Problem 1 (Main) (5pts)**.** Solve by Fourier method

$$
\begin{aligned}
&u_{tt} - 4u_{xx} = 0, && -\pi < x < \pi,\\
&u|_{x=-\pi} = u|_{x=\pi} = 0, &&\\
&u|_{t=0} = 0, && u_t|_{t=0} = \pi - |x|.
\end{aligned}
$$

Solution. Observing that interval $[-\pi, \pi]$ is symmetric and the problem is invariant with respect to $x \mapsto -x$, we conclude that $u(x,t)$ must be also invariant with respect to $x \mapsto -x$ (that means, $u(x,t)$ is an even function) and we can consider problem on $[0, \pi]$ with $u_x|_{x=0} = 0$.
Plugging $u(x,t) = X(x)T(t)$ into equation and boundary conditions and separating variables we get

$$
\begin{aligned}
&X'' + \lambda X = 0,\\
&X'(0) = X(\pi) = 0,\\
&T'' + 4\lambda T = 0.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\lambda_n = (n + \tfrac{1}{2})^2, \qquad X_n = \cos((n + \tfrac{1}{2})x),\\
&T_n = A_n \cos((2n+1)t) + B_n \sin((2n+1)t),\\
&u_n = \Big[A_n \cos((2n+1)t) + B_n \sin((2n+1)t)\Big] \cos((n + \tfrac{1}{2})x)
\end{aligned}
$$

with $n = 0, 1, 2, \ldots$. Therefore the general solution is

$$
u(x,t) = \sum_{n=0}^{\infty} \Big[A_n \cos((2n+1)t) + B_n \sin((2n+1)t)\Big] \cos((n + \tfrac{1}{2})x).
$$

Plugging into initial conditions we get

$$
\begin{aligned}
u(x,0) &= \sum_{n=0}^{\infty} A_n \cos((n + \tfrac{1}{2})x) = 0,\\
u_t(x,0) &= \sum_{n=0}^{\infty} (2n+1) B_n \cos((n + \tfrac{1}{2})x) = \pi - x,
\end{aligned}
$$

and therefore $A_n = 0$ and

$$
\begin{aligned}
B_n =& \frac{2}{(2n+1)\pi} \int_0^{\pi} (\pi - x) \cos((n + \tfrac{1}{2})x)\, dx = \frac{4}{(2n+1)^2\pi} \int_0^{\pi} (\pi - x) d \sin((n + \tfrac{1}{2})x) =\\
& \frac{4}{(2n+1)^2\pi} \Big[(\pi - x)\sin((n + \tfrac{1}{2})x)\big|_{x=0}^{x=\pi} + \int_0^{\pi} \sin((n + \tfrac{1}{2})x)\, dx\Big] =\\
& - \frac{8}{(2n+1)^3\pi} \cos((n + \tfrac{1}{2})x)\big|_{x=0}^{x=\pi} = \frac{8}{(2n+1)^3\pi}.
\end{aligned}
$$

Finally,

$$
u(x,t) = \sum_{n=0}^{\infty} \frac{8}{(2n+1)^3\pi} \sin((2n+1)t) \cos((n + \tfrac{1}{2})x).
$$

□

Problem 1 (Late) (5pts)**.** Solve by Fourier method

$$\begin{aligned} &u_{tt} - u_{xx} = 0, && 0 < x < \pi, \\ &u|_{x=0} = u|_{x=\pi} = 0, \\ &u|_{t=0} = 0, && u_t|_{t=0} = 1 \end{aligned}$$

Solution. Plugging $u(x,t) = X(x)T(t)$ into equation and boundary conditions and separating variables we get

$$\begin{aligned} &X'' + \lambda X = 0, \\ &X(0) = X(\pi) = 0, \\ &T'' + \lambda T = 0. \end{aligned}$$

Then

$$\begin{aligned} &\lambda_n = n^2, && X_n = \sin(nx), \\ &T_n = A_n \cos(nt) + B_n \sin(nt), \\ &u_n = \Big[A_n \cos(nt) + B_n \sin(nt)\Big] \cos(nx) \end{aligned}$$

with $n = 1, 2, 3, \ldots$. Therefore the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} \Big[A_n \cos(nt) + B_n \sin(nt)\Big] \sin(nx).$$

Plugging into initial conditions we get

$$\begin{aligned} u(x,0) &= \sum_{n=0}^{\infty} A_n \sin(nx) = 0, \\ u_t(x,0) &= \sum_{n=0}^{\infty} nB_n \sin(nx) = 1. \end{aligned}$$

Therefore $A_n = 0$ and

$$B_n = \frac{2}{n\pi} \int_0^{\pi} \sin(nx)\,dx = -\frac{2}{n^2\pi} \cos(nx)\big|_{x=0}^{x=\pi} = \frac{2}{n^2\pi}\big[1 - \cos(n\pi)\big].$$

This vanishes for n even, and for $n = 2m+1$ odd equals

$$\frac{4}{(2m+1)^2\pi}$$

Finally,

$$u(x,t) = \sum_{n=0}^{\infty} \frac{4}{(2n+1)^2\pi} \sin((2n+1)t) \sin((2n+1)x).$$

□

Problem 1 (Early) (5pts)**.** Solve by Fourier method

$$\begin{aligned} &u_t - u_{xx} = 0, && 0 < x < \pi, \\ &u|_{x=0} = u_x|_{x=\pi} = 0, \\ &u|_{t=0} = \sin(x) \end{aligned}$$

Solution. Plugging $u(x,t) = X(x)T(t)$ into equation and boundary conditions and separating variables we get

$$\begin{aligned} &X'' + \lambda X = 0, \\ &X(0) = X'(\pi) = 0, \\ &T' + \lambda T = 0. \end{aligned}$$

Then

$$\begin{aligned} &\lambda_n = (n + \tfrac{1}{2})^2, && X_n = \sin((n + \tfrac{1}{2})x), \\ &T_n = A_n e^{-(n+\frac{1}{2})^2 t}, \\ &u_n = A_n e^{-(n+\frac{1}{2})^2 t} \sin((n + \tfrac{1}{2})x) \end{aligned}$$

with $n = 0, 1, 2, \ldots$. Therefore the general solution is

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-(n+\frac{1}{2})^2 t} \sin((n + \tfrac{1}{2})x).$$

Plugging into initial conditions we get

$$u(x,0) = \sum_{n=0}^{\infty} A_n \sin((n + \tfrac{1}{2})x) = \sin(x).$$

Therefore

$$A_n = \frac{2}{\pi} \int_0^{\pi} \sin(x) \sin\left(\frac{2n+1}{2}x\right)\, dx = \frac{1}{\pi}\left(\int_0^{\pi} \cos\left(\frac{2n-1}{2}x\right)\, dx - \int_0^{\pi} \cos\left(\frac{2n+3}{2}x\right)\, dx\right)$$

$$= \frac{2}{\pi}\left(\frac{1}{2n-1}\underbrace{\sin\left(\frac{2n-1}{2}\pi\right)}_{(-1)^{n+1}} - \frac{1}{2n+3}\underbrace{\sin\left(\frac{2n+3}{2}\pi\right)}_{(-1)^{n+1}}\right) = \frac{8}{\pi}\frac{(-1)^{n+1}}{(2n-1)(2n+3)}.$$

Finally,

$$u(x,t) = \frac{8}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n-1)(2n+3)} e^{-(2n+1)^2 t/4} \sin((n + \tfrac{1}{2})x).$$

□

Problem 1 (Deferred) (5pts)**.** Solve by Fourier method

$$\begin{aligned} &4u_{tt} - u_{xx} = 0, && 0 < x < \pi, \\ &u_x|_{x=0} = u_x|_{x=\pi} = 0, && \\ &u|_{t=0} = \sin(x), && u_t|_{t=0} = 0. \end{aligned}$$

Solution. Plugging $u(x,t) = X(x)T(t)$ into equation and boundary conditions and separating variables we get

$$\begin{aligned} &X'' + \lambda X = 0, \\ &X'(0) = X'(\pi) = 0, \\ &4T'' + \lambda T = 0. \end{aligned}$$

Then

$$\begin{aligned} &\lambda_n = n^2, \qquad\qquad X_n = \cos(nx), \\ &T_n = A_n \cos\left(\frac{n}{2}t\right) + B_n \sin\left(\frac{n}{2}t\right), \\ &u_n = \left[A_n \cos\left(\frac{n}{2}t\right) + B_n \sin\left(\frac{n}{2}t\right)\right] \cos(nx) \end{aligned}$$

with $n = 1, 2, 3, \ldots$ and also $\lambda_0 = 0$, $X_0 = \frac{1}{2}$, $T_0 = A_0 + B_0 t$, $u_0 = \frac{1}{2}(A_0 + B_0 t)$. Therefore the general solution is

$$u(x,t) = \tfrac{1}{2}(A_0 + B_0 t) + \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n}{2}t\right) + B_n \sin\left(\frac{n}{2}t\right)\right] \cos(nx).$$

The initial condition $u_t|_{t=0} = 0$ implies that all of the $B_n = 0$. The other initial condition gives

$$\sin(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} \cos(nx).$$

So

$$A_0 = \frac{2}{\pi} \int_0^{\pi} \sin(x)dx = \frac{4}{\pi},$$

and

$$\begin{aligned} A_n &= \frac{2}{\pi} \int_0^{\pi} \sin(x)\cos(nx)\,dx = \frac{1}{\pi}\left(\int_0^{\pi} \sin((n+1)x)\,dx - \int_0^{\pi} \sin((n-1)x)\,dx\right) \\ &= \frac{1}{\pi}\left(\left[\frac{\cos((n-1)x)}{n-1}\right]_0^{\pi} - \left[\frac{\cos((n+1)x)}{n+1}\right]_0^{\pi}\right) = \frac{1}{\pi}\left(\frac{(-1)^{n-1} - 1}{n-1} - \frac{(-1)^{n+1} - 1}{n+1}\right). \end{aligned}$$

This vanishes for odd n, and for $n = 2m$ even it equals

$$-\frac{4}{(4m^2 - 1)\pi}.$$

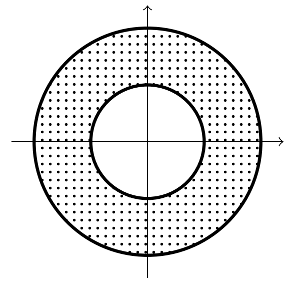
Hence we obtain

$$u(x,t) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \cos(nt)\cos(2nx).$$

□

Problem 2 (Main) (5pts)**.** In the ring $\{(r,\theta)\colon 1 < r \le 2,\ -\pi \le \theta < \pi\}$ find solution

$$\begin{aligned}&\Delta u = 0,\\ &u_r|_{r=1} = 0,\\ &u|_{r=2} = |\sin(\theta)|.\end{aligned}$$



Solution. In polar coordinates $\Delta u = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta}$. Plugging $u(r,\theta) = R(r)\Theta(\theta)$ into equation we get $(R'' + r^{-1}R')\Theta + r^{-2}\Theta'' = 0$ and separating variables we get $\frac{R''+rR'}{R} + \frac{\Theta''}{\Theta} = 0$, and therefore

$$\begin{aligned}&\Theta'' + \lambda\Theta = 0,\\ &\Theta \quad 2\pi - \text{periodic},\\ &R'' + rR' - \lambda R = 0.\end{aligned}$$

Then

$$\begin{aligned}&\lambda_0 = 0, && \Theta_0 = \frac{1}{2}, &&\\ &\lambda_n = n^2, && \Theta_{n,1} = \cos(n\theta), && \Theta_{n,2} = \sin(n\theta),\\ &R_0 = A_0 + B_0\ln(r) && R_n = A_n r^n + Br^{-n}, && n = 1, 2, \ldots\end{aligned}$$

but since problem is invariant with respect to $\theta \mapsto -\theta$ we may reject odd Θ_n. Therefore

$$u(r,\theta) = \frac{1}{2}(A_0 + B_0\ln(r)) + \sum_{n=1}^{\infty}\Bigl[A_n r^n + B_n r^{-n}\Bigr]\cos(n\theta).$$

Plugging to boundary conditions we get

$$\begin{aligned}u_r(1,\theta) &= \frac{1}{2}B_0 + \sum_{n=1}^{\infty} n(A_n - B_n)\cos(n\theta) = 0, \implies B_0 = 0,\ A_n = B_n,\\ u(2,\theta) &= \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n(2^n + 2^{-n})\cos(n\theta) = |\sin(\theta)|\end{aligned}$$

and $A_0 = \frac{2}{\pi}\int_0^{\pi}\sin(\theta)\,d\theta = \frac{4}{\pi}$,

$$A_n(2^n + 2^{-n}) = \frac{2}{\pi}\int_0^{\pi}\sin(\theta)\cos(n\theta)\,d\theta = \frac{1}{\pi}\int_0^{\pi}\Bigl[\sin((n+1)\theta) - \sin((n-1)\theta)\Bigr]\,d\theta$$

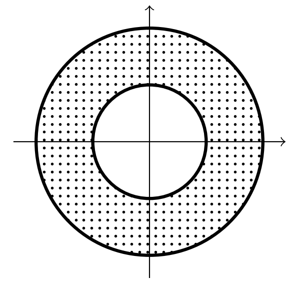
is 0 for odd n and for even n is equal to $-\frac{4}{(n^2-1)\pi}$. Thus $A_n = 0$ for odd n and $A_{2m} = -\dfrac{4}{(4m^2-1)(2^{2m}+2^{-2m})}$. Finally

$$u(r,\theta) = \frac{2}{\pi} - \sum_{m=1}^{\infty}\frac{4}{(4m^2-1)(2^{2m}+2^{-2m})}\Bigl[r^{2m} + r^{-2m}\Bigr]\cos(2m\theta).$$

□

Problem 2 (Late) (5pts)**.** In the ring $\{(r,\theta)\colon 1 < r \le 2,\ -\pi \le \theta < \pi\}$ find solution

$$\begin{aligned}&\Delta u = 0,\\ &u|_{r=1} = 0,\\ &u|_{r=2} = \theta^2.\end{aligned}$$



Solution. In polar coordinates $\Delta u = u_{rr}+r^{-1}u_r+r^{-2}u_{\theta\theta}$. Plugging $u(r,\theta) = R(r)\Theta(\theta)$ into equation we get $(R''+r^{-1}R')\Theta + r^{-2}\Theta'' = 0$ and separating variables we get $\frac{R''+rR'}{R} + \frac{\Theta''}{\Theta} = 0$, and therefore

$$\begin{aligned}&\Theta'' + \lambda\Theta = 0,\\ &\Theta \quad 2\pi - \text{periodic},\\ &R'' + rR' - \lambda R = 0.\end{aligned}$$

Then

$$\begin{aligned}&\lambda_0 = 0, && \Theta_0 = \frac{1}{2}, &&\\ &\lambda_n = n^2, && \Theta_{n,1} = \cos(n\theta), && \Theta_{n,2} = \sin(n\theta),\\ &R_0 = A_0 + B_0\ln(r) && R_n = A_n r^n + B r^{-n}, && n = 1,2,\ldots\end{aligned}$$

but since problem is invariant with respect to $\theta \mapsto -\theta$ we may reject odd Θ_n. Therefore

$$u(r,\theta) = \frac{1}{2}(A_0 + B_0\ln(r)) + \sum_{n=1}^{\infty}\Bigl[A_n r^n + B_n r^{-n}\Bigr]\cos(n\theta).$$

Plugging to boundary conditions we get

$$\begin{aligned}u(1,\theta) &= \frac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n + B_n)\cos(n\theta) = 0, \implies A_0 = 0,\ A_n = -B_n,\\ u(2,\theta) &= \frac{1}{2}B_0\ln(2) + \sum_{n=1}^{\infty}A_n(2^n - 2^{-n})\cos(n\theta) = \theta^2\end{aligned}$$

and $B_0 = \frac{2}{\pi\ln(2)}\int_0^\pi \theta^2\,d\theta = \frac{\pi}{\ln(2)}$,

$$\begin{aligned}A_n(2^n - 2^{-n}) =& \frac{2}{\pi}\int_0^\pi \theta^2\cos(n\theta)\,d\theta = \frac{2}{n\pi}\int_0^\pi \theta^2\,d\sin(n\theta) = -\frac{4}{n\pi}\int_0^\pi \theta\sin(n\theta)\,d\theta =\\ &-\frac{4}{n^2\pi}\int_0^\pi \theta\,d\cos(n\theta) = -\frac{4}{n^2\pi}\Bigl[\theta\cos(n\pi) - \int_0^\pi \cos(n\theta)\,d\theta\Bigr] = \frac{4}{n^2\pi}(-1)^{n-1}.\end{aligned}$$

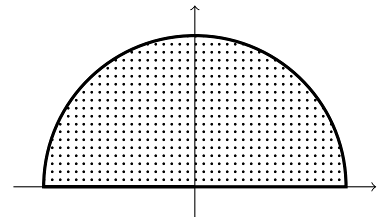
Finally

$$u(r,\theta) = \frac{\pi}{2\ln(2)}\ln(r) - \sum_{n=1}^{\infty}\frac{4}{n^2(2^n-2^{-n})\pi}(-1)^{n-1}\Bigl[r^n + r^{-n}\Bigr]\cos(n\theta).$$

□

Problem 2 (Early) (5pts)**.** In the half-disk $\{(r,\theta)\colon r\le 2,\ 0<\theta<\pi\}$ find solution

$$\begin{aligned}&\Delta u=0,\\&u|_{\theta=0}=u|_{\theta=\pi}=0,\\&u|_{r=2}=\theta(\pi-\theta).\end{aligned}$$



Solution. In polar coordinates $\Delta u=u_{rr}+r^{-1}u_r+r^{-2}u_{\theta\theta}$. Plugging $u(r,\theta)=R(r)\Theta(\theta)$ into equation we get $(R''+r^{-1}R')\Theta+r^{-2}\Theta''=0$ and separating variables we get $\frac{R''+rR'}{R}+\frac{\Theta''}{\Theta}=0$, and therefore

$$\begin{aligned}&\Theta''+\lambda\Theta=0,\\&\Theta(\quad 2\pi-\text{periodic})0)=\Theta(\pi)=0,\\&R''+rR'-\lambda R=0.\end{aligned}$$

Then

$$\begin{aligned}&\lambda_n=n^2, &&\Theta_n=\sin(n\theta),\\&R_n=A_nr^n+Br^{-n}, &&n=1,2,\ldots\end{aligned}$$

but since solution must be regular as $r=0$ we must reject r^{-n}. Therefore

$$u(r,\theta)=\sum_{n=1}^{\infty}A_nr^n\sin(n\theta).$$

Plugging to boundary conditions we get

$$u(2,\theta)=\sum_{n=1}^{\infty}2^nA_n\sin(n\theta)=\theta(\pi-\theta)$$

and

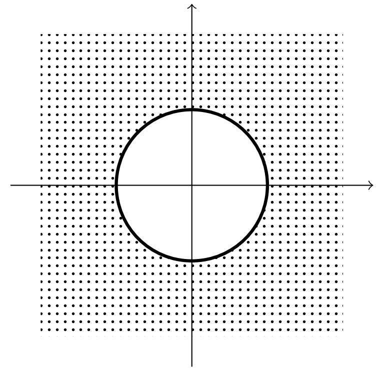
$$\begin{aligned}A_n=&\frac{2}{2^n\pi}\int_0^{\pi}\theta(\pi-\theta)\sin(n\theta)\,d\theta=-\frac{2}{2^nn\pi}\int_0^{\pi}\theta(\pi-\theta)\,d\cos(n\theta)=\\&\frac{2}{2^nn\pi}\int_0^{\pi}(\pi-2\theta)\cos(n\theta)\,d\theta=\frac{2}{2^nn^2\pi}\int_0^{\pi}(\pi-2\theta)\,d\sin(n\theta)=\\&-\frac{4}{2^nn^2\pi}\int_0^{\pi}\sin(n\theta)\,d\theta\end{aligned}$$

which is 0 for even n and $-\frac{8}{2^nn^3\pi}$ for odd n. Finally

$$u(r,\theta)=-\sum_{m=0}^{\infty}\frac{1}{(2m+1)^3 2^{2m-2}\pi}r^{2m}\sin(2m\theta).$$

□

Problem 2 (Deferred) (5pts)**.** In the disk exterior $\{(r,\theta)\colon r>1,\ -\pi\le \theta<\pi\}$ find solution



$$\begin{aligned}
&\Delta u = 0,\\
&u|_{r=1} = |\theta|,\\
&\max|u| < \infty.
\end{aligned}$$

Solution. In polar coordinates $\Delta u = u_{rr}+r^{-1}u_r+r^{-2}u_{\theta\theta}$. Plugging $u(r,\theta)=R(r)\Theta(\theta)$ into equation we get $(R''+r^{-1}R')\Theta + r^{-2}\Theta''=0$ and separating variables we get $\frac{R''+rR'}{R}+\frac{\Theta''}{\Theta}=0$, and therefore

$$\begin{aligned}
&\Theta''+\lambda\Theta=0,\\
&\Theta \quad 2\pi-\text{periodic},\\
&R''+rR'-\lambda R=0.
\end{aligned}$$

Then

$$\begin{aligned}
&\lambda_0=0, && \Theta_0=\frac{1}{2}, &&\\
&\lambda_n=n^2, && \Theta_{n,1}=\cos(n\theta), && \Theta_{n,2}=\sin(n\theta),\\
&R_0=A_0\ln(r)+B_0 && R_n=A_nr^n+Br^{-n}, && n=1,2,\ldots
\end{aligned}$$

but must solution must be bounded we must reject $\ln(r)$ and r^n.

$$u(r,\theta)=\frac{1}{2}B_0+\sum_{n=1}^{\infty} r^{-n}[A_n\cos(n\theta)+B_n\sin(n\theta)].$$

Plugging to boundary conditions we get

$$u(1,\theta)=\frac{1}{2}A_0+\sum_{n=1}^{\infty}[A_n\cos(n\theta)+B_n\sin(n\theta)]=|\theta|,$$

and since $|\theta$ is even function $B_n=0$ and $A_0=\frac{2}{\pi}\int_0^{\pi}\theta\,d\theta=\pi$,

$$A_n=\frac{2}{\pi}\int_0^{2\pi}\theta\cos(n\theta)\,d\theta=\frac{2}{n\pi}\int_0^{\pi}\theta\,d\sin(n\theta)=-\frac{2}{n\pi}\int_0^{\pi}\sin(n\theta)\,d\theta,$$

which is 0 for even n and $-\frac{4}{n^2\pi}$ or odd n. Finally

$$u(r,\theta)=\frac{\pi}{2}-\sum_{m=0}^{\infty}\frac{4}{(2m+1)^2\pi}r^{-2m-1}\cos((2m+1)\theta).$$

□

Problem 3 (Main) (5pts)**.** Find the solution $u(x,t)$ to

$$\begin{aligned} &u_{tt} = -4u_{xxxx} && -\infty < x < \infty, \\ &u|_{t=0} = \begin{cases} 1 & |x| < 2, \\ 0 & |x| \geq 2, \end{cases} && u_t|_{t=0} = 0, \\ &\max|u| < \infty. \end{aligned}$$

Solution. Making partial Fourier transform $F_{x\to k}u = \hat{u}$ we get

$$\begin{aligned} &\hat{u}_{tt} = -4k^4\hat{u}, \\ &\hat{u}|_{t=0} = \hat{g}, && \hat{u}|_{t=0} = \hat{h} \end{aligned}$$

with

$$\hat{g} = \frac{1}{2\pi}\int_{-2}^{2} e^{-ixk}\,dx = \frac{1}{-2ik\pi}\bigl[e^{-2ik} - e^{2ik}\bigr] = \frac{\sin(2k)}{k\pi}$$

and $\hat{h} = 0$. Then

$$\hat{u}(k,t) = \frac{\sin(2k)}{k\pi}\cos(2k^2t)$$

and

$$\begin{aligned} u(x,t) = &\int_{-\infty}^{\infty} \frac{\sin(2k)}{k\pi}\cos(2k^2t)e^{ikx}\,dk = \\ &\int_{0}^{\infty} \frac{2\sin(2k)}{k\pi}\cos(2k^2t)\cos(kx)\,dk. \end{aligned}$$

□

Problem 3 (Late) (5pts)**.** Find the solution $u(x,t)$ to

$$\begin{aligned}
&u_t = u_{xx} && -\infty < x < \infty,\ t > 0,\\
&u|_{t=0} = \begin{cases} 1 - x^2 & |x| < 1,\\ 0 & |x| \geq 1,\end{cases}\\
&\max |u| < \infty.
\end{aligned}$$

Solution. Making partial Fourier transform $F_{x\to k} u = \hat{u}$ we get

$$\begin{aligned}
\hat{u}_t &= -k^2\hat{u},\\
\hat{u}|_{t=0} &= \hat{g},
\end{aligned}$$

with

$$\begin{aligned}
\hat{g} =& \frac{1}{2\pi}\int_{-1}^{1} (1-x^2)e^{-ixk}\,dx = \frac{1}{-ik\pi}\int_{-1}^{1} xe^{-ixk}\,dx = \frac{1}{-k^2\pi}\Bigl[xe^{-ixk}\bigr|_{x=-1}^{x=1} - \int_{-1}^{1} e^{-ixk}\,dx\Bigr] =\\
&\frac{1}{k^2\pi}\Bigl[-\cos(k) + k^{-1}\sin(k)\Bigr] = \frac{1}{k^3\pi}\bigl[\sin(k) - k\cos(k)\bigr].
\end{aligned}$$

Then

$$\begin{aligned}
\hat{u}(k,t) =& \frac{1}{k^3\pi}\bigl[\sin(k) - k\cos(k)\bigr]e^{-k^2 t}\\
u(x,t) =& \int_{-\infty}^{\infty} \frac{2}{k^3\pi}\bigl[k\cos(k) - \sin(k)\bigr]e^{-k^2 t}e^{ikx}\,dk =\\
&\int_{0}^{\infty} \frac{2}{k^3\pi}\bigl[\sin(k) - k\cos(k)\bigr]e^{-k^2 t}\cos(kx)\,dk.
\end{aligned}$$

□

Problem 3 (Early) (5pts)**.** Find the solution $u(x,t)$ to

$$\begin{aligned} &u_t = 4u_{xx} && -\infty < x < \infty,\ t > 0,\\ &u|_{t=0} = e^{-|x|}\\ &\max|u| < \infty. \end{aligned}$$

Solution. Making partial Fourier transform $F_{x\to k}u = \hat{u}$ we get

$$\begin{aligned} &\hat{u}_t = -k^2\hat{u},\\ &\hat{u}|_{t=0} = \hat{g}, \end{aligned}$$

with

$$\begin{aligned} \hat{g} =& \frac{1}{2\pi}\Bigl[\int_{-\infty}^{0} e^{-ixk+x}\,dx + \int_0^{\infty} e^{-ixk-x}\,dx\Bigr] =\\ & \frac{1}{2\pi}\bigl[(1-ik)+(1+ik)\bigr] = \frac{1}{(1+k^2)\pi}. \end{aligned}$$

Then

$$\hat{u}(k,t) = \frac{1}{(1+k^2)\pi}e^{-4k^2t}$$

and

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{(1+k^2)\pi}e^{-4k^2t}e^{ikx}\,dk = \int_0^{\infty} \frac{2}{(1+k^2)\pi}e^{-4k^2t}\cos(kx)\,dk.$$

□

Problem 3 (Deferred) (5pts)**.** Find the solution $u(x,t)$ to

$$u_{tt} = u_{xx} - 4u \qquad -\infty < x < \infty,$$
$$u|_{t=0} = 0, \qquad u_t|_{t=0} = e^{-x^2/2}.$$

Solution. Making partial Fourier transform $F_{x\to k} u = \hat{u}$ we get

$$\hat{u}_{tt} = -(k^2+4)\hat{u},$$
$$\hat{u}|_{t=0} = 0, \qquad \hat{u}_t|_{t=0} = \hat{h}$$

with $\hat{h} = \frac{1}{\sqrt{2\pi}} e^{-k^2/2}$ (standard F.T.) Then

$$\hat{u}(k,t) = \frac{1}{\sqrt{2\pi(k^2+4)}} e^{-k^2/2} \sin(\sqrt{k^2+4}\,t)$$

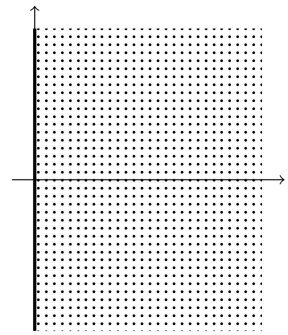
and

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(k^2+4)}} e^{-k^2/2} \sin(\sqrt{k^2+4}\,t) e^{ikx}\, dk =$$
$$\int_0^{\infty} \frac{2}{\sqrt{2\pi(k^2+4)}} e^{-k^2/2} \sin(\sqrt{k^2+4}\,t) \cos(kx)\, dk.$$

□

Problem 4 (Main) (5pts)**.** In the half-plane $\{(x,y)\colon x>0,\ -\infty<y<\infty\}$ find solution

$$\begin{aligned}
&\Delta u = 0,\\
&u|_{x=0} = e^{-|y|},\\
&\max|u| < \infty.
\end{aligned}$$



Solution. Making partial Fourier transform $Fy\to ku = \hat{u}$ we get

$$\begin{aligned}
&\hat{u}_{xx} - k^2\hat{u} = 0,\\
&\hat{u}|_{x=0} = \hat{g}(k)
\end{aligned}$$

with $\hat{g}(k) = \frac{1}{\pi(k^2+1)}$. Then $\hat{u} = A(k)e^{-|k|x} + B(k)e^{|k|x}$ and $B(k)=0$ since the corresponding term is growing with respect to $x\to+\infty$. So $\hat{u} = A(k)e^{-|k|x}$ and from the boundary condition we conclude that $A(k) = \hat{g}(k) = \frac{1}{\pi(k^2+1)}$. So

$$\hat{u}(x,k) = \frac{1}{\pi(k^2+1)}e^{-|k|x}$$

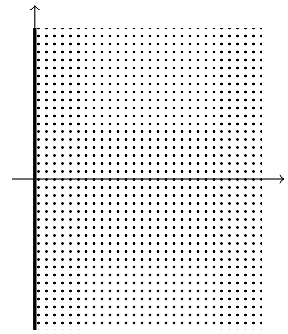
and

$$\begin{aligned}
u(x,y) &= \int_{-\infty}^{\infty} \frac{1}{\pi(k^2+1)} e^{-|k|x+iky}\,dk\\
&= \int_{0}^{\infty} \frac{2}{\pi(k^2+1)} e^{-|k|x}\cos(ky)\,dk.
\end{aligned}$$

□

Problem 4 (Late) (5pts)**.** In the half-plane $\{(x,y)\colon x>0,\ -\infty<y<\infty\}$ find solution

$$\Delta u - 4u = 0,$$
$$u|_{x=0} = \begin{cases} 1-|y|, & |y|\le 1,\\ 0, & |y|\ge 1,\end{cases}$$
$$\max|u|<\infty.$$



Solution. Making partial Fourier transform $Fy\to ku=\hat{u}$ we get

$$\hat{u}_{xx}-(k^2+4)\hat{u}=0,$$

and the Fourier transform of the boundary condition is

$$\begin{aligned}\hat{h}(k) =&\frac{1}{2\pi}\int_{-1}^{1}(1-|y|)e^{-iky}\,dy = \frac{1}{-2ik\pi}\int_{-1}^{1}(1-|y|)\,de^{-iky}\\ =&\frac{1}{2ik\pi}\int_{-1}^{1}e^{-iky}\,d(1-|y|) = \frac{1}{2k^2\pi}\Bigl[\int_{-1}^{0}e^{-iky}\,dy-\int_{0}^{1}e^{-iky}\,dy\Bigr]\\ =&\frac{1}{2k^2\pi}\bigl[(1-e^{ik})-(e^{-ik}-1)\bigr] = \frac{1}{k^2\pi}(1-\cos(k)).\end{aligned}$$

Then

$$\hat{u} = A(k)e^{-\sqrt{k^2+4}|x|}+B(k)e^{+\sqrt{k^2+4}|x|}$$

and boundedness implies that $B(k)=0$. So $A(k)=\hat{h}(k)$, and

$$\hat{u}(x,k)=\frac{1}{k^2\pi}(1-\cos(k))e^{-\sqrt{k^2+4}|x|}.$$

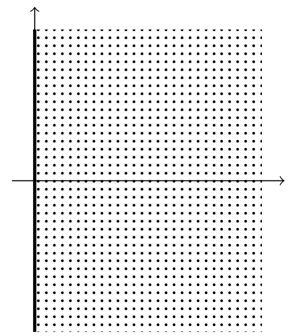
So

$$u(x,y)=\int_{-\infty}^{\infty}\frac{1}{k^2\pi}(1-\cos(k))e^{-\sqrt{k^2+4}|x|}e^{iky}\,dk.$$

□

Problem 4 (Early) (5pts)**.** In the half-plane $\{(x,y)\colon x>0,\ -\infty<y<\infty\}$ find solution

$$\begin{aligned}
&\Delta u - u = 0,\\
&u|_{x=0} = \begin{cases} \cos(y), & |y|\le \frac{\pi}{2},\\ 0, & |y|\ge \frac{\pi}{2},\end{cases}\\
&\max|u| < \infty.
\end{aligned}$$



Solution. Making partial Fourier transform $Fy \to ku = \hat{u}$ we get

$$\hat{u}_{xx} - (k^2+1)\hat{u} = 0,$$

with Fourier transform of the boundary condition given by

$$\begin{aligned}
\hat{h}(k) =& \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \cos(y)e^{-iky}\,dy = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos(y)\cos(ky)\,dy\\
=&\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\bigl[\cos((k+1)y)+\cos((k-1)y)\bigr]\,dy\\
=&\frac{1}{2\pi}\Bigl[\frac{1}{k+1}\sin((k+1)\pi/2)+\frac{1}{k-1}\sin((k-1)\pi/2)\Bigr]\\
=&\frac{1}{2\pi}\cos(k\pi)\Bigl[\frac{1}{k+1}-\frac{1}{k-1}\Bigr] = -\frac{1}{(k^2-1)\pi}\cos(k\pi).
\end{aligned}$$

Then

$$\hat{u} = A(k)e^{-\sqrt{k^2+1}|x|} + B(k)e^{+\sqrt{k^2+1}|x|}$$

and boundedness implies that $B(k)=0$. So $A(k)=\hat{h}(k)$, and

$$\hat{u}(x,k) = -\frac{1}{(k^2-1)\pi}\cos(k\pi)e^{-\sqrt{k^2+1}|x|}.$$

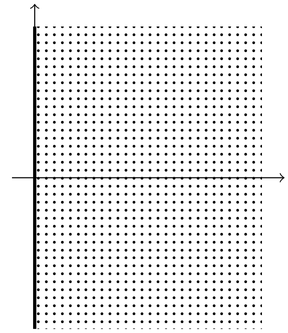
Finally,

$$u(x,y) = -\int_{-\infty}^{\infty}\frac{1}{(k^2-1)\pi}\cos(k\pi)e^{-\sqrt{k^2+1}|x|}e^{iky}\,dk.$$

□

Problem 4 (Deferred) (5pts)**.** In the half-plane $\{(x,y)\colon x>0,\ -\infty<y<\infty\}$ find solution

$$\begin{aligned}&\Delta u=0,\\ &u|_{x=0}=\begin{cases}1-y^2, & |y|\le 1,\\ 0, & |y|\ge 1,\end{cases}\\ &\max|u|<\infty.\end{aligned}$$



Solution. Making partial Fourier transform $F_{y\to k}u=\hat{u}$ we get

$$\hat{u}_{xx}=k^2\hat{u},$$

with Fourier transform of the boundary condition given by

$$\hat{h}=\frac{1}{2\pi}\int_{-1}^{1}(1-y^2)e^{-iyk}\,dx=\frac{1}{-ik\pi}\int_{-1}^{1}ye^{-iyk}\,dx=\frac{1}{-k^2\pi}\Bigl[ye^{-iyk}\bigr|_{y=-1}^{y=1}-\int_{-1}^{1}e^{-iyk}\,dy\Bigr]=\frac{1}{k^2\pi}\Bigl[-\cos(k)+k^{-1}\sin(k)\Bigr]=\frac{1}{k^3\pi}\bigl[\sin(k)-k\cos(k)\bigr].$$

Then

$$\hat{u}=A(k)e^{-k|x|}+B(k)e^{k|x|}$$

and boundedness implies that $B(k)=0$. So $A(k)$ is equal to the Fourier transform of the boundary condition, and

$$\hat{u}(x,k)=\frac{1}{k^3\pi}\bigl[\sin(k)-k\cos(k)\bigr]e^{-k|x|}.$$

Finally,

$$u(x,y)=\int_{-\infty}^{\infty}\frac{1}{k^3\pi}\bigl[\sin(k)-k\cos(k)\bigr]e^{-k|x|}e^{iky}\,dk.$$

□

Bonus Problem (Main) (3pts)**.** Suppose that

$$e^{-\frac{x^2}{2}} = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|} f(y)dy.$$

Solve explicitly for $f(x)$. (I.e. do *not* leave your answer in integral form!)

Solution. Equation could be rewritten as

$$f * g = h$$

with $g(x) = \frac{1}{2}e^{-|x|}$, $h(x) = e^{-\frac{x^2}{2}}$.
Making Fourier transform we get $\hat{f}(k)\hat{g}(k) = \hat{h}(k)$. Since

$$\hat{g}(k) = \frac{1}{2(k^2+1)\pi}, \qquad \hat{h}(k) = \frac{1}{\sqrt{2\pi}}e^{-\frac{k^2}{2}}$$

we get

$$\hat{f}(k) = 2\pi(k^2+1) \times \frac{1}{\sqrt{2\pi}}e^{-\frac{k^2}{2}}.$$

Then

$$f(x) = 2\pi(-\partial_x^2 + 1)e^{-\frac{x^2}{2}} = 2\pi(x^2-1)e^{-\frac{x^2}{2}}.$$

□

Bonus Problem (Late) (3pts)**.** Suppose that

$$e^{-\frac{x^2}{4}} = \int_{-\infty}^{0} f(x-y)e^{\frac{1}{2}y}dy.$$

Solve explicitly for $f(x)$. (I.e. do *not* leave your answer in integral form!)

Solution. Equation could be rewritten as

$$f * g = h$$

with $g(x) = \begin{cases} e^x & x < 0, \\ 0 & x \geq 0, \end{cases} \quad h(x) = e^{-\frac{x^2}{4}}.$

Making Fourier transform we get $\hat{f}(k)\hat{g}(k) = \hat{h}(k)$. Since

$$\hat{g}(k) = \frac{1}{2(1-ki)\pi}, \qquad \hat{h}(k) = \frac{1}{\sqrt{\pi}}e^{-k^2},$$

we get

$$\hat{f}(k) = 2\pi(1-ki) \times \frac{1}{\sqrt{2\pi}}e^{-k^2}.$$

Then

$$f(x) = 2\pi(1-\partial_x)e^{-\frac{x^2}{4}} = \pi(2+x)e^{-\frac{x^2}{4}}.$$

□

Bonus Problem (Early) (3pts)**.** Suppose that

$$e^{-x^2} = \int_0^\infty f(x-y)e^{-y}dy.$$

Solve explicitly for $f(x)$. (I.e. do *not* leave your answer in integral form!)

Solution. Equation could be rewritten as

$$f * g = h$$

with $g(x) = \begin{cases} 0 & x < 0, \\ e^x & x \geq 0 \end{cases}$, $h(x) = e^{-x^2}$.
Making Fourier transform we get $\hat{f}(k)\hat{g}(k) = \hat{h}(k)$. Since

$$\hat{g}(k) = \frac{1}{2(1+ki)\pi}, \qquad \hat{h}(k) = \frac{1}{\sqrt{4\pi}}e^{-\frac{k^2}{4}},$$

we get

$$\hat{f}(k) = 2\pi(1+ki) \times \frac{1}{\sqrt{4\pi}}e^{-\frac{k^2}{4}}.$$

Then

$$f(x) = 2\pi(1+\partial_x)e^{-x^2} = 2\pi(1-2x)e^{-x^2}.$$

□

Bonus Problem (Deferred) (3pts)**.** Suppose that

$$e^{-\frac{x^2}{2}} = \frac{1}{\sqrt{6\pi}} \int_{-\infty}^{\infty} e^{-2(x-y)^2} f(y)dy.$$

Solve explicitly for $f(x)$. (I.e. do *not* leave your answer in integral form!)

Solution. Equation could be rewritten as

$$f * g = h$$

with $g(x) = \frac{1}{\sqrt{6\pi}} e^{-2x^2}$, $h(x) = e^{-\frac{x^2}{2}}$.
Making Fourier transform we get $\hat{f}(k)\hat{g}(k) = \hat{h}(k)$. Since

$$\hat{g}(k) = \frac{1}{\sqrt{6\pi}} \times \frac{1}{\sqrt{8\pi}} e^{-\frac{k^2}{8}}, \qquad \hat{h}(k) = \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{2}},$$

we get

$$\hat{f}(k) = 2\sqrt{6\pi} e^{-\frac{3k^2}{8}} = 8\pi \times \frac{1}{\sqrt{2\pi a}} e^{-\frac{k^2}{2a}}, \qquad a = \tfrac{4}{3}.$$

Then

$$f(x) = 8\pi e^{-\frac{ax^2}{2}} = 8\pi e^{-\frac{2x^2}{3}}.$$

□