Solutions to Problem 1

Day, Problem 1A

Problem 1 (5pts)**.** Consider the first order equation:

$$(x^2+1)\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = e^x \cos(x). \tag{1.1}$$

(a) Solve for and sketch the characteristic curves for this equation. (Your sketch should be large and clear – make it at least 1/3 of the page!)

(b) Consider the IVP $u(x,0) = g(x)$ for this PDE:

(i) Does this IVP have a solution on the domain $-\infty < x < \infty$, $t > 0$? If so, is the solution unique? If not, would some extra constraint on g guarantee existence of a solution? Justify your answer.

(ii) Does this IVP have a solution on the domain $0 \le x < \infty$, $t > 0$? If so, is the solution unique? If not, would some extra constraint on g guarantee existence of a solution? Justify your answer.

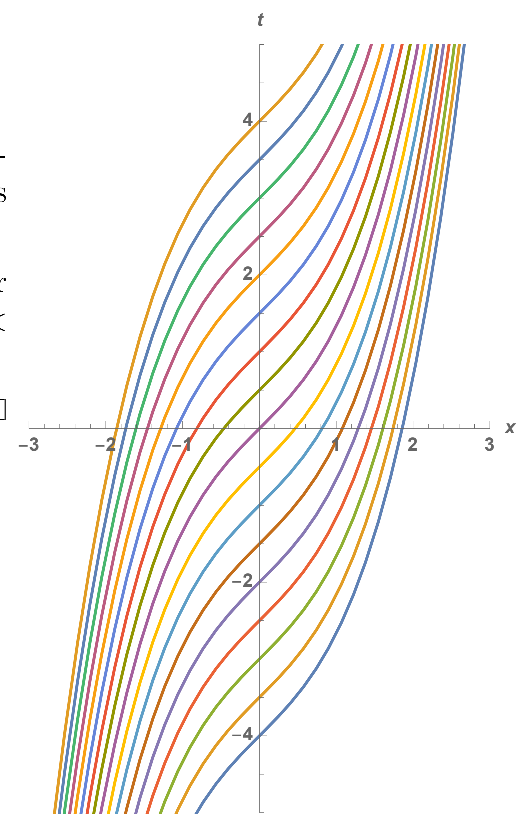
Solution. (a) $t - x - \frac{1}{3}x^3 = c$

(b) IVP:

(i) Unique solution: each characteristic intersects $t = 0$ exactly once, and characteristics fill out the plane.

(ii) Non-unique solution: need more data for characteristics which intersect $t = 0$ at $x < 0$.

□



Day, Problem 1B

Problem 1 (5pts)**.** Consider the first order equation

$$\frac{\partial u}{\partial t} + e^{-t}\frac{\partial u}{\partial x} = \tanh(t^2). \tag{1.1}$$

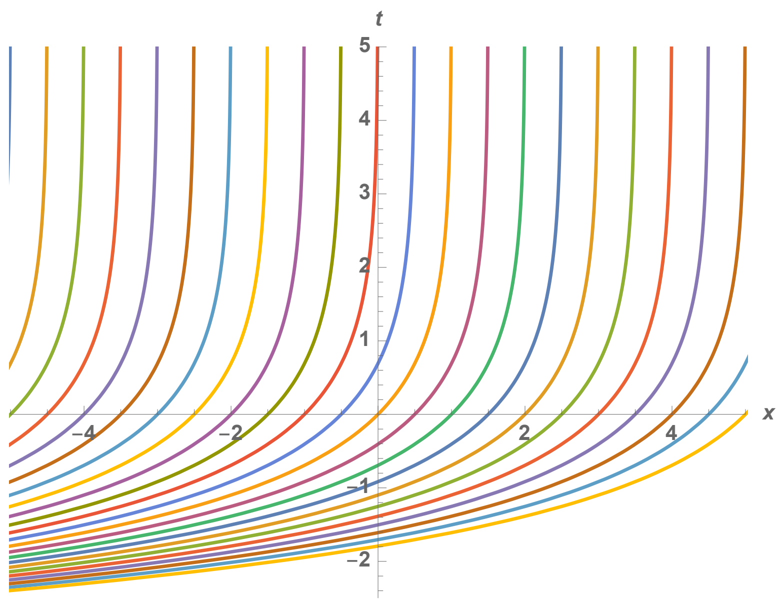
(a) Solve for and sketch the characteristic curves for this equation. (Your sketch should be large and clear – make it at least 1/3 of the page!)

(b) Consider the IVP $u(x, 0) = g(x)$ for this PDE:

(i) Does this IVP have a solution on the domain $-\infty < x < \infty$, $t > 0$? If so, is the solution unique? If not, would some extra constraint on g guarantee existence of a solution? Justify your proofer.

(ii) Does this IVP have a solution on the domain $0 \leq x < \infty$, $t > 0$? If so, is the solution unique? If not, would some extra constraint on g guarantee existence of a solution? Justify your proofer.

Solution. (a) $x + e^{-t} = c$



(b) IVP:

(i) Unique solution: each characteristic intersects $t = 0$ exactly once, and characteristics fill out the plane.

(ii) Non-unique solution: need more data for characteristics which intersect $t = 0$ at $x < 0$.

□

Day, Problem 1C

Problem 1 (5pts)**.** Consider the first order equation

$$(x^2 - 1)\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0. \tag{1.1}$$

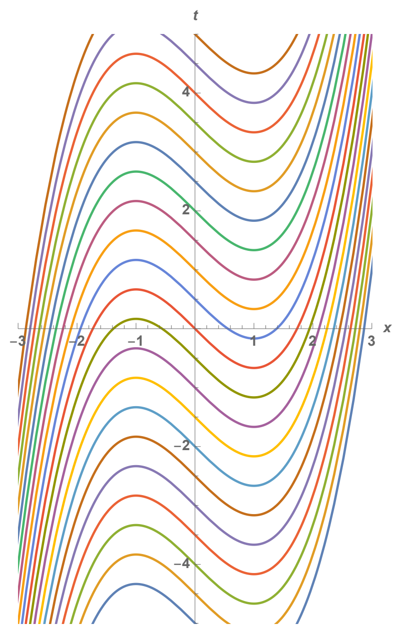
(a) Solve for and sketch the characteristic curves for this equation. (Your sketch should be large and clear – make it at least 1/3 of the page!)

(b) Consider the IVP $u(x, 0) = g(x)$ for this PDE:

(i) Does this IVP have a solution on the domain $-\infty < x < \infty$, $t > 0$? If so, is the solution unique? If not, would some extra constraint on g guarantee existence of a solution? Justify your proofer.

(ii) Does this IVP have a solution on the domain $0 \leq x < \infty$, $t > 0$? If so, is the solution unique? If not, would some extra constraint on g guarantee existence of a solution? Justify your proofer.

Solution. (a) $t + x - \frac{1}{3}x^3 = c$



(b) IVP:

(i) Solution may not exist: some characteristics intersect $t = 0$ multiple times and constraint the initial data. Including this constraint would then give unique solution.

(ii) Solution may not exist and fails uniqueness: even if constraints on initial data coming from characteristics intersecting $t = 0$ multiple times are imposed (for existence), still need more data for characteristics which intersect $t = 0$ only at $x < 0$ (for uniqueness).

□

Day, Problem 1D

Problem 1 (5pts)**.** Consider the first order equation

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = (1+t)x^2 e^{t^2}. \tag{1.1}$$

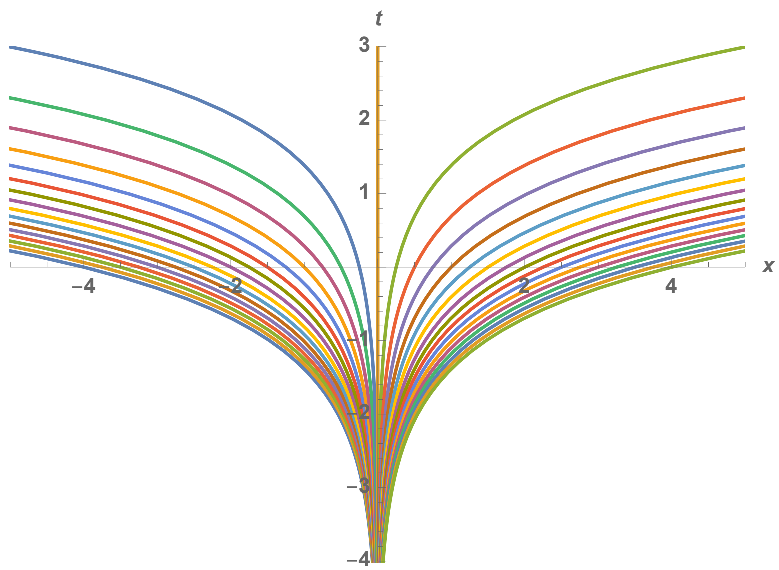
(a) Solve for and sketch the characteristic curves for this equation. (Your sketch should be large and clear – make it at least 1/3 of the page!)

(b) Consider the IVP $u(x,0) = g(x)$ for this PDE:

(i) Does this IVP have a solution on the domain $-\infty < x < \infty$, $t > 0$? If so, is the solution unique? If not, would some extra constraint on g guarantee existence of a solution? Justify your proofer.

(ii) Does this IVP have a solution on the domain $0 \leq x < \infty$, $t > 0$? If so, is the solution unique? If not, would some extra constraint on g guarantee existence of a solution? Justify your proofer.

Solution. (a) $x = ce^t$



(b) IVP:

(i) Unique solution: each characteristic intersects $t = 0$ exactly once, and characteristics fill out the plane.

(ii) Unique solution: each point in the region $t > 0, x > 0$ lies on a characteristic that intersects $t = 0$ at a point in $x > 0$.

□

Day, Problem 1E

Problem 1 (5pts)**.** Consider the first order equation

$$x\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = x\sinh(t + x^2). \tag{1.1}$$

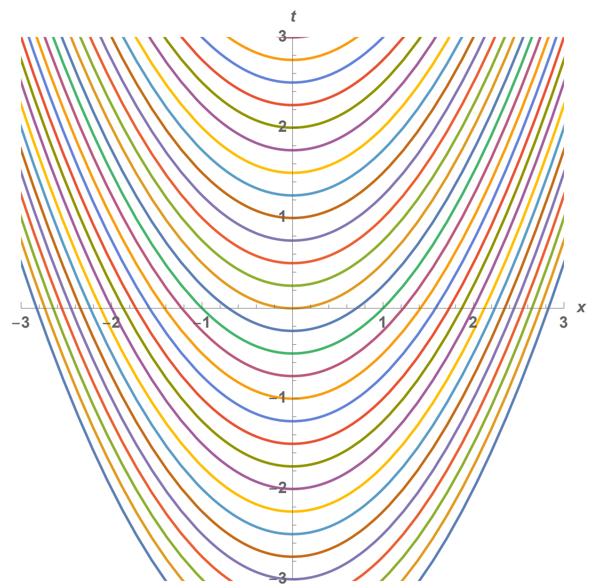
(a) Solve for and sketch the characteristic curves for this equation. (Your sketch should be large and clear – make it at least 1/3 of the page!)

(b) Consider the IVP $u(x, 0) = g(x)$ for this PDE:

(i) Does this IVP have a solution on the domain $-\infty < x < \infty$, $t > 0$? If so, is the solution unique? If not, would some extra constraint on g guarantee existence of a solution? Justify your proofer.

(ii) Does this IVP have a solution on the domain $0 \leq x < \infty$, $t > 0$? If so, is the solution unique? If not, would some extra constraint on g guarantee existence of a solution? Justify your proofer.

Solution. (a) $t - \frac{x^2}{2} = c$



(b) IVP:

(i) Solution may not exist: some characteristics intersect $t = 0$ multiple times and constraint the initial data. Including this constraint would then give a non-unique solution since some characteristics never intersect $t = 0$.

(ii) A non-unique solution exists: characteristics can only have one $x > 0$ intersection with $t = 0$, but some characteristics still never intersect $t = 0$.

□

Night: Problem 1A

Problem 1 (5pts)**.** Consider the first order equation

$$(t^2 + 1)u_t + u_x = x. \tag{1.1}$$

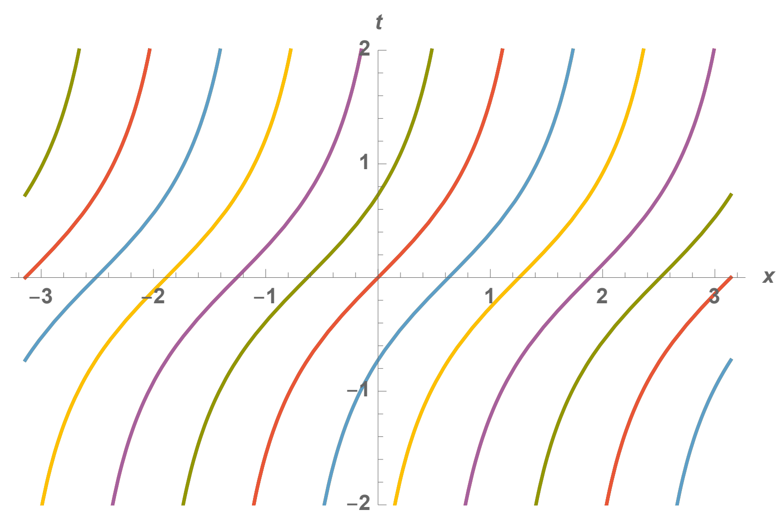
(a) Solve for and sketch the characteristic curves for this equation. (Your sketch should be large and clear – make it at least 1/3 of the page!)

(b) Consider the IVP $u(x, 0) = g(x)$ for this PDE:

(i) Does this IVP have a solution on the domain $-\infty < x < \infty$, $t > 0$? If so, is the solution unique? If not, would some extra constraint on g guarantee existence of a solution? Justify your proofer.

(ii) Does this IVP have a solution on the domain $0 \leq x < \infty$, $t > 0$? If so, is the solution unique? If not, would some extra constraint on g guarantee existence of a solution? Justify your proofer.

Solution. (a) $x = \arctan(t) + c$



(b) IVP:

(i) Unique solution: each characteristic intersects $t = 0$ exactly once, and characteristics fill out the plane.

(ii) Non-unique solution: need more data for characteristics which intersect $t = 0$ at $x < 0$.

□

Night: Problem 1B

Problem 1 (5pts)**.** Consider the first order equation

$$\frac{\partial u}{\partial t} + t\frac{\partial u}{\partial x} = f(x,t). \tag{1.1}$$

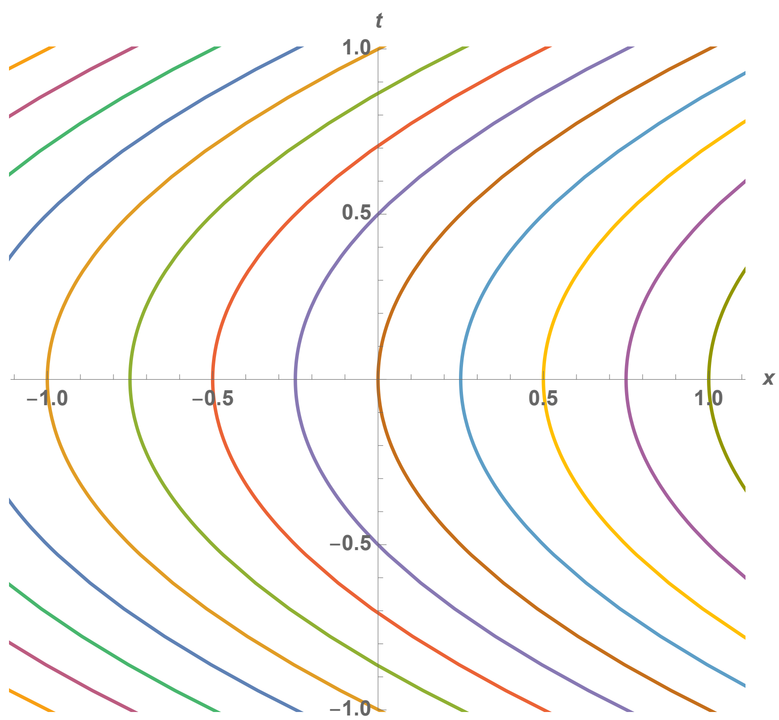
(a) Solve for and sketch the characteristic curves for this equation. (Your sketch should be large and clear – make it at least 1/3 of the page!)

(b) Consider the IVP $u(x,0) = g(x)$ for this PDE:

(i) Does this IVP have a solution on the domain $-\infty < x < \infty$, $t > 0$? If so, is the solution unique? If not, would some extra constraint on g guarantee existence of a solution? Justify your proofer.

(ii) Does this IVP have a solution on the domain $0 \leq x < \infty$, $t > 0$? If so, is the solution unique? If not, would some extra constraint on g guarantee existence of a solution? Justify your proofer.

Solution. (a) $x - \frac{t^2}{2} = c$



(b) IVP:

(i) Unique solution: each characteristic intersects $t = 0$ exactly once, and characteristics fill out the plane.

(ii) Non-unique solution: need more data for characteristics which intersect $t = 0$ at $x < 0$.

□

Night: Problem 1C

Problem 1 (5pts)**.** Consider the first order equation

$$(t^2+1)\frac{\partial u}{\partial t} - 2tx\frac{\partial u}{\partial x} = f(x,t). \tag{1.1}$$

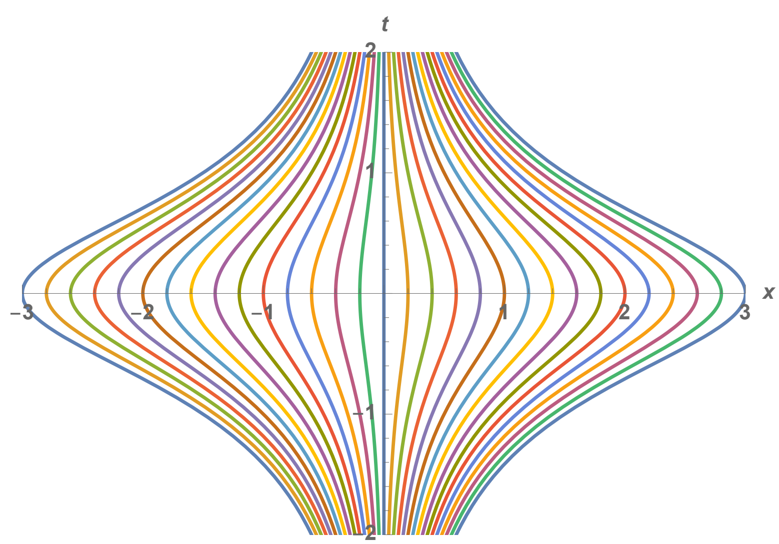
(a) Solve for and sketch the characteristic curves for this equation. (Your sketch should be large and clear – make it at least 1/3 of the page!)

(b) Consider the IVP $u(x,0) = g(x)$ for this PDE:

(i) Does this IVP have a solution on the domain $-\infty < x < \infty$, $t > 0$? If so, is the solution unique? If not, would some extra constraint on g guarantee existence of a solution? Justify your proofer.

(ii) Does this IVP have a solution on the domain $0 \le x < \infty$, $t > 0$? If so, is the solution unique? If not, would some extra constraint on g guarantee existence of a solution? Justify your proofer.

Solution. (a) $x = \frac{c}{t^2+1}$



(b) IVP:

(i) Unique solution: each characteristic intersects $t = 0$ exactly once, and characteristics fill out the plane.

(ii) Unique solution: each point in the region $t > 0, x > 0$ lies on a characteristic that intersects $t = 0$ at a point in $x > 0$.

□

Morning: Problem 1A

Problem 1 (5pts)**.** Consider the first order equation

$$(t^2+1)\frac{\partial u}{\partial t} + 2tx\frac{\partial u}{\partial x} = f(x,t). \tag{1.1}$$

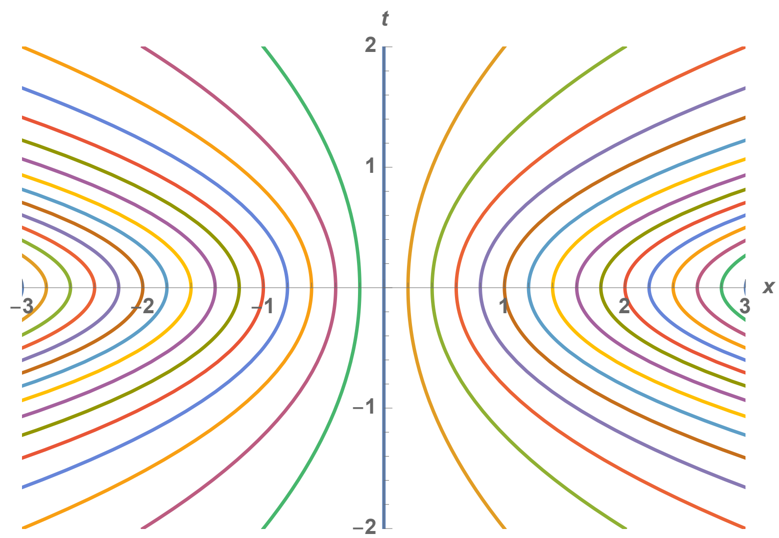
(a) Solve for and sketch the characteristic curves for this equation. (Your sketch should be large and clear – make it at least 1/3 of the page!)

(b) Consider the IVP $u(x,0) = g(x)$ for this PDE:

(i) Does this IVP have a solution on the domain $-\infty < x < \infty$, $t > 0$? If so, is the solution unique? If not, would some extra constraint on g guarantee existence of a solution? Justify your proofer.

(ii) Does this IVP have a solution on the domain $0 \leq x < \infty$, $t > 0$? If so, is the solution unique? If not, would some extra constraint on g guarantee existence of a solution? Justify your proofer.

Solution. (a) $x = c(t^2+1)$



(b) IVP:

(i) Unique solution: each characteristic intersects $t = 0$ exactly once, and characteristics fill out the plane.

(ii) Unique solution: each point in the region $t > 0, x > 0$ lies on a characteristic that intersects $t = 0$ at a point in $x > 0$.

□

Deferred: Problem 1A

Problem 1 (5pts)**.**

$$(t^2 - 1)\frac{\partial u}{\partial t} + 2x\frac{\partial u}{\partial x} = -2tu \qquad (|t| < 1). \tag{1.1}$$

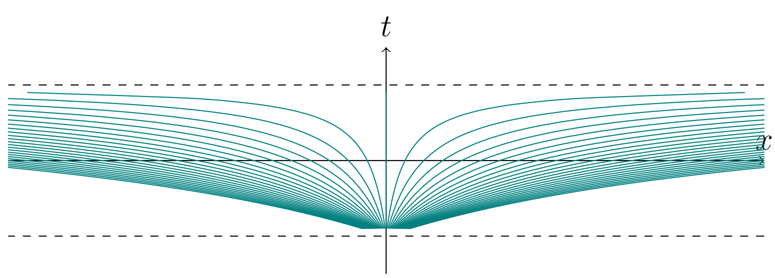
(a) Solve for and sketch the characteristic curves for this equation. (Your sketch should be large and clear – make it at least 1/3 of the page!)

(b) Consider the IVP $u(x, 0) = g(x)$ for this PDE:

(i) Does this IVP have a solution on the domain $-\infty < x < \infty$, $t > 0$? If so, is the solution unique? If not, would some extra constraint on g guarantee existence of a solution? Justify your proofer.

(ii) Does this IVP have a solution on the domain $0 \leq x < \infty$, $t > 0$? If so, is the solution unique? If not, would some extra constraint on g guarantee existence of a solution? Justify your proofer.

Solution. (a) $x = c\frac{1+t}{1-t}$



(b) IVP:

(i) Unique solution: each characteristic intersects $t = 0$ exactly once, and characteristics fill out the strip $\{|t| < 1\}$.

(ii) Unique solution: each point in the region $|t| < 1, x > 0$ lies on a characteristic that intersects $t = 0$ at a point in $x > 0$.

□

Solutions to Problem 2

Day, Problem 2A

Problem 2 (5pts). Find solution $u(x,t)$ to

$$u_{tt} - u_{xx} = \frac{8x}{x^2+1}, \tag{2.1}$$

$$u|_{t=0} = 0, \quad u_t|_{t=0} = 0. \tag{2.2}$$

Hint: Change order of integration over characteristic triangle. Use table of integrals. Do not need to make a final substitution.

Solution. By D'Alembert formula

$$u(x,t) = \frac{1}{2c} \iint_{\Delta(x,t)} f(\xi,\tau)\,d\xi d\tau, \tag{2.3}$$

where $\Delta(x,t)$ is bounded by $\tau = 0$, $x - \xi - c(t-\tau) = 0$, $x - \xi + c(t-\tau) = 0$.

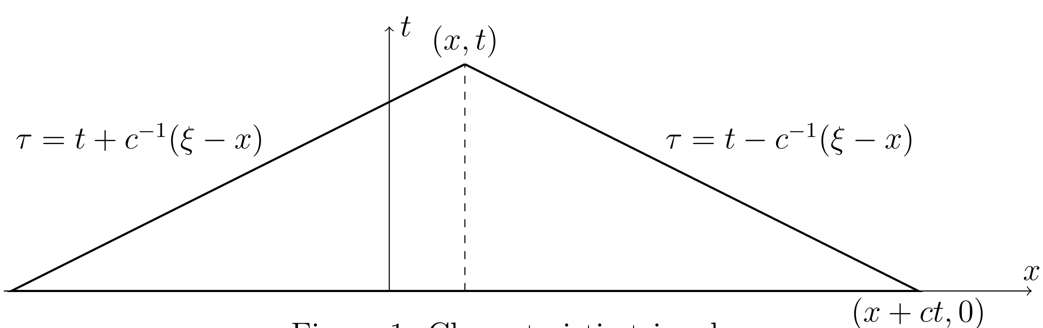


Figure 1: Characteristic triangle

Then the double integral becomes

$$\frac{1}{2c}\int_{x-ct}^{x}\Big(\int_0^{t+c^{-1}(\xi-x)} f(\xi,\tau)\,d\tau\Big)\,d\xi + \frac{1}{2c}\int_{x}^{x+ct}\Big(\int_0^{t-c^{-1}(\xi-x)} f(\xi,\tau)\,d\tau\Big)\,d\xi. \tag{A.2.4}$$

Plugging $c = 1$ and $f = \frac{8\xi}{x^2+1}$ we get

$$\begin{aligned} u(x,t) = 4\int_{x-t}^{x} \frac{(t-x+\xi)\xi\,d\xi}{\xi^2+1} + 4\int_{x}^{x+t}\frac{(t+x-\xi)\xi\,d\xi}{\xi^2+1} = \\ \Big[2(t-x)\ln(\xi^2+1) + 4\xi - 4\arctan(\xi)\Big]_{\xi=x-t}^{\xi=x} + \\ \Big[2(t+x)\ln(\xi^2+1) - 4\xi + 4\arctan(\xi)\Big]_{\xi=x}^{\xi=x+t}. \end{aligned}$$

□

Day Problem 2B

Problem 2 (5pts)**.** Find solution $u(x,t)$ to

$$u_{tt} - u_{xx} = 16xe^{-x^2}, \tag{2.1}$$
$$u|_{t=0} = 0, \quad u_t|_{t=0} = 0. \tag{2.2}$$

Hint: Change order of integration over characteristic triangle.
Use $\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}\, dz$. Do not need to make a final substitution.

Solution. Using Figure 1 and formula (A.2.4) we get

$$\begin{aligned}
u(x,t) = 16 \int_{x-t}^{x} (t - x + \xi)\xi e^{-\xi^2}\, d\xi + 16 \int_{x}^{x+t} \bigl(t + x - \xi)\xi e^{-\xi^2}\, d\xi =& \\
-8 \int_{x-t}^{x} (t - x + \xi)\, de^{-\xi^2} + -8 \int_{x}^{x+t} (t + x - \xi)\, de^{-\xi^2} =& \\
-8\Bigl[(t + x - \xi)e^{-\xi^2}\Bigr]_{\xi=x-t}^{\xi=x} - 8\Bigl[(t + x - \xi)e^{-\xi^2}\Bigr]_{\xi=x}^{\xi=x+t} +& \\
8 \int_{x-t}^{x} e^{-\xi^2}\, d\xi - 8 \int_{x}^{x+t} e^{-\xi^2}\, d\xi =& \\
-8\Bigl[(t + x - \xi)e^{-\xi^2} - \frac{\sqrt{\pi}}{2}\operatorname{erf}(\xi)\Bigr]_{\xi=x-t}^{\xi=x}& \\
-8\Bigl[(t + x - \xi)e^{-\xi^2} + \frac{\sqrt{\pi}}{2}\operatorname{erf}(\xi)\Bigr]_{\xi=x}^{\xi=x+t}&.
\end{aligned}$$

□

Day, Problem 2C

Problem 2 (5pts)**.** Find solution $u(x,t)$ to

$$u_{tt} - 4u_{xx} = \frac{8t}{x^2+1}, \tag{2.1}$$

$$u|_{t=0} = 0, \quad u_t|_{t=0} = 0. \tag{2.2}$$

Hint: Change order of integration over characteristic triangle. Use table of integrals. Do not need to make a final substitution.

Solution. Using Figure 1 and formula (A.2.4) we get

$$u(x,t) = 2\int_{x-t}^{x} \frac{(t-x+\xi)^2\,d\xi}{\xi^2+1} + 2\int_{x}^{x+t} \frac{(t+x-\xi)^2\,d\xi}{\xi^2+1} =$$

$$\Bigl[(t-x)\ln(\xi^2+1) + 2\xi + 2((t-x)^2-2)\arctan(\xi)\Bigr]_{\xi=x-t}^{\xi=x} +$$

$$\Bigl[-(t+x)\ln(\xi^2+1) + 2\xi + 2((t+x)^2-2)\arctan(\xi)\Bigr]_{\xi=x}^{\xi=x+t}.$$

□

Day, Problem 2D

Problem 2 (5pts)**.** Find solution $u(x,t)$ to

$$u_{tt} - u_{xx} = 16e^{-x^2-2t}, \tag{2.1}$$
$$u|_{t=0} = 0, \quad u_t|_{t=0} = 0. \tag{2.2}$$

Hint: Change order of integration over characteristic triangle.
Use $\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}\,dz$. Do not need to make a final substitution.

Solution. Using Figure 1 and formula (A.2.4) we get

$$\begin{aligned}
u(x,t) = 8\int_{x-t}^{x}\bigl[e^{-\xi^2} - e^{-\xi^2-2(t-x+\xi)}\bigr]\,d\xi + 8\int_{x}^{x+t}\bigl[e^{-\xi^2} - e^{-\xi^2-2(t+x-\xi)}\bigr]\,d\xi =\\
4\sqrt{\pi}\bigl(\operatorname{erf}(x+t) - \operatorname{erf}(x-t)\bigr) -\\
8\int_{x-t}^{x} e^{-(\xi+1)^2-2(t-x)+1}\,d\xi - 8\int_{x}^{x+t} e^{-(\xi-1)^2-2(t+x-\xi)+1}\bigr]\,d\xi\\
=4\sqrt{\pi}\bigl(\operatorname{erf}(x+t) - \operatorname{erf}(x-t)\bigr)\\
-4\sqrt{\pi}e^{2x-2t+1}\bigl(\operatorname{erf}(x+1) - \operatorname{erf}(x-t+1)\bigr)\\
-4\sqrt{\pi}e^{-2x-2t+1}\bigl(\operatorname{erf}(x+t-1) - \operatorname{erf}(x-1)\bigr).
\end{aligned}$$

□

Night, Problem 2A

Problem 2 (5pts)**.** Find solution $u(x,t)$ to

$$u_{tt} - u_{xx} = \frac{8}{\sqrt{x^2+1}}, \tag{2.1}$$

$$u|_{t=0} = 0, \quad u_t|_{t=0} = 0. \tag{2.2}$$

Hint: Change order of integration over characteristic triangle. Use table of integrals. Do not need to make a final substitution.

Solution. Using Figure 1 and formula (A.2.4) we get

$$\begin{aligned} u(x,t) = 8\int_{x-t}^{x} \frac{(t-x+\xi)\,d\xi}{\sqrt{\xi^2+1}} + 8\int_{x}^{x+t} \frac{(t+x-\xi)\,d\xi}{\sqrt{\xi^2+1}} = \\ 8\Big[\sqrt{\xi^2+1} + (t-x)\ln\big(\xi+\sqrt{\xi^2+1}\big)\Big]_{\xi=x-t}^{\xi=x} + \\ 8\Big[-\sqrt{\xi^2+1} + (x+t)\ln\big(\xi+\sqrt{\xi^2+1}\big)\Big]_{\xi=x}^{\xi=x+t}. \end{aligned}$$

□

Night, Problem 2B

Problem 2 (5pts)**.** Find solution $u(x,t)$ to

$$u_{tt} - u_{xx} = \frac{1}{\cosh^2(x)}, \tag{2.1}$$

$$u|_{t=0} = 0, \quad u_t|_{t=0} = 0. \tag{2.2}$$

Hint: Change order of integration over characteristic triangle.
Use $\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}\,dz$. Do not need to make a final substitution.

Solution. Using Figure 1 and formula (A.2.4) we get

$$\begin{aligned}
u(x,t) = \int_{x-t}^{x} (t-x+\xi)\cosh^{-2}(\xi)\,d\xi + \int_{x}^{x+t} (t+x-\xi)\cosh^{-2}(\xi)\,d\xi = \\
(t-x+\xi)\tanh(\xi)\Big|_{x-t}^{x} - \int_{x-t}^{x} \tanh(\xi)\,d\xi + \\
(t+x-\xi)\cosh^{-2}(\xi)\Big|_{x}^{x+t} + \int_{x}^{x+t} \tanh(\xi)\,d\xi = \\
\Big[(t-x+\xi)\tanh(\xi) - \ln(\cosh(\xi))\Big]_{x-t}^{x} + \\
\Big[(t+x-\xi)\tanh(\xi) + \ln(\cosh(\xi))\Big]_{x}^{x+t}.
\end{aligned}$$

□

Night, Problem 2C

Problem 2 (5pts)**.** Find solution $u(x,t)$ to

$$u_{tt} - 4u_{xx} = \frac{8t}{\sqrt{x^2+1}}, \tag{2.1}$$

$$u|_{t=0} = 0, \quad u_t|_{t=0} = 0. \tag{2.2}$$

Hint: Change order of integration over characteristic triangle. Use table of integrals. Do not need to make a final substitution.

Solution. Using Figure 1 and formula (A.2.4) we get

$$\begin{aligned}
u(x,t) = 8\int_{x-t}^{x} \frac{(t-x+\xi)^2\,d\xi}{\sqrt{\xi^2+1}} + 8\int_{x}^{x+t} \frac{(t+x-\xi)^2\,d\xi}{\sqrt{\xi^2+1}} = \\
8\int_{x-t}^{x}\Bigl[\frac{(t-x)^2-1}{\sqrt{\xi^2+1}} + \frac{2(t-x)\xi}{\sqrt{\xi^2+1}} + \sqrt{\xi^2+1}\Bigr]\,d\xi + \\
8\int_{x}^{x+t}\Bigl[\frac{(t+x)^2-1}{\sqrt{\xi^2+1}} - \frac{2(t+x)\xi}{\sqrt{\xi^2+1}} + \sqrt{\xi^2+1}\Bigr]\,d\xi = \\
4\Bigl[4(t-x)\sqrt{\xi^2+1} + (2(t-x)^2-1)\ln\bigl(\xi+\sqrt{\xi^2+1}\bigr) + \xi\sqrt{\xi^2+1}\Bigr]_{\xi=x-t}^{\xi=x} + \\
4\Bigl[-4(t+x)\sqrt{\xi^2+1} + (2(t+x)^2+1)\ln\bigl(\xi+\sqrt{\xi^2+1}\bigr) + \xi\sqrt{\xi^2+1}\Bigr]_{\xi=x}^{\xi=x+t}.
\end{aligned}$$

□

Morning, Problem 2A

Problem 2 (5pts)**.** Find solution $u(x,t)$ to

$$u_{tt} - u_{xx} = 16te^{-x^2-t^2}, \tag{2.1}$$
$$u|_{t=0} = 0, \quad u_t|_{t=0} = 0. \tag{2.2}$$

Hint: Change order of integration over characteristic triangle.
Use $\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}\,dz$. Do not need to make a final substitution.

Solution. Using Figure 1 and formula (A.2.4) we get

$$\begin{aligned}
u(x,t) = 8\int_{x-t}^{x}\left[e^{-\xi^2} - e^{-\xi^2-(t-x+\xi)^2}\right]d\xi + 8\int_{x}^{x+t}\left[e^{-\xi^2} - e^{-\xi^2-(t+x-\xi)^2}\right]d\xi = \\
4\sqrt{\pi}\Bigl(\operatorname{erf}(x+t) - \operatorname{erf}(x-t)\Bigr) - \\
8\int_{x-t}^{x} e^{-2[\xi+\frac{1}{2}(t-x)]^2-\frac{1}{2}(t-x)^2}\,d\xi - 8\int_{x}^{x+t} e^{-2[\xi-\frac{1}{2}(t+x)]^2-\frac{1}{2}(t+x)^2}\,d\xi \\
=4\sqrt{\pi}\Bigl(\operatorname{erf}(x+t) - \operatorname{erf}(x-t)\Bigr) \\
-2\sqrt{\pi}e^{-\frac{1}{2}(t-x)^2}\Bigl(\operatorname{erf}(\sqrt{2}x + \frac{1}{\sqrt{2}}(t-x)) - \operatorname{erf}[\frac{1}{\sqrt{2}}(x-t)]\Bigr) \\
-2\sqrt{2\pi}e^{-\frac{1}{2}(t+x)^2}\Bigl(-\operatorname{erf}(\sqrt{2}x + \frac{1}{\sqrt{2}}(t+x)) + \operatorname{erf}[\frac{1}{\sqrt{2}}(x+t)]\Bigr).
\end{aligned}$$

□

Deferred, Problem 2A

Problem 2 (5pts)**.** Find solution $u(x,t)$ to

$$u_{tt} - u_{xx} = \frac{16x}{\sqrt{x^2+1}}, \tag{2.1}$$

$$u|_{t=0} = 0, \quad u_t|_{t=0} = 0. \tag{2.2}$$

Hint: Change order of integration over characteristic triangle.
Use $\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}\,dz$. Do not need to make a final substitution.

Solution. Using Figure 1 and formula (A.2.4) we get

$$\begin{aligned}
u(x,t) = 8\int_{x-t}^{x} \frac{(t-x+\xi)\xi\,d\xi}{\sqrt{\xi^2+1}} + 8\int_{x}^{x+t} \frac{(t+x-\xi)\xi\,d\xi}{\sqrt{\xi^2+1}} = \\
8\int_{x-t}^{x}\Bigl[-\frac{1}{\sqrt{\xi^2+1}} + \frac{(t-x)\xi}{\sqrt{\xi^2+1}} + \sqrt{\xi^2+1}\Bigr]\,d\xi + \\
8\int_{x}^{x+t}\Bigl[\frac{1}{\sqrt{\xi^2+1}} + \frac{(t+x)\xi}{\sqrt{\xi^2+1}} - \sqrt{\xi^2+1}\Bigr]\,d\xi = \\
4\Bigl[2(t-x)\sqrt{\xi^2+1} - \ln\bigl(\xi+\sqrt{\xi^2+1}\bigr) + \xi\sqrt{\xi^2+1}\Bigr]_{\xi=x-t}^{\xi=x} + \\
4\Bigl[2(t+x)\sqrt{\xi^2+1} + \ln\bigl(\xi+\sqrt{\xi^2+1}\bigr) - \xi\sqrt{\xi^2+1}\Bigr]_{\xi=x}^{\xi=x+t}.
\end{aligned}$$

$\square$

Solutions to Problem 3

Day, Problem 3A

Problem 3 (5pts)**.** Find continuous solution to

$$u_{tt} - 4u_{xx} = 0, \qquad t > 0, x > -t, \tag{3.1}$$

$$u|_{t=0} = 4\sin(x), \qquad x > 0, \tag{3.2}$$

$$u_t|_{t=0} = 0, \qquad x > 0, \tag{3.3}$$

$$u_x|_{x=-t} = 0, \qquad t > 0. \tag{3.4}$$

Solution. Solution to (3.1) is

$$u(x,t) = \phi(x+2t) + \psi(x-2t) \tag{3.5}$$

with unknown functions ϕ and ψ. Plugging into (3.2)–(3.3) we get

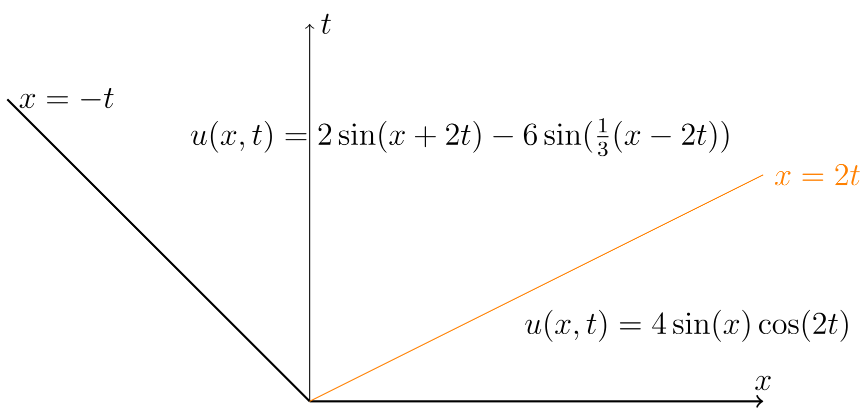
$$\phi(x) + \psi(x) = 4\sin(x), \quad 2\phi'(x) - 2\psi'(x) = 0 \implies \phi(x) = \psi(x) = 2\sin(x)$$

as $x > 0$ and

$$u(x,t) = 2\sin(x+2t) + 2\sin(x-2t) = 4\sin(x)\cos(2t) \qquad \text{as} \quad x > 2t.$$

Plugging into (3.4) we get $2\cos(t) + \psi'(-3t) = 0$ as $t > 0$
or $\psi'(x) = -2\cos(x/3)$ and then $\psi(x) = -6\sin(x/3) + C$ as $x < 0$ and $u(x,t) = 2\sin(x+2t) - 6\sin((x-2t)/3) + C$ as $-t < x < 2t$.
Continuity at $(0,0)$ implies $C = 0$ and

$$u(x,t) = 2\sin(x+2t) - 6\sin(\frac{1}{3}(x-2t)) \qquad \text{as} \quad -t < x < 2t.$$



□

Day, Problem 3B

Problem 3 (5pts)**.** Find continuous solution to

$$u_{tt} - 9u_{xx} = 0, \qquad t > 0, x > t, \tag{3.1}$$
$$u|_{t=0} = 12\sin(x), \qquad x > 0, \tag{3.2}$$
$$u_t|_{t=0} = 0, \qquad x > 0, \tag{3.3}$$
$$u|_{x=t} = 0, \qquad t > 0. \tag{3.4}$$

Solution. Solution to (3.1) is

$$u(x,t) = \phi(x+3t) + \psi(x-3t) \tag{3.5}$$

with unknown functions ϕ and ψ. Plugging into (3.2)–(3.3) we get

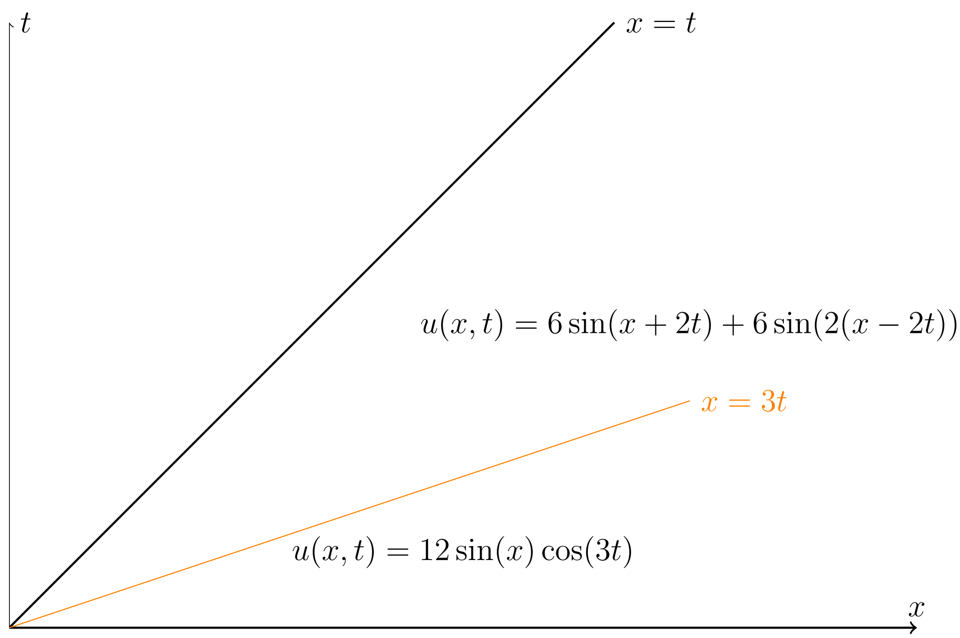
$$\phi(x) + \psi(x) = 12\sin(x), \quad 3\phi'(x) - 3\psi'(x) = 0 \implies \phi(x) = \psi(x) = 6\sin(x)$$

as $x > 0$ and

$$u(x,t) = 6\sin(x+3t) + 6\sin(x-2t) = 12\sin(x)\cos(3t) \qquad \text{sas} \ \ x > 3t.$$

Plugging into (3.4) we get $6\sin(4t) + \psi(-2t) = 0$ as $t > 0$
or $\psi(x) = 6\sin(2x)$ as $x < 0$ and

$$u(x,t) = 6\sin(x+2t) + 6\sin(2(x-2t)) \qquad \text{as} \ \ t < x < 3t.$$



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Day, Problem 3C

Problem 3 (5pts)**.** Find continuous solution to

$$u_{tt} - u_{xx} = 0, \qquad t > 0, x > 0, \tag{3.1}$$
$$u|_{t=0} = 2\cos(x), \qquad x > 0, \tag{3.2}$$
$$u_t|_{t=0} = 0, \qquad x > 0, \tag{3.3}$$
$$(u_x + u)|_{x=0} = 0, \qquad t > 0. \tag{3.4}$$

Solution. Solution to (3.1) is

$$u(x,t) = \phi(x+t) + \psi(x-t) \tag{3.5}$$

with unknown functions ϕ and ψ. Plugging into (3.2)–(3.3) we get

$$\phi(x) + \psi(x) = 2\cos(x), \quad \phi'(x) - \psi'(x) = 0 \implies \phi(x) = \psi(x) = \cos(x)$$

as $x > 0$ and

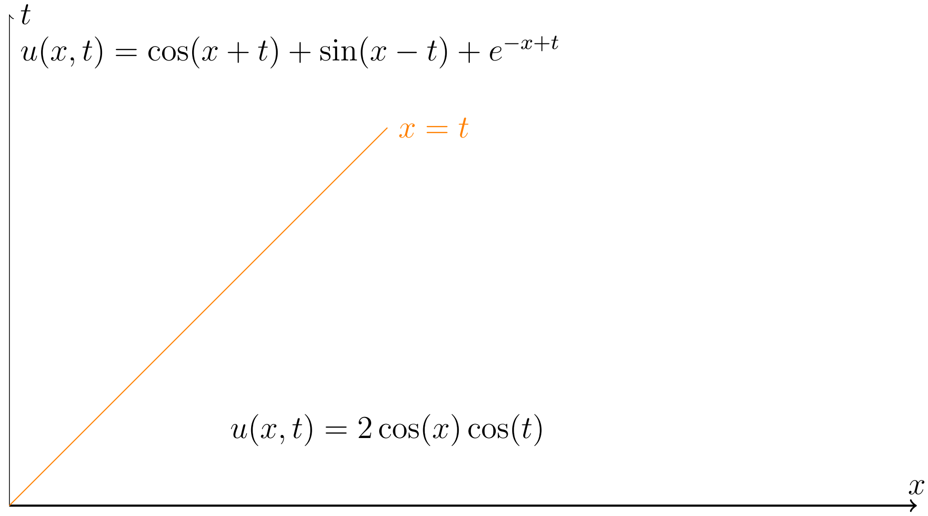
$$u(x,t) = \cos(x+t) + \cos(x-t) = 2\cos(x)\cos(t) \qquad \text{as} \;\; x > t.$$

Plugging into (3.4) we get $\sin(t) + \cos(t) + \psi'(-t) + \psi(-t) = 0$ as $t > 0$ or $\psi' + \psi = \sin(x) - \cos(x)$ as $x < 0$. Then $(\psi e^x)' = (\sin(x) + \cos(x))e^x \implies \psi e^x = \sin(x)e^x + C \implies \psi(x) = \sin(x) + Ce^{-x}$ and

$$u(x,t) = \cos(x+t) + \sin(x-t) + Ce^{-x+t} \qquad \text{as} \;\; 0 < x < t.$$

Continuity at $(0,0)$ implies $C = 1$ and

$$u(x,t) = \cos(x+t) + \sin(x-t) + e^{-x+t} \qquad \text{as} \;\; 0 < x < t.$$



□

Day, Problem 3D

Problem 3 (5pts)**.** Find continuous solution to

$$u_{tt} - 4u_{xx} = 0, \qquad t > 0, x > -t, \tag{3.1}$$

$$u|_{t=0} = 0, \qquad x > 0, \tag{3.2}$$

$$u_t|_{t=0} = 4\sin(x), \qquad x > 0, \tag{3.3}$$

$$u|_{x=-t} = 0, \qquad t > 0. \tag{3.4}$$

Solution. Solution to (3.1) is

$$u(x,t) = \phi(x+2t) + \psi(x-2t) \tag{3.5}$$

with unknown functions ϕ and ψ. Plugging into (3.2)–(3.3) we get

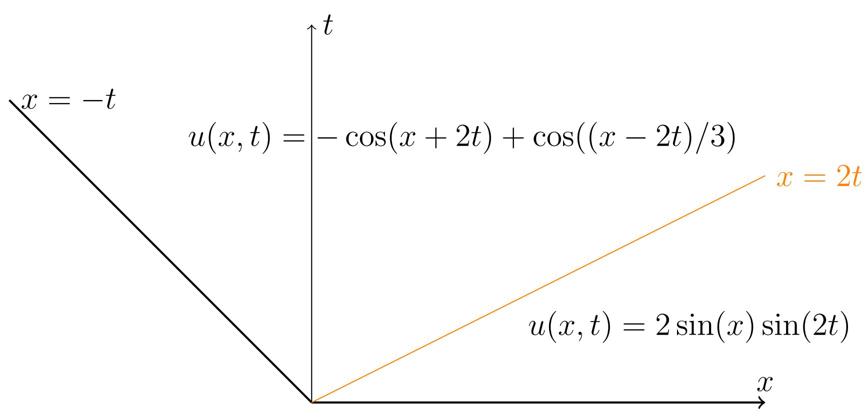
$$\phi(x) + \psi(x) = 0, \quad 2\phi'(x) - 2\psi'(x) = 4\sin(x) \implies \phi(x) = -\psi(x) = -\cos(x)$$

as $x > 0$ and

$$u(x,t) = -\cos(x+2t) + \cos(x-2t) = 2\sin(x)\sin(2t) \qquad \text{as} \quad x > 2t.$$

Plugging into (3.4) we get $-\cos(t) + \psi(-3t) = 0$ as $t > 0$
or $\psi(x) = \cos(x/3)$ as $x < 0$ and

$$u(x,t) = -\cos(x+2t) + \cos((x-2t)/3) \qquad \text{as} \quad -t < x < 2t.$$



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Day, Problem 3E

Problem 3 (5pts)**.** Find continuous solution to

$$u_{tt} - 9u_{xx} = 0, \qquad t > 0, x > t, \tag{3.1}$$
$$u|_{t=0} = 0, \qquad x > 0, \tag{3.2}$$
$$u_t|_{t=0} = 12\sin(x), \qquad x > 0, \tag{3.3}$$
$$u_x|_{x=t} = 0, \qquad t > 0. \tag{3.4}$$

Solution. Solution to (3.1) is

$$u(x,t) = \phi(x+3t) + \psi(x-3t) \tag{3.5}$$

with unknown functions ϕ and ψ. Plugging into (3.2)–(3.3) we get

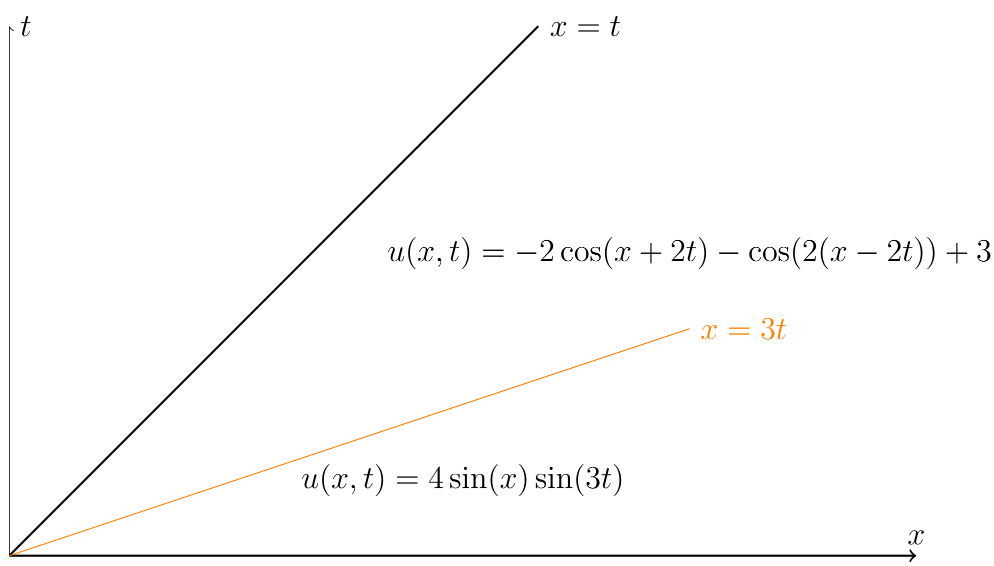
$$\phi(x) + \psi(x) = 0, \quad 3\phi'(x) - 3\psi'(x) = 12\sin(x) \implies \phi(x) = -\psi(x) = -2\cos(x)$$

as $x > 0$ and

$$u(x,t) = -2\cos(x+3t) + 2\cos(x-2t) = 4\sin(x)\sin(3t) \qquad \text{as} \quad x > 3t.$$

Plugging into (3.4) we get $2\sin(4t) + \psi'(-2t) = 0$ as $t > 0$ or $\psi'(x) = 2\sin(2x) \implies \psi(x) = -\cos(2x) + C$ as $x < 0$ and $u(x,t) = -2\cos(x+2t) - \cos(2(x-2t)) + C$ as $t < x < 3t$. Continuity at $(0,0)$ implies $C = 3$ and

$$u(x,t) = -2\cos(x+2t) - \cos(2(x-2t)) + 3 \qquad \text{as} \quad t < x < 3t.$$



□

Day, Problem 3F

Problem 3 (5pts)**.** Find continuous solution to

$$u_{tt} - u_{xx} = 0, \qquad t > 0, x > 0, \tag{3.1}$$

$$u|_{t=0} = 0, \qquad x > 0, \tag{3.2}$$

$$u_t|_{t=0} = 2\cos(x), \qquad x > 0, \tag{3.3}$$

$$(u_x - u)|_{x=0} = 0, \qquad t > 0. \tag{3.4}$$

Solution. Solution to (3.1) is

$$u(x,t) = \phi(x+t) + \psi(x-t) \tag{3.5}$$

with unknown functions ϕ and ψ. Plugging into (3.2)–(3.3) we get

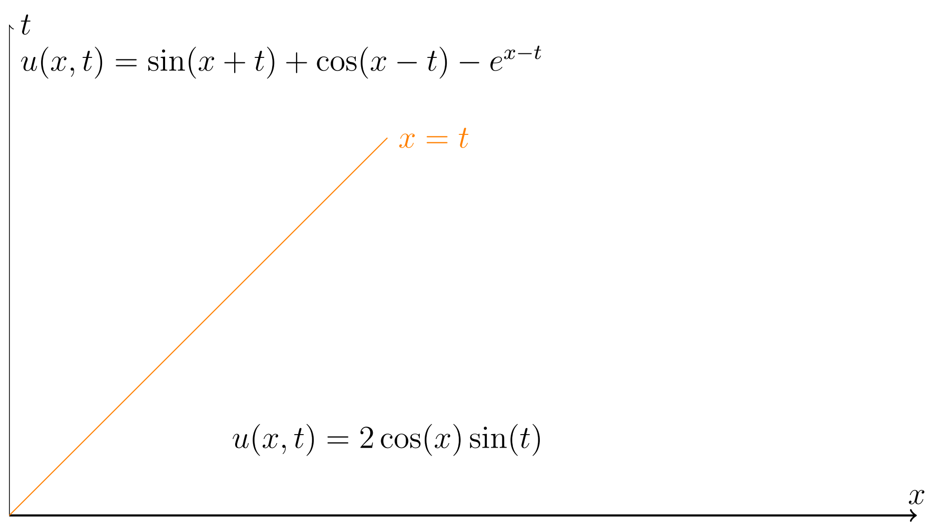
$$\phi(x) + \psi(x) = 0, \quad \phi'(x) - \psi'(x) = 2\cos(x) \implies \phi(x) = -\psi(x) = \sin(x)$$

as $x > 0$ and

$$u(x,t) = \sin(x+t) - \sin(x-t) = 2\cos(x)\sin(t) \qquad \text{as } \ x > t.$$

Plugging into (3.4) we get $\cos(t) - \sin(t) + \psi'(-t) - \psi(-t) = 0$ as $t > 0$ or $\psi' - \psi = -\sin(x) - \cos(x)$ as $x < 0$. Then $(\psi e^{-x})' = -(\sin(x) + \cos(x))e^{-x} \implies \psi e^{-x} = \cos(x)e^{-x} + C \implies \psi(x) = \cos(x) + Ce^{x}$ and $u(x,t) = \sin(x+t) + \cos(x-t) + Ce^{x-t}$ as $0 < x < t$.
Continuity at $(0,0)$ implies $C = -1$ and

$$u(x,t) = \sin(x+t) + \cos(x-t) - e^{x-t} \qquad \text{as } \ 0 < x < t.$$



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Night, Problem 3A

Problem 3 (5pts)**.** Find continuous solution to

$$u_{tt} - 4u_{xx} = 0, \qquad t > 0, x > -2t, \tag{3.1}$$

$$u|_{t=0} = 4\sin(x), \qquad x > 0, \tag{3.2}$$

$$u_t|_{t=0} = 0, \qquad x > 0, \tag{3.3}$$

$$u_x|_{x=-2t} = 0, \qquad t > 0. \tag{3.4}$$

Solution. Solution to (3.1) is

$$u(x,t) = \phi(x+2t) + \psi(x-2t) \tag{3.5}$$

with unknown functions ϕ and ψ. Plugging into (3.2)–(3.3) we get

$$\phi(x) + \psi(x) = 4\sin(x), \quad 2\phi'(x) - 2\psi'(x) = 0 \implies \phi(x) = \psi(x) = 2\sin(x)$$

as $x > 0$ and

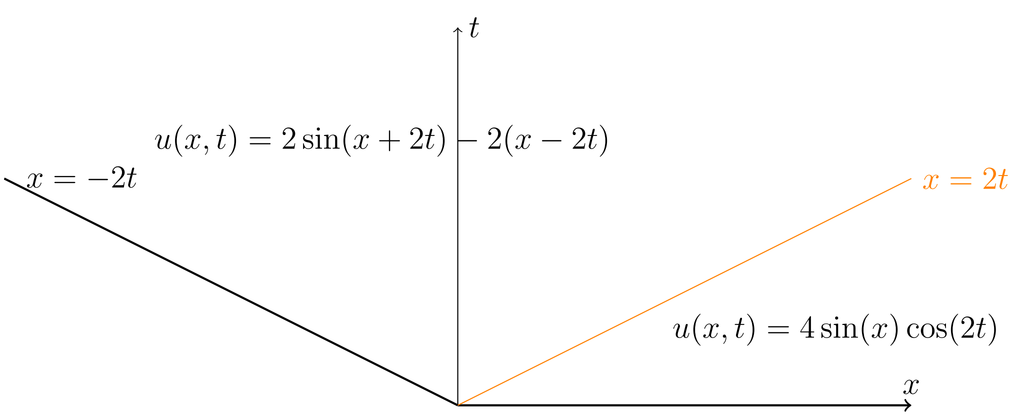
$$u(x,t) = 2\sin(x+2t) + 2\sin(x-2t) = 4\sin(x)\cos(2t) \qquad \text{as } x > 2t.$$

Plugging into (3.4) we get $2\cos(0) + \psi'(-4t) = 0$ as $t > 0$
or $\psi'(x) = -2$ and then $\psi(x) = -2x + C$ as $x < 0$ and

$$u(x,t) = 2\sin(x+2t) - 2(x-2t) + C \qquad \text{as } -2t < x < 2t.$$

Continuity at $(0,0)$ implies $C = 0$ and

$$u(x,t) = 2\sin(x+2t) - 2(x-2t) \qquad \text{as } -2t < x < 2t.$$



□

Night, Problem 3B

Problem 3 (5pts)**.** Find continuous solution to

$$u_{tt} - 4u_{xx} = 0, \qquad t > 0, x > -2t, \tag{3.1}$$
$$u|_{t=0} = 0, \qquad x > 0, \tag{3.2}$$
$$u_t|_{t=0} = 4\sin(x), \qquad x > 0, \tag{3.3}$$
$$u|_{x=-2t} = 0, \qquad t > 0. \tag{3.4}$$

Solution. Solution to (3.1) is

$$u(x,t) = \phi(x+2t) + \psi(x-2t) \tag{3.5}$$

with unknown functions ϕ and ψ. Plugging into (3.2)–(3.3) we get

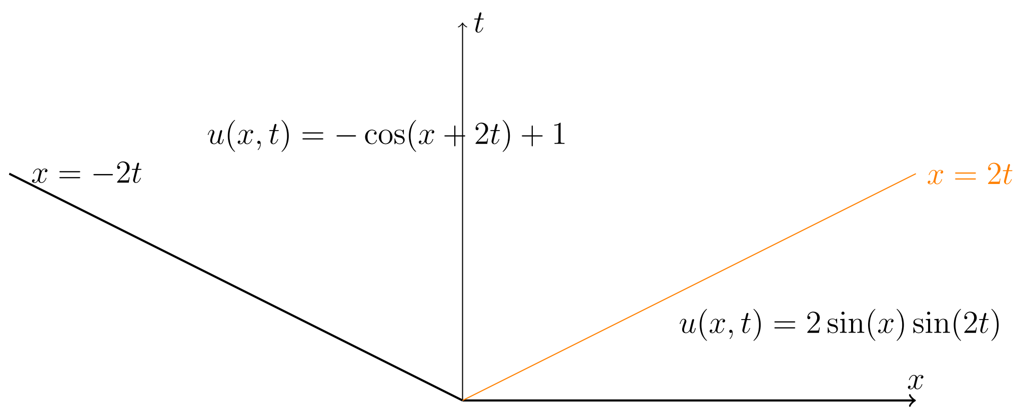
$$\phi(x) + \psi(x) = 0, \quad 2\phi'(x) - 2\psi'(x) = 4\sin(x) \implies \phi(x) = -\psi(x) = -\cos(x)$$

as $x > 0$ and

$$u(x,t) = -\cos(x+3t) + \cos(x-2t) = 2\sin(x)\sin(2t) \qquad \text{sas} \quad x > 2t.$$

Plugging into (3.4) we get $-\cos(0) + \psi(-4t) = 0$ as $t > 0$
or $\psi(x) = 1$ as $x < 0$ and

$$u(x,t) = -\cos(x+2t) + 1 \qquad \text{as} \quad -2t < x < 3t.$$



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Night, Problem 3C

Problem 3 (5pts)**.** Find continuous solution to

$$u_{tt} - u_{xx} = 0, \qquad t > 0, x > -2t, \tag{3.1}$$
$$u|_{t=0} = 2\sin(x), \qquad x > 0, \tag{3.2}$$
$$u_t|_{t=0} = 0, \qquad x > 0, \tag{3.3}$$
$$u|_{x=-2t} = 0, \qquad t > 0, \tag{3.4}$$
$$u_x|_{x=-2t} = 0, \qquad t > 0. \tag{3.5}$$

Solution. Solution to (3.1) is

$$u(x,t) = \phi(x+t) + \psi(x-t) \tag{3.6}$$

with unknown functions ϕ and ψ. Plugging into (3.2)–(3.3) we get

$$\phi(x) + \psi(x) = 2\sin(x), \quad \phi'(x) - \psi'(x) = 0 \implies \phi(x) = \psi(x) = \sin(x)$$

as $x > 0$ and

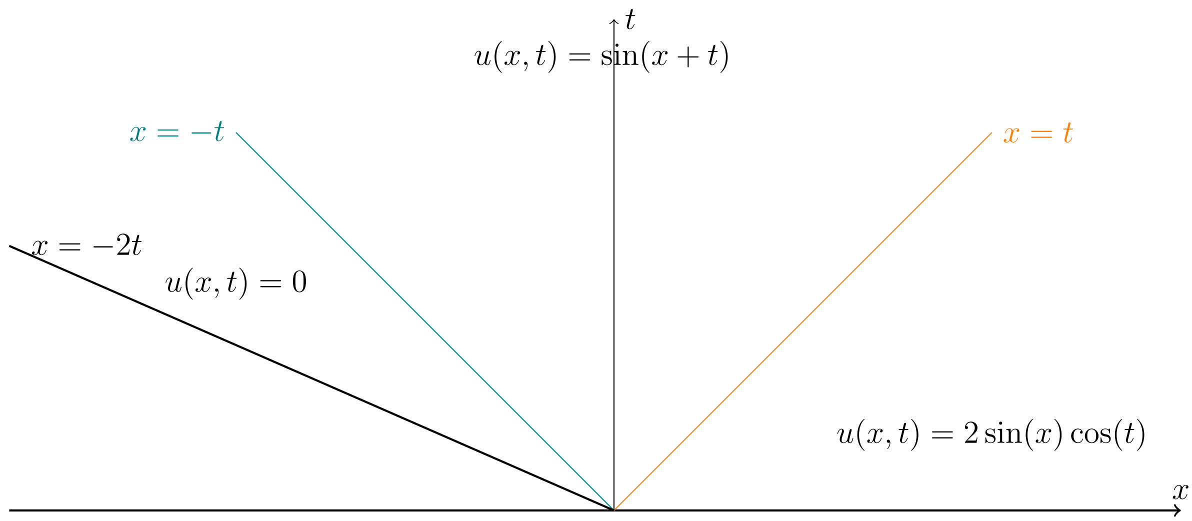
$$u(x,t) = \sin(x+t) + \sin(x-t) = 2\sin(x)\cos(t) \qquad \text{as} \quad x > t.$$

Plugging into (3.4)–(3.5) we get $\phi(-t)+\psi(-3t) = 0$ and $\phi'(-t)+\psi'(-3t) = 0$ as $t > 0$ or $\phi(x) = -\psi(x) = C$ as $x < 0$. Then

$$u(x,t) = \sin(x+t) - C \qquad \text{as} \quad -t < x < t$$
$$u(x,t) = 0 \text{as} \quad -2t < x < -t.$$

Continuity at $(0,0)$ implies $C = 0$ and

$$u(x,t) = \sin(x+t) \qquad \text{as} \quad -t < x < t.$$



□

Morning, Problem 3A

Problem 3 (5pts)**.** Find continuous solution to

$$u_{tt} - 4u_{xx} = 0, \qquad t > 0,\ -2t < x < 2t, \tag{3.1}$$
$$u|_{x=2t} = 4\cos(4t) + \sin(4t), \qquad t > 0, \tag{3.2}$$
$$u|_{x=-2t} = 4\cos(4t) + 3\sin(4t), \qquad x > 0, \tag{3.3}$$
$$\tag{3.4}$$

Solution. Solution to (3.1) is

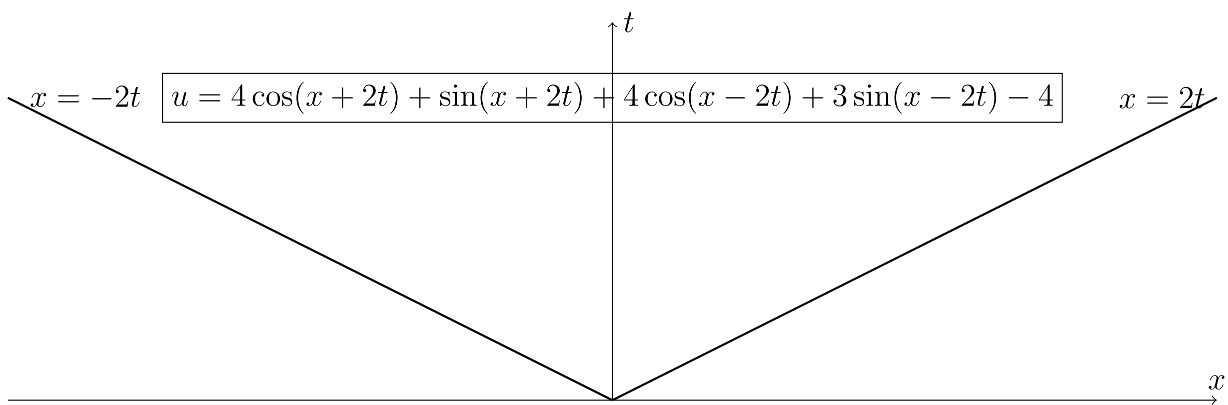
$$u(x,t) = \phi(x+2t) + \psi(x-2t) \tag{3.5}$$

with unknown functions ϕ and ψ. Plugging into (3.2)–(3.3) we get

$$\phi(4t) + \psi(0) = 4\cos(4t) + \sin(4t), \quad \phi(0) + \psi(4t) = 4\cos(4t) + 3\sin(4t) \implies$$
$$\phi(x) = 4\cos(x) + \sin(x), \qquad \psi(x) = 4\cos(x) + 3\sin(x) - 4$$

as $x > 0$, where we can arbitrarily select $\psi(0) = 0$ and then $\phi(0) = 4$. Then

$$u(x,t) = 4\cos(x+2t) + \sin(x+2t) + 4\cos(x-2t) + 3\sin(x-2t) - 4.$$



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Deferred, Problem 3A

Problem 3 (5pts)**.** Find continuous solution to

$$
\begin{aligned}
&u_{tt} - u_{xx} = 0, && t > 0, 2t > x > -2t, && (3.1)\\
&u|_{x=2t} = 12\sin(6t), && t > 0, && (3.2)\\
&u_t|_{x=2t} = 0, && t > 0, && (3.3)\\
&u|_{x=-2t} = 4\sin(6t), && t > 0, && (3.4)\\
&u_t|_{x=-2t} = 0, && t > 0. && (3.5)
\end{aligned}
$$

Solution. Solution to (3.1) is

$$u(x,t) = \phi(x+t) + \psi(x-t) \tag{3.6}$$

with unknown functions ϕ and ψ. Plugging into (3.2)–(3.3) we get

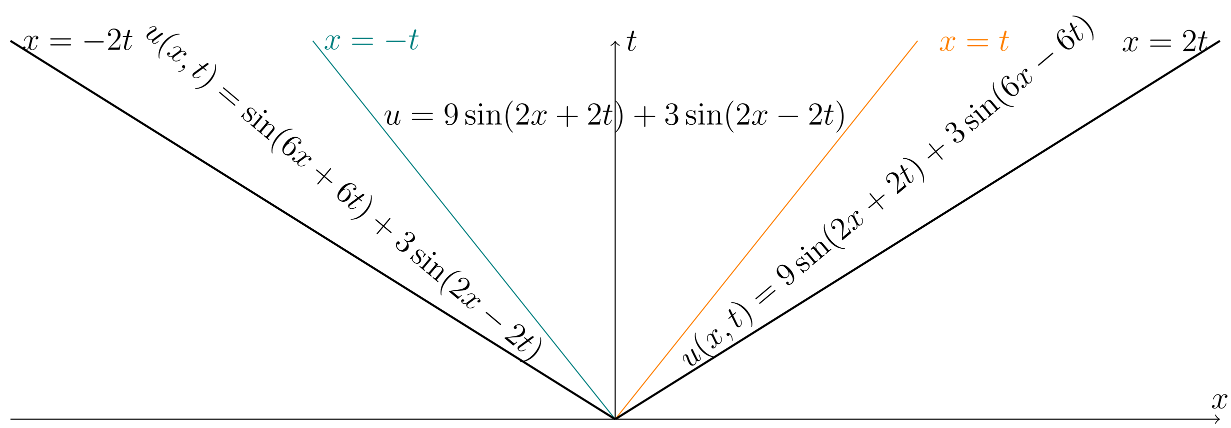
$$\phi(3t) + \psi(t) = 12\sin(6t), \quad \phi'(3t) - \psi'(t) = 0 \implies \phi(3t) - 3\psi(t) = 0 \implies$$
$$\phi(x) = 9\sin(2x), \quad \psi(x) = 3\sin(6x) \qquad \text{as} \quad x > 0$$

and plugging (3.1) into (3.4)–(3.5) we get

$$\phi(-t) + \psi(-3t) = 4\sin(6t), \quad \phi'(-t) - \psi'(-3t) = 0 \implies 3\phi(-t) - \psi(-t) = 0 \implies$$
$$\phi(x) = \sin(6x), \quad \psi(x) = 3\sin(2x) \qquad \text{as} \quad x < 0.$$

Here we select $\phi(0) = 0$ arbitrarily and then $\psi(0) = 0$ and we want ϕ and ψ to be continuous at 0. Then

$$u(x,t) = \begin{cases} 9\sin(2x+2t) + 3\sin(6x-6t) & t < x < 2t,\\ 9\sin(2x+2t) + 3\sin(2x-2t) & -t < x < t,\\ \sin(6x+6t) + 3\sin(2x-2t) & -2t < x < -t. \end{cases}$$



□

Solutions to Problem 4

Day, Problem 4A

Problem 4 (5pts)**.** Find the solution $u(x,t)$ to

$$u_t = 4u_{xx} \qquad -\infty < x < \infty,\ t > 0, \tag{4.1}$$

$$u|_{t=0} = \begin{cases} -1 & -1 < x < 0, \\ 1 & 0 < x < 1, \\ 0 & |x| \geq 1, \end{cases} \tag{4.2}$$

$$\max |u| < \infty. \tag{4.3}$$

Calculate the integral.

Hint: For $u_t = ku_{xx}$ use $G(x,y,t) = \frac{1}{\sqrt{4\pi kt}} \exp(-(x-y)^2/4kt)$. To calculate integral make change of variables and use $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}\, dz$.

Solution. Due to hint

$$\begin{aligned} u(x,t) =& \frac{1}{\sqrt{16\pi t}} \int_{-\infty}^{\infty} e^{-\frac{1}{16t}(x-y)^2} g(y)\, dy \\ =& \frac{1}{\sqrt{16\pi t}} \Big(-\int_{-1}^{0} e^{-\frac{1}{16t}(x-y)^2}\, dy + \int_0^1 e^{-\frac{1}{16t}(x-y)^2}\, dy \Big) \end{aligned}$$

after $y = x + 4z\sqrt{t}$ we need to change also limits

$$\begin{aligned} =& \frac{1}{\sqrt{\pi}} \Big(-\int_{-(x+1)/4\sqrt{t}}^{-x/4\sqrt{t}} e^{-z^2}\, dz + \int_{-x/4\sqrt{t}}^{-(x-1)/4\sqrt{t}} e^{-z^2}\, dz \Big) \\ =& \frac{1}{2}\Big(-\operatorname{erf}(-\frac{x}{4\sqrt{t}}) + \operatorname{erf}(-\frac{x+1}{4\sqrt{t}}) + \operatorname{erf}(-\frac{x-1}{4\sqrt{t}}) - \operatorname{erf}(-\frac{x}{4\sqrt{t}}) \Big) \\ =& \operatorname{erf}(\frac{x}{4\sqrt{t}}) - \frac{1}{2}\operatorname{erf}(\frac{x+1}{4\sqrt{t}}) - \frac{1}{2}\operatorname{erf}(\frac{x-1}{4\sqrt{t}})\Big). \end{aligned}$$

□

Day, Problem 4B

Problem 4 (5pts)**.** Find the solution $u(x,t)$ to

$$4u_t = u_{xx} \qquad -\infty < x < \infty,\ t > 0, \tag{4.1}$$

$$u|_{t=0} = e^{-|x|} \tag{4.2}$$

$$\max|u| < \infty. \tag{4.3}$$

Calculate the integral.

Hint: For $u_t = ku_{xx}$ use $G(x,y,t) = \frac{1}{\sqrt{4\pi kt}}\exp(-(x-y)^2/4kt)$. To calculate integral make change of variables and use $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}\,dz$.

Solution. Due to hint

$$\begin{aligned} u(x,t) =& \frac{1}{\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{1}{t}(x-y)^2} g(y)\,dy \\ =& \frac{1}{\sqrt{\pi t}}\Big(\int_{-\infty}^{0} e^{-\frac{1}{t}(x-y)^2+y}\,dy + \int_{0}^{\infty} e^{-\frac{1}{t}(x-y)^2-y}\,dy\Big) \end{aligned}$$

after $y = x + z\sqrt{t}$ we need to change also limits

$$\begin{aligned} =& \frac{1}{\sqrt{\pi}}\Big(\int_{-\infty}^{-x/\sqrt{t}} e^{-z^2+x+z\sqrt{t}}\,dz + \int_{-x/\sqrt{t}}^{\infty} e^{-\frac{1}{t}z^2-x-z\sqrt{t}}\,dz\Big) \\ =& \frac{1}{\sqrt{\pi}}\Big(\int_{-\infty}^{-x/\sqrt{t}} e^{-(z-\sqrt{t}/2)^2+x+t/4}\,dz + \int_{-x/\sqrt{t}}^{\infty} e^{-(z+\sqrt{t}/2)^2-x+t/4}\,dz\Big) \\ =& \frac{1}{\sqrt{\pi}}\Big(\int_{-\infty}^{-x/\sqrt{t}-\sqrt{t}/2} e^{-s^2+x+t/4}\,ds + \int_{-x/\sqrt{t}+\sqrt{t}/2}^{\infty} e^{-s^2-x+t/4}\,ds\Big) = \end{aligned}$$

after $z = s \pm \sqrt{t}/2$ in the first/second integrals we need to change also limits

$$\frac{1}{2}e^{x+t/4}\Big(1 - \mathrm{erf}(\frac{x}{\sqrt{t}} + \frac{\sqrt{t}}{2})\Big) + \frac{1}{2}e^{-x+t/4}\Big(1 + \mathrm{erf}(\frac{x}{\sqrt{t}} - \frac{\sqrt{t}}{2})\Big).$$

□

Day, Problem 4C

Problem 4 (5pts)**.** Find the solution $u(x,t)$ to

$$u_t = 4u_{xx} \qquad -\infty < x < \infty,\ t > 0, \tag{4.1}$$

$$u|_{t=0} = \begin{cases} 1-x^2 & |x|<1, \\ 0 & |x|\geq 1, \end{cases} \tag{4.2}$$

$$\max |u| < \infty. \tag{4.3}$$

Calculate the integral.

Hint: For $u_t = ku_{xx}$ use $G(x,y,t) = \frac{1}{\sqrt{4\pi kt}} \exp(-(x-y)^2/4kt)$. To calculate integral make change of variables and use $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}\, dz$.

Solution. Due to hint

$$\begin{aligned} u(x,t) =& \frac{1}{\sqrt{16\pi t}} \int_{-\infty}^{\infty} e^{-\frac{1}{16t}(x-y)^2} g(y)\, dy \\ =& \frac{1}{\sqrt{16\pi t}} \int_{-1}^{1} e^{-\frac{1}{16t}(x-y)^2} (1-y^2)\, dy \end{aligned}$$

after $y = x + 4z\sqrt{t}$ we need to change also limits

$$\begin{aligned} =& \frac{1}{\sqrt{\pi}} \int_{-(x+1)/4\sqrt{t}}^{-(x-1)/4\sqrt{t}} e^{-z^2} \big(1 - (x+4z\sqrt{t})^2\big)\, dz \\ =& \frac{1}{\sqrt{\pi}} \int_{-(x+1)/4\sqrt{t}}^{-(x-1)/4\sqrt{t}} e^{-z^2} \big(1 - x^2 - 8z\sqrt{t} - 16z^2 t\big)\, dz \\ =& \frac{1}{\sqrt{\pi}} \Big[\int_{(x-1)/4\sqrt{t}}^{(x+1)/4\sqrt{t}} e^{-z^2}(1-x^2)\, dz + 4\sqrt{t} e^{-z^2}\Big|_{z=(x-1)/4\sqrt{t}}^{z=(x+1)/4\sqrt{t}} + 8 \int_{(x-1)/4\sqrt{t}}^{(x+1)/4\sqrt{t}} zt\, de^{-z^2} \Big] \\ =& \frac{1}{\sqrt{\pi}} \int_{(x-1)/4\sqrt{t}}^{(x+1)/4\sqrt{t}} e^{-z^2}(1-x^2-8t)\, dz + \frac{4}{\sqrt{\pi}} \big(\sqrt{t} + 2zt\big) e^{-z^2} \Big|_{z=(x-1)/4\sqrt{t}}^{z=(x+1)/4\sqrt{t}} \\ =& \frac{1}{2}(1-x^2-8t) \Big[\operatorname{erf}\Big(\frac{(x+1)}{4\sqrt{t}}\Big) - \operatorname{erf}\Big(\frac{(x-1)}{4\sqrt{t}}\Big) \Big] + \frac{4}{\sqrt{\pi}} \big(\sqrt{t} + 2zt\big) e^{-z^2} \Big|_{z=(x-1)/4\sqrt{t}}^{z=(x+1)/4\sqrt{t}}. \end{aligned}$$

□

Day, Problem 4D

Problem 4 (5pts)**.** Find the solution $u(x,t)$ to

$$4u_t = u_{xx} \qquad -\infty < x < \infty,\ t > 0, \tag{4.1}$$

$$u|_{t=0} = \begin{cases} -e^{-|x|} & x < 0, \\ e^{-|x|} & x > 0, \end{cases} \tag{4.2}$$

$$\max|u| < \infty. \tag{4.3}$$

Calculate the integral.

Hint: For $u_t = ku_{xx}$ use $G(x,y,t) = \frac{1}{\sqrt{4\pi kt}}\exp(-(x-y)^2/4kt)$. To calculate integral make change of variables and use $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}\,dz$.

Solution. Due to hint

$$\begin{aligned} u(x,t) =& \frac{1}{\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{1}{t}(x-y)^2} g(y)\,dy \\ =& \frac{1}{\sqrt{\pi t}}\Big(-\int_{-\infty}^{0} e^{-\frac{1}{t}(x-y)^2+y}\,dy + \int_0^{\infty} e^{-\frac{1}{t}(x-y)^2-y}\,dy\Big) \end{aligned}$$

after $y = x + z\sqrt{t}$ we need to change also limits

$$\begin{aligned} =& \frac{1}{\sqrt{\pi}}\Big(-\int_{-\infty}^{-x/\sqrt{t}} e^{-z^2+x+z\sqrt{t}}\,dz + \int_{-x/\sqrt{t}}^{\infty} e^{-\frac{1}{t}z^2-x-z\sqrt{t}}\,dz\Big) \\ =& \frac{1}{\sqrt{\pi}}\Big(-\int_{-\infty}^{-x/\sqrt{t}} e^{-(z-\sqrt{t}/2)^2+x+t/4}\,dz + \int_{-x/\sqrt{t}}^{\infty} e^{-(z+\sqrt{t}/2)^2-x+t/4}\,dz\Big) \\ =& \frac{1}{\sqrt{\pi}}\Big(-\int_{-\infty}^{-x/\sqrt{t}-\sqrt{t}/2} e^{-s^2+x+t/4}\,ds + \int_{-x/\sqrt{t}+\sqrt{t}/2}^{\infty} e^{-s^2-x+t/4}\,ds\Big) = \end{aligned}$$

after $z = s \pm \sqrt{t}/2$ in the first/second integrals we need to change also limits

$$\frac{1}{2}e^{x+t/4}\Big(-1 + \operatorname{erf}(\frac{x}{\sqrt{t}} + \frac{\sqrt{t}}{2})\Big) + \frac{1}{2}e^{-x+t/4}\Big(1 + \operatorname{erf}(\frac{x}{\sqrt{t}} - \frac{\sqrt{t}}{2})\Big).$$

□

Day, Problem 4E

Problem 4 (5pts)**.** Find the solution $u(x,t)$ to

$$u_t = 4u_{xx} \qquad -\infty < x < \infty,\ t > 0, \tag{4.1}$$

$$u|_{t=0} = \begin{cases} 1-|x| & |x|<1, \\ 0 & |x|\geq 1, \end{cases} \tag{4.2}$$

$$\max|u| < \infty. \tag{4.3}$$

Calculate the integral.

Hint: For $u_t = ku_{xx}$ use $G(x,y,t) = \frac{1}{\sqrt{4\pi kt}}\exp(-(x-y)^2/4kt)$. To calculate integral make change of variables and use $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}\,dz$.

Solution. Due to hint

$$\begin{aligned} u(x,t) =& \frac{1}{\sqrt{16\pi t}}\int_{-\infty}^{\infty} e^{-\frac{1}{16t}(x-y)^2} g(y)\,dy \\ =& \frac{1}{\sqrt{16\pi t}}\Big(\int_{-1}^{0} e^{-\frac{1}{16t}(x-y)^2}(1+y)\,dy + \int_0^1 e^{-\frac{1}{16t}(x-y)^2}(1-y)\,dy\Big) \end{aligned}$$

after $y = x + 4z\sqrt{t}$ we need to change also limits

$$\begin{aligned} =& \frac{1}{\sqrt{\pi}}\Big(\int_{-(x+1)/4\sqrt{t}}^{-x/4\sqrt{t}} e^{-z^2}\big(1+x+4z\sqrt{t}\big)\,dz + \int_{-x/4\sqrt{t}}^{-(x-1)/4\sqrt{t}} e^{-z^2}\big(1-x-4z\sqrt{t}\big)\,dz\Big) \\ =& \frac{1}{2}(1+x)\Big(\operatorname{erf}(\frac{x+1}{4\sqrt{t}}) - \operatorname{erf}(\frac{x}{4\sqrt{t}})\Big) + \frac{1}{2}(1-x)\Big(\operatorname{erf}(\frac{x}{4\sqrt{t}}) - \operatorname{erf}(\frac{x-1}{4\sqrt{t}})\Big) \\ &+ \frac{2\sqrt{t}}{\sqrt{\pi}}\Big(e^{-z^2}\Big|_{z=x/4\sqrt{t}}^{z=(x+1)/4\sqrt{t}} + e^{-z^2}\Big|_{z=x/4\sqrt{t}}^{z=(x-1)/4\sqrt{t}}\Big). \end{aligned}$$

□

Night, Problem 4A

Problem 4 (5pts)**.** Find the solution $u(x,t)$ to

$$u_t = 9u_{xx} \qquad -\infty < x < \infty,\ t>0, \tag{4.1}$$

$$u|_{t=0} = \begin{cases} -1 & x<-1,\\ x & |x|\le 1,\\ 1 & x\ge 1,\end{cases} \tag{4.2}$$

$$\max|u| < \infty. \tag{4.3}$$

Calculate the integral.

Hint: For $u_t = ku_{xx}$ use $G(x,y,t) = \frac{1}{\sqrt{4\pi kt}}\exp(-(x-y)^2/4kt)$. To calculate integral make change of variables and use $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}\,dz$.

Solution. Due to hint

$$\begin{aligned} u(x,t) =& \frac{1}{\sqrt{36\pi t}}\int_{-\infty}^{\infty} e^{-\frac{1}{36t}(x-y)^2} g(y)\,dy\\ =& \frac{1}{\sqrt{36\pi t}}\Big(-\int_{-\infty}^{-1} e^{-\frac{1}{36t}(x-y)^2}\,dy + \int_{-1}^{1} ye^{-\frac{1}{36t}(x-y)^2}\,dy + \int_{1}^{\infty} e^{-\frac{1}{36t}(x-y)^2}\,dy\Big)\end{aligned}$$

after $y = x + 9z\sqrt{t}$ we need to change also limits

$$\begin{aligned} =& \frac{1}{\sqrt{\pi}}\Big(-\int_{-\infty}^{(-1-x)/9\sqrt{t}} e^{-z^2}\,dz + \int_{(-1-x)/9\sqrt{t}}^{(1-x)/9\sqrt{t}} \big(x+9z\sqrt{t}\big)e^{-z^2}\,dz + \int_{(1-x)/9\sqrt{t}}^{\infty} e^{-z^2}\,dz\Big)\\ =& -\frac{1}{2} - \frac{1}{2}\operatorname{erf}\big((-1-x)/9\sqrt{t}\big) + \frac{1}{2}x\Big(\operatorname{erf}\big((1-x)/9\sqrt{t}\big) - \operatorname{erf}\big((-1-x)/9\sqrt{t}\big)\Big)\\ &+ \frac{1}{\sqrt{\pi}}\int_{(-1-x)/9\sqrt{t}}^{(1-x)/9\sqrt{t}} 9z\sqrt{t}e^{-z^2}\,dz + \frac{1}{2} - \frac{1}{2}\operatorname{erf}\big((1-x)/9\sqrt{t}\big)\\ =& \frac{1}{2}(x+1)\operatorname{erf}\big((x+1)/9\sqrt{t}\big) - \frac{1}{2}(x-1)\operatorname{erf}\big((x-1)/9\sqrt{t}\big) + + \frac{9\sqrt{t}}{2\sqrt{\pi}}e^{-z^2}\Big|_{(x-1)/9\sqrt{t}}^{(x+1)/9\sqrt{t}}.\end{aligned}$$

□

Night, Problem 4B

Problem 4 (5pts)**.** Find the solution $u(x,t)$ to

$$4u_t = u_{xx} \qquad -\infty < x < \infty,\ t > 0, \tag{4.1}$$

$$u|_{t=0} = xe^{-2|x|} \tag{4.2}$$

$$\max|u| < \infty. \tag{4.3}$$

Calculate the integral.

Hint: For $u_t = ku_{xx}$ use $G(x,y,t) = \frac{1}{\sqrt{4\pi kt}}\exp(-(x-y)^2/4kt)$. To calculate integral make change of variables and use $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}\,dz$.

Solution. Due to hint

$$\begin{aligned}
u(x,t) =& \frac{1}{\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{1}{t}(x-y)^2} g(y)\,dy\\
=& \frac{1}{\sqrt{\pi t}}\Bigl(\int_{-\infty}^{0} ye^{-\frac{1}{t}(x-y)^2+2y}\,dy + \int_0^{\infty} ye^{-\frac{1}{1t}(x-y)^2-2y}\,dy\Bigr)
\end{aligned}$$

after $y = x + z\sqrt{t}$ we need to change also limits

$$\begin{aligned}
=& \frac{1}{\sqrt{\pi}}\Bigl(\int_{-\infty}^{-x/\sqrt{t}} (x+z\sqrt{t})e^{-z^2+2x+2z\sqrt{t}}\,dz + \int_{-x/\sqrt{t}}^{-\infty}(x+z\sqrt{t})e^{-z^2-2x-2z\sqrt{t}}\,dz\Bigr) =\\
=& \frac{1}{\sqrt{\pi}}\Bigl(\int_{-\infty}^{-x/\sqrt{t}} (x+z\sqrt{t})e^{-(z-\sqrt{t})^2+2x+t}\,dz + \int_{-x/\sqrt{t}}^{\infty}(x+z\sqrt{t})e^{-(z+\sqrt{t})^2-2x+t}\,dz\Bigr)
\end{aligned}$$

after $z = w \pm \sqrt{t}$ in the first/second integrals we need to change also limits

$$\begin{aligned}
=& \frac{1}{\sqrt{\pi}}\Bigl(\int_{-\infty}^{-x/\sqrt{t}-\sqrt{t}} (x+w\sqrt{t}+t)e^{-w^2+2x+t}\,dw +\\
& \int_{-x/\sqrt{t}+\sqrt{t}}^{\infty} (x+w\sqrt{t}-t)e^{-w^2-2x+t}\,dw\Bigr)\\
=& \ \frac{1}{2}(x+t)e^{2x+t}\Bigl(1+\operatorname{erf}\bigl(-x/\sqrt{t}-\sqrt{t}\bigr)\Bigr)\\
&+\frac{1}{2}(x-t)e^{-x+t}\Bigl(1-\operatorname{erf}\bigl(-x/\sqrt{t}-\sqrt{t}\bigr)\Bigr)\\
&+\frac{\sqrt{t}}{2\sqrt{\pi}}\Bigl(-e^{-w^2+2x+t}\Big|_{w=-x/\sqrt{t}-\sqrt{t}} + e^{-w^2-2x+t}\Big|_{w=-x/\sqrt{t}+\sqrt{t}}\Bigr)
\end{aligned}$$

(where the last line $= 0$). □

Night, Problem 4C

Problem 4 (5pts)**.** Find the solution $u(x,t)$ to

$$u_t = 4u_{xx} \qquad -\infty < x < \infty,\ t > 0, \tag{4.1}$$

$$u|_{t=0} = xe^{-x^2} \tag{4.2}$$

$$\max|u| < \infty. \tag{4.3}$$

Calculate the integral.

Hint: For $u_t = ku_{xx}$ use $G(x,y,t) = \frac{1}{\sqrt{4\pi kt}}\exp(-(x-y)^2/4kt)$. To calculate integral make change of variables and use $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}\,dz$.

Solution. Due to hint

$$\begin{aligned}
u(x,t) =& \frac{1}{\sqrt{16\pi t}}\int_{-\infty}^{\infty} e^{-\frac{1}{16t}(x-y)^2} g(y)\,dy = \frac{1}{\sqrt{16\pi t}}\int_{-\infty}^{\infty} e^{-\frac{1}{16t}(x-y)^2-y^2} y\,dy\\
=& \frac{1}{\sqrt{16\pi t}}\int_{-\infty}^{\infty} \exp\bigl[-(\frac{1}{16t}+1)y^2 + \frac{1}{8t}xy - \frac{1}{16t}x^2\bigr] y\,dy\\
=& \frac{1}{\sqrt{16\pi t}}\int_{-\infty}^{\infty} \exp\bigl[-\frac{(16t+1)}{16t}(y - \frac{x}{16t+1})^2 - \frac{x^2}{16t+1}\bigr] y\,dy
\end{aligned}$$

and changing variables $y = z + \frac{x}{16t+1}$

$$\begin{aligned}
=& \frac{1}{\sqrt{16\pi t}}\int_{-\infty}^{\infty} \exp\bigl[-\frac{(16t+1)}{16t}z^2 - \frac{x^2}{16t+1}\bigr]\bigl(z + \frac{x}{(16t+1)}\bigr)\,dz\\
=& \frac{x}{\sqrt{16\pi t}(16t+1)} e^{-\frac{x^2}{16t+1}} \int_{-\infty}^{\infty} e^{-\frac{(16t+1)}{16t}z^2}\,dz\\
=& \frac{x}{\sqrt{16\pi t}(16t+1)} e^{-\frac{x^2}{16t+1}} \times \frac{\sqrt{16t}}{\sqrt{16t+1}}\int_{-\infty}^{\infty} e^{-w^2}\\
=& \frac{x}{(16t+1)^{3/2}} e^{-x^2/(16t+1)}.
\end{aligned}$$

$\square$

Morning, Problem 4A

Problem 4 (5pts)**.** Find the solution $u(x,t)$ to

$$4u_t = u_{xx} \qquad -\infty < x < \infty,\ t > 0, \tag{4.1}$$

$$u|_{t=0} = |x|e^{-2|x|} \tag{4.2}$$

$$\max |u| < \infty. \tag{4.3}$$

Calculate the integral.

Hint: For $u_t = ku_{xx}$ use $G(x,y,t) = \frac{1}{\sqrt{4\pi kt}} \exp(-(x-y)^2/4kt)$. To calculate integral make change of variables and use $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}\, dz$.

Solution. Due to hint

$$\begin{aligned} u(x,t) =& \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{1}{t}(x-y)^2} g(y)\, dy \\ =& \frac{1}{\sqrt{\pi t}} \Big(- \int_{-\infty}^{0} y e^{-\frac{1}{t}(x-y)^2 + 2y}\, dy + \int_0^{\infty} y e^{-\frac{1}{t}(x-y)^2 - 2y}\, dy \Big) \end{aligned}$$

after $y = x + z\sqrt{t}$ we need to change also limits

$$\begin{aligned} =& \frac{1}{\sqrt{\pi}} \Big(- \int_{-\infty}^{-x/\sqrt{t}} (x + z\sqrt{t}) e^{-z^2 + 2x + 2z\sqrt{t}}\, dz + \int_{-x/\sqrt{t}}^{-\infty} (x + z\sqrt{t}) e^{-z^2 - 2x - 2z\sqrt{t}}\, dz \Big) = \\ =& \frac{1}{\sqrt{\pi}} \Big(- \int_{-\infty}^{-x/\sqrt{t}} (x + z\sqrt{t}) e^{-(z-\sqrt{t})^2 + 2x + t}\, dz + \int_{-x/\sqrt{t}}^{\infty} (x + z\sqrt{t}) e^{-(z+\sqrt{t})^2 - 2x + t}\, dz \Big) \end{aligned}$$

after $z = w \pm \sqrt{t}$ in the first/second integrals we need to change also limits

$$\begin{aligned} =& \frac{1}{\sqrt{\pi}} \Big(- \int_{-\infty}^{-x/\sqrt{t}-\sqrt{t}} \big(x + w\sqrt{t} + t\big) e^{-w^2 + 2x + t}\, dw + \\ & \int_{-x/\sqrt{t}+\sqrt{t}}^{\infty} \big(x + w\sqrt{t} - t\big) e^{-w^2 - 2x + t}\, dw \Big) \\ =& -\frac{1}{2}(x+t)e^{2x+t}\Big(1 + \operatorname{erf}\big(-x/\sqrt{t} - \sqrt{t}\big)\Big) \\ &+ \frac{1}{2}(x-t)e^{-2x+t}\Big(1 - \operatorname{erf}\big(-x/\sqrt{t} - \sqrt{t}\big)\Big) \\ &+ \frac{\sqrt{t}}{2\sqrt{\pi}} \Big(e^{-w^2+2x+t}\Big|_{w=-x/\sqrt{t}-\sqrt{t}} + e^{-w^2-2x+t}\Big|_{w=-x/\sqrt{t}+\sqrt{t}} \Big) \end{aligned}$$

(where the last line $= \frac{\sqrt{t}}{\sqrt{\pi}} e^{-x^2/t}$). ∎

Deferred, Problem 4A

Problem 4 (5pts)**.** Find the solution $u(x,t)$ to

$$u_t = 4u_{xx} \qquad -\infty < x < \infty,\ t > 0, \tag{4.1}$$

$$u|_{t=0} = e^{-x^2} \tag{4.2}$$

$$\max |u| < \infty. \tag{4.3}$$

Calculate the integral.

Hint: For $u_t = ku_{xx}$ use $G(x,y,t) = \frac{1}{\sqrt{4\pi kt}} \exp(-(x-y)^2/4kt)$. To calculate integral make change of variables and use $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}\,dz$.

Solution. Due to hint

$$\begin{aligned}
u(x,t) =& \frac{1}{\sqrt{16\pi t}} \int_{-\infty}^{\infty} e^{-\frac{1}{16t}(x-y)^2} g(y)\,dy = \frac{1}{\sqrt{16\pi t}} \int_{-\infty}^{\infty} e^{-\frac{1}{16t}(x-y)^2 - y^2}\,dy \\
=& \frac{1}{\sqrt{16\pi t}} \int_{-\infty}^{\infty} \exp\bigl[-(\frac{1}{16t}+1)y^2 + \frac{1}{8t}xy - \frac{1}{16t}x^2\bigr]\,dy \\
=& \frac{1}{\sqrt{16\pi t}} \int_{-\infty}^{\infty} \exp\bigl[-\frac{(16t+1)}{16t}(y - \frac{x}{16t+1})^2 - \frac{x^2}{16t+1}\bigr]\,dy
\end{aligned}$$

and changing variables $y = z + \frac{16tx}{16t+1}$

$$\begin{aligned}
=& \frac{1}{\sqrt{16\pi t}} \int_{-\infty}^{\infty} \exp\bigl[-\frac{(16t+1)}{16t} z^2 - \frac{x^2}{16t+1}\bigr]\,dz \\
=& \frac{x}{\sqrt{16\pi t}(16t+1)} e^{-\frac{x^2}{16t+1}} \int_{-\infty}^{\infty} e^{-\frac{(16t+1)}{16t}z^2}\,dz \\
=& \frac{1}{\sqrt{16\pi t}} e^{-\frac{x^2}{16t+1}} \times \frac{\sqrt{16t}}{\sqrt{16t+1}} \int_{-\infty}^{\infty} e^{-w^2} \\
=& \frac{1}{\sqrt{16t+1}} e^{-x^2/(16t+1)}.
\end{aligned}$$

□

Solutions to Problem 5

Day, Problem 5A

Problem 5 (2pts, bonus)**.** Consider the PDE with boundary conditions:

$$u_{tt} - c^2 u_{xx} + a u_t = 0, \qquad 0 < x < L, \tag{5.1}$$
$$u|_{x=0} = 0, \tag{5.2}$$
$$u_x|_{x=L} = 0, \tag{5.3}$$

where $a > 0$ are constant. Consider

$$E(t) := \frac{1}{2} \int_0^L \left(u_t^2 + c^2 u_x^2 \right) dx \tag{5.4}$$

and check, what *exactly* holds:
(a) $\frac{dE}{dt} \leq 0$ (b) $\frac{dE}{dt} = 0$ (c) $\frac{dE}{dt} \geq 0$.

Hint: Calculate $\frac{dE}{dt}$, substitute u_{tt} from equation and integrate by parts with respect to x as needed, taking into account boundary conditions (5.2)–(5.3).

Solution.

$$\begin{aligned} \frac{dE}{dt} &= \int_0^L \left(u_t u_{tt} + c^2 u_x u_{tx} \right) dx = \int_0^L \left(c^2 u_t u_{xx} - a u_t^2 + c^2 u_x u_{tx} \right) dx \\ &= \int_0^L \left(c^2 (u_t u_x)_x - a u_t^2 \right) dx = c^2 u_t u_x \Big|_{x=0}^{x=L} - a \int_0^L u_t^2 \, dx \end{aligned}$$

with the first term equal 0 due to boundary conditions and the second ≤ 0.

Answer: (a) $\frac{dE}{dt} \leq 0$. □

Day, Problem 5B

Problem 5 (2pts, bonus)**.** Consider the PDE with boundary conditions:

$$u_{tt} - c^2 u_{xx} = 0, \qquad 0 < x < L, \tag{5.1}$$
$$(u_x + a u_t)|_{x=0} = 0, \tag{5.2}$$
$$u|_{x=L} = 0, \tag{5.3}$$

where $a > 0$ are constant. Consider

$$E(t) := \frac{1}{2}\int_0^L \left(u_t^2 + c^2 u_x^2\right) dx \tag{5.4}$$

and check, what *exactly* holds:

(a) $\frac{dE}{dt} \le 0$ (b) $\frac{dE}{dt} = 0$ (c) $\frac{dE}{dt} \ge 0$.

Hint: Calculate $\frac{dE}{dt}$, substitute u_{tt} from equation and integrate by parts with respect to x as needed, taking into account boundary conditions (5.2)–(5.3).

Solution.

$$\begin{aligned} \frac{dE}{dt} &= \int_0^L \left(u_t u_{tt} + c^2 u_x u_{tx}\right) dx = \int_0^L \left(c^2 u_t u_{xx} + c^2 u_x u_{tx}\right) dx \\ &= \int_0^L (c^2 (u_t u_x)_x \, dx = a c^2 u_t^2\big|_{x=0} \end{aligned}$$

due to boundary conditions.

Answer: (c) $\frac{dE}{dt} \ge 0$. □

Day, Problem 5C

Problem 5 (2pts, bonus)**.** Consider the PDE with boundary conditions:

$$
\begin{aligned}
&u_{tt} - c^2 u_{xx} + a u_t = 0, && 0 < x < L, && (5.1)\\
&u|_{x=0} = 0, && && (5.2)\\
&u_x|_{x=L} = 0, && && (5.3)
\end{aligned}
$$

where $a > 0$ are constant. Consider

$$E(t) := \frac{1}{2} e^{2at} \int_0^L \left(u_t^2 + c^2 u_x^2\right) dx \tag{5.4}$$

and check, what *exactly* holds:

(a) $\frac{dE}{dt} \le 0$ (b) $\frac{dE}{dt} = 0$ (c) $\frac{dE}{dt} \ge 0$.

Hint: Calculate $\frac{dE}{dt}$, substitute u_{tt} from equation and integrate by parts with respect to x as needed, taking into account boundary conditions (5.2)–(5.3).

Solution.

$$
\begin{aligned}
\frac{dE}{dt} =& e^{2at}\Big[\int_0^L \left(u_t u_{tt} + c^2 u_x u_{tx}\right) dx + a \int_0^L \left(u_t^2 + c^2 u_x^2\right) dx\Big]\\
=& e^{2at}\Big[\int_0^L \left(c^2 u_t u_{xx} - a u_t^2 + c^2 u_x u_{tx}\right) dx + a \int_0^L \left(u_t^2 + c^2 u_x^2\right) dx\Big]\\
=& e^{2at}\Big[\int_0^L (c^2 (u_t u_x)_x \, dx + a \int_0^L c^2 u_x^2 \, dx\Big] = c^2 e^{2at} u_t u_x\Big|_{x=0}^{x=L} + a c^2 e^{2at} \int_0^L u_x^2 \, dx
\end{aligned}
$$

with the first term equal 0 due to boundary conditions and the second ≥ 0.

Answer: (a) $\frac{dE}{dt} \le 0$. □

Day, Problem 5D

Problem 5 (2pts, bonus)**.** Consider the PDE with boundary conditions:

$$u_{tt} - c^2 u_{xx} = 0, \qquad x > 0, \tag{5.1}$$

$$u|_{x=0} = 0. \tag{5.2}$$

Consider

$$E(t) := \frac{1}{2}\int_0^\infty \Big(t(u_t^2 + c^2 u_x^2) + 2x u_t u_x \Big)\, dx \tag{5.3}$$

and check, what *exactly* holds:

(a) $\frac{dE}{dt} \le 0$ (b) $\frac{dE}{dt} = 0$ (c) $\frac{dE}{dt} \ge 0$.

Hint: Calculate $\frac{dE}{dt}$, substitute u_{tt} from equation and integrate by parts with respect to x as needed, taking into account boundary condition (5.2). Assume that u and its derivatives decay at infinity.

Solution.

$$\begin{aligned}
\frac{dE}{dt} &= \int_0^L \Big(t(u_t u_{tt} + c^2 u_x u_{tx}) + \frac{1}{2}(u_t^2 + c^2 u_x^2) + x u_{tt} u_x + x u_t u_{tx} \Big)\, dx \\
&= \int_0^\infty \Big(c^2 t(u_t u_{xx} + u_x u_{tx}) + c^2 x u_{xx} u_x + x u_t u_{tx} + \frac{1}{2}(u_t^2 + c^2 u_x^2) \Big)\, dx \\
&= \int_0^\infty \Big(c^2 t(u_t u_x)_x + \frac{1}{2} x (u_t^2 + c^2 u_x^2)_x + \frac{1}{2}(u_t^2 + c^2 u_x^2) \Big)\, dx \\
&= \int_0^\infty \big(c^2 t u_t u_x + \frac{1}{2} x (u_t^2 + c^2 u_x^2) \big)_x\, dx = -\big(c^2 t u_t u_x + \frac{1}{2} x (u_t^2 + c^2 u_x^2) \big)\Big|_{x=0} = 0
\end{aligned}$$

due to boundary conditions.

Answer: (b) $\frac{dE}{dt} = 0$. □

Night, Problem 5A

Problem 5 (2pts, bonus)**.** Consider the PDE with boundary conditions:

$$u_{tt} - c^2 u_{xx} + 2au^3 = 0, \qquad 0 < x < L, \tag{5.1}$$
$$u|_{x=0} = 0, \tag{5.2}$$
$$u_x|_{x=L} = 0, \tag{5.3}$$

where $a > 0$ are constant. Consider

$$E(t) \coloneqq \frac{1}{2}\int_0^L \left(u_t^2 + c^2 u_x^2 + au^4\right) dx \tag{5.4}$$

and check, what *exactly* holds:

(a) $\frac{dE}{dt} \le 0$ (b) $\frac{dE}{dt} = 0$ (c) $\frac{dE}{dt} \ge 0$.

Hint: Calculate $\frac{dE}{dt}$, substitute u_{tt} from equation and integrate by parts with respect to x as needed, taking into account boundary conditions (5.2)–(5.3).

Solution.

$$\begin{aligned}\frac{dE}{dt} &= \int_0^L \left(u_t u_{tt} + c^2 u_x u_{tx} + 2au_t u^3\right) dx = \int_0^L \left(c^2 u_t u_{xx} + c^2 u_x u_{tx}\right) dx \\ &= \int_0^L \left(c^2 (u_t u_x)_x\right) dx = c^2 u_t u_x \Big|_{x=0}^{x=L} = 0\end{aligned}$$

due to boundary conditions.

Answer: (b) $\frac{dE}{dt} = 0$. ☐

Night, Problem 5B

Problem 5 (2pts, bonus)**.** Consider nonnegative solutions($u \geq 0$)

$$u_t - a(uu_x)_x = 0, \qquad -\infty < x < \infty, \tag{5.1}$$

where $a > 0$ are constant. Consider

$$E(t) := \frac{1}{2}\int_0^L u^2\,dx \tag{5.2}$$

and check, what *exactly* holds:

(a) $\frac{dE}{dt} \leq 0$ (b) $\frac{dE}{dt} = 0$ (c) $\frac{dE}{dt} \geq 0$.

Hint: Calculate $\frac{dE}{dt}$, substitute u_t from equation and integrate by parts with respect to x as needed. Assume that u and its derivatives decay at infinity.

Solution.

$$\begin{aligned}\frac{dE}{dt} &= \int_{-\infty}^{\infty}\big(u_t u\big)\,dx = \int_{-\infty}^{\infty}\big(au(uu_x)_x\big)\,dx \\ &= -\int_{-\infty}^{\infty} auu_x^2 \leq 0.\end{aligned}$$

Answer: (a) $\frac{dE}{dt} \leq 0$. □

Night, Problem 5C

Problem 5 (2pts, bonus)**.** Consider the PDE with boundary conditions:

$$u_{ttxx} - c^2 u = 0, \qquad 0 < x < L, \tag{5.1}$$

$$u|_{x=0} = 0, \tag{5.2}$$

$$u_x|_{x=L} = 0, \tag{5.3}$$

where $a > 0$ are constant. Consider

$$E(t) := \frac{1}{2}\int_0^L \left(u_{xt}^2 + c^2 u\right) dx \tag{5.4}$$

and check, what *exactly* holds:

(a) $\frac{dE}{dt} \leq 0$ (b) $\frac{dE}{dt} = 0$ (c) $\frac{dE}{dt} \geq 0.$

Hint: Calculate $\frac{dE}{dt}$, substitute u_{tt} from equation and integrate by parts with respect to x as needed, taking into account boundary conditions (5.2)–(5.3).

Solution.

$$\frac{dE}{dt} = \int_0^L \left(u_{ttx}u_{tx} + c^2 u_t u)\right) dx = \int_0^L \left(-u_{ttxx}u_t + c^2 u_t u)\right) dx = 0$$

where we integrated by parts.

Answer: (b) $\frac{dE}{dt} = 0.$ ◻

Morning, Problem 5A

Problem 5 (2pts, bonus)**.** Consider the PDE with boundary conditions:

$$u_{tt} + c^2 u_{xxxx} = 0, \qquad 0 < x < L, \tag{5.1}$$

$$u|_{x=0} = u_x x = 0 = u|_{x=L} = u_x|_{x=L} 0. \tag{5.2}$$

Consider

$$E(t) := \frac{1}{2}\int_0^\infty \bigl(u_t^2 + c^2 u_{xx}^2\bigr)\,dx \tag{5.3}$$

and check, what *exactly* holds:

(a) $\frac{dE}{dt} \le 0$ (b) $\frac{dE}{dt} = 0$ (c) $\frac{dE}{dt} \ge 0$.

Hint: Calculate $\frac{dE}{dt}$, substitute u_{tt} from equation and integrate by parts with respect to x as needed, taking into account boundary conditions (5.2).

Solution.

$$\begin{aligned}\frac{dE}{dt} &= \int_0^L \bigl(u_t u_{tt} + c^2 u_{txx} u_{xx}\bigr)\,dx = c^2\int_0^L \bigl(-u_t u_{xxxx} + c u_{txx} u_{xx}\bigr)\,dx\\ &= c^2\int_0^L \bigl(u_{tx} u_{xxx} + 2u_{txx} u_{xx}\bigr)\,dx = c^2\int_0^L \bigl(-u_{txx} u_{xx} + u_{txx} u_{xx}\bigr)\,dx = 0\end{aligned}$$

due to boundary conditions (we integrated by parts twice).

Answer: (b) $\frac{dE}{dt} = 0$. □

Deferred, Problem 5A

Problem 5 (2pts, bonus)**.** Consider the PDE with boundary conditions:

$$u_t + uu_x + u_{xxx} = 0, \qquad -\infty < x < \infty, \tag{5.1}$$

where $a > 0$ are constant. Consider

$$E(t) := \frac{1}{2}\int_0^L u^2\,dx \tag{5.2}$$

and check, what *exactly* holds:

(a) $\frac{dE}{dt} \le 0$ (b) $\frac{dE}{dt} = 0$ (c) $\frac{dE}{dt} \ge 0.$

Hint: Calculate $\frac{dE}{dt}$, substitute u_{tt} from equation and integrate by parts with respect to x as needed. Assume that u and its derivatives decay at infinity.

Solution.

$$\begin{aligned}\frac{dE}{dt} &= \int_{-\infty}^{\infty} 2u_t u\,dx = -\int_{-\infty}^{\infty} 2\big(u_x u^2 + uu_{xxx}\big)\,dx\\ &= -\int_{-\infty}^{\infty} 2\big(\frac{1}{3}(u^3)_x + uu_{xxx}\big)\,dx = \int_{-\infty}^{\infty} 2u_{xx}u_x\,dx = \int_{-\infty}^{\infty}(u_x^2)_x = 0.\end{aligned}$$

Answer: (b) $\frac{dE}{dt} = 0$. □