2$ (we will see this condition when we discuss Koszul rings). The nontrivial 2-extension is realized by the exact sequence

$$0 \to L(0)\langle 2\rangle \to M^{\vee}(0)\langle 2\rangle \to M(0) \to L(0) \to 0.$$

Definition 2.2 ([BGS96], Definition 4.1.4.)**.** *For an integer d a **degree** d **Tate twist** on a mixed category $\mathcal{M}$ is an automorphism $\langle d\rangle$ of $\mathcal{M}$ written $M \mapsto M\langle d\rangle$ with the property that $w(M(\langle d\rangle)) = w(M) + d$ for all $M \in \mathrm{Irr}\mathcal{M}$.*

The graded version of category $\mathcal{O}$ comes with a degree 1 Tate twist $\langle 1\rangle$ (shifting the degrees up by 1).

Definition 2.3. *A **degrading functor** is a pair (ν, ε) where ν is an exact faithful functor $\nu : \mathcal{M} \to \mathcal{C}$ from a mixed categoty $\mathcal{M}$ to an artinian category $\mathcal{C}$ that sends semisimple objects to semisimple objects and a ε is choice of natural isomorphisms $\varepsilon : \nu(M) \mapsto \nu(M\langle d\rangle)$ for every object M.*

Forgetting the gradings in our graded version of category $\mathcal{O}$ is a degrading functor.

Definition 2.4. *A **grading** on an artinian category $\mathcal{C}$ is a pair $(\mathcal{M}, (\nu, \varepsilon))$ where $\mathcal{M}$ is a mixed category with a Tate twist and $(\nu, \varepsilon) : \mathcal{M} \to \mathcal{C}$ is a degrading functor such that*

(a) Any irreducible object in $\mathcal{C}$ is isomorphic to $\nu(M)$ for some $M \in \mathcal{M}$ and

(b) The map

$$\nu^i_{M,N} : \bigoplus_i \mathrm{Ext}^i_{\mathcal{M}}(M, N\langle nd\rangle) \to \mathrm{Ext}^i_{\mathcal{C}}(\nu M, \nu N)$$

is bijective for all M, N, i.

Definition 2.5. *Two gradings $(\mathcal{M}, (\nu, \varepsilon))$ and $(\mathcal{M}', (\nu', \varepsilon'))$ on $\mathcal{C}$ are **equivalent** if there is a functor $\phi : \mathcal{M} \to \mathcal{M}'$ that is pure (preserves weights), is compatible with Tate twists, and an isomorphism $\nu'\phi = \nu$ compatible with $\varepsilon, \varepsilon'$.*

References

[BGS96] Alexander Beilinson, Victor Ginzburg, and Wolfgang Soergel. Koszul duality patterns in representation theory. *J. Amer. Math. Soc.*, 9(2):473–527, 1996. 8

[HS97] P. J. Hilton and U. Stammbach. *A course in homological algebra*, volume 4 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1997. 3

[Hum08] James E. Humphreys. *Representations of semisimple Lie algebras in the BGG category $\mathcal{O}$*, volume 94 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008. 1

[ML63] Saunders Mac Lane. *Homology*. Die Grundlehren der mathematischen Wissenschaften, Bd. 114. Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963. 3

[Soe90] Wolfgang Soergel. Kategorie $\mathcal{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe. *J. Amer. Math. Soc.*, 3(2):421–445, 1990. 4, 5, 6

[Str03] Catharina Stroppel. Category $\mathcal{O}$: gradings and translation functors. *J. Algebra*, 268(1):301–326, 2003. 4, 6, 7