Problem 1 (4pt)**.** Solve by Fourier method

$$u_{tt} - u_{xx} = 0 \qquad 0 < x < \pi, \tag{1.1}$$
$$u|_{x=0} = 0, \qquad (u_x + \alpha u)|_{x=\pi} = 0 \tag{1.2}$$
$$u|_{t=0} = \sin(x), \qquad u_t|_{t=0} = 0 \tag{1.3}$$

with $\alpha \in \mathbb{R}$.
HINT: We know that λ_n are real but since we do not know the sign of α we do not know if it all $\lambda_n \geq 0$; so you must consider the case of some of $\lambda_n < 0$.
NOTE: Only find equations for eigenvalues.

Solution. Separation of variables leads to

$$X'' + \lambda X = 0, \tag{1.4}$$
$$X(0) = 0, \qquad (X' + \alpha X)(\pi) = 0 \tag{1.5}$$
$$T'' + \lambda T = 0. \tag{1.6}$$

(a) Consider $\lambda = k^2 > 0$. Then $X = A\cos(kx) + B\sin(kx)$ and plugging to boundary conditions we get $A = 0$, $B\bigl(k\cos(k\pi) + \alpha\sin(k\pi)\bigr) = 0$, and to have a nontrivial solution we need $k\cos(k\pi) + \alpha\sin(k\pi) = 0$ which for $\alpha \neq 0$ is equivalent to

$$k = -\alpha\tan(k\pi). \tag{1.7}$$

Solving graphically

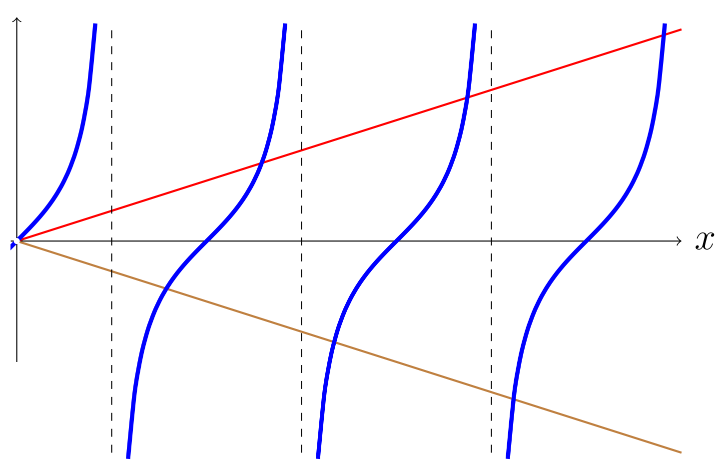

Figure 1: Brown line for $\alpha > 0$, red line for $\alpha < 0$.

(b) If $\lambda = 0$, then $X = A + Bx$; plugging into boundary conditions we get $A = 0$, $B(1 + \alpha\pi) = 0$ and we have nontrivial solutions only for $\alpha = -1/\pi$.

(c) Let $\lambda = -k^2 < 0$. Then $X = A\cosh(kx) + B\sinh(kx)$; plugging into boundary conditions we get $A = 0$, $B\bigl(k\cosh(k\pi) + \alpha\sinh(k\pi)\bigr) = 0$, and we have nontrivial solutions only for $k\cosh(k\pi) + \alpha\sinh(k\pi) = 0$, which is equivalent to

$$k = -\alpha\tanh(k\pi). \tag{1.8}$$

It has only one solution and only for $\alpha < -1/\pi$.

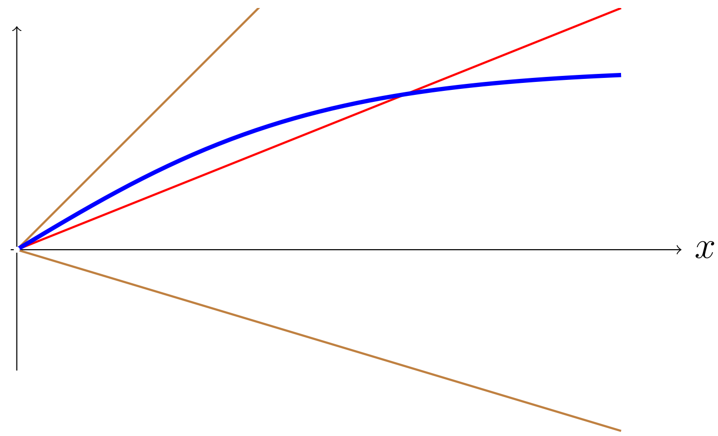

Figure 2: Brown line for $\alpha > -1/\pi$, red line for $\alpha < -1/\pi$.

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Problem 2 (4pt)**.** Solve

$$u_{xx} + u_{yy} = 0 \qquad -\infty < x < \infty,\ 0 < y < \infty, \tag{2.1}$$

$$(u_y + \alpha u)|_{y=0} = g(x) = \begin{cases} 1 & |x| < 1, \\ 0 & |x| > 1 \end{cases} \tag{2.2}$$

$$\max |u| < \infty. \tag{2.3}$$

HINT: Use partial Fourier transform with respect to x. Write solution as a Fourier integral without calculating it.
Find restriction to α , so that there will be no singularities.

Solution. After partial Fourier transform

$$-k^2\hat{u} + \hat{u}_{yy} = 0 \qquad 0 < y < \infty, \tag{2.4}$$

$$(\hat{u}_y + \alpha\hat{u})|_{y=0} = \hat{g}(k). \tag{2.5}$$

One can calculate easily $\hat{g}(k) = \frac{\sin(k)}{\pi k}$.
Solving (2.4) we get $\hat{u} = A(k)e^{-|k|y} + B(k)e^{|k|y}$ and $B(k) = 0$ due to (2.3) and plugging to boundary condition we get $A(k)(-|k| + \alpha) = \frac{\sin(k)}{\pi k} \implies A(k) = \frac{\sin(k)}{\pi k(\alpha - |k|)}$,

$$\hat{u} = \frac{\sin(k)}{\pi k(\alpha - |k|)} e^{-|k|y}. \tag{2.6}$$

and

$$u(x, y) = \int_{-\infty}^{\infty} \frac{\sin(k)}{\pi k(\alpha - |k|)} e^{-|k|y + ikx}\, dk. \tag{2.7}$$

There will be singularity for $|k| = \alpha$ which is excluded by $\alpha < 0$. □

Problem 3 (4pt)**.** Using Fourier method find eigenvalues and eigenfunctions of Laplacian in the rectangle $\{0 < x < a, 0 < y < b\}$ with the boundary conditions:

$$u_{xx} + u_{yy} = -\lambda u \qquad 0 < x < a,\ 0 < y < b, \tag{3.1}$$

$$u_x|_{x=0} = u_x|_{x=a} = u|_{y=0} = u|_{y=b} = 0. \tag{3.2}$$

Solution. Separating variables $u = X(x)Y(y)$ we arrive to

$$\frac{X''}{X} + \frac{Y''}{Y} + \lambda = 0 \tag{3.3}$$

Then

$$X'' + \mu X = 0, \tag{3.4}$$

$$X'(0) = X'(a) = 0 \tag{3.5}$$

and

$$Y'' + \nu X = 0, \tag{3.6}$$

$$Y(0) = Y(b) = 0 \tag{3.7}$$

and

$$\lambda = \mu + \nu. \tag{3.8}$$

Next

$$\mu_m = \frac{\pi^2 m^2}{a^2}, \qquad X_m = \cos\left(\frac{\pi m x}{a}\right), \qquad m = 0, 1, 2, \ldots, \tag{3.9}$$

$$\nu_n = \frac{\pi^2 n^2}{b^2}, \qquad Y_n = \sin\left(\frac{\pi n y}{b}\right), \qquad n = 1, 2, \ldots, \tag{3.10}$$

and finally

$$\lambda_{mn} = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right), \quad u_{mn} = \cos\left(\frac{\pi m x}{a}\right)\sin\left(\frac{\pi n y}{b}\right), \tag{3.11}$$

with $m = 0, 1, 2, \ldots$, and $n = 1, 2, \ldots$. □

Problem 4 (4pt)**.** Consider Laplace equation in the disc with a cut

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \qquad r < 9,\, 0 < \theta < 2\pi \tag{4.1}$$

with the Dirichlet boundary conditions as $\theta = 0$ and $\theta = 2\pi$

$$u|_{\theta=0} = u|_{\theta=2\pi} = 0 \tag{4.2}$$

and the Neumann boundary condition as $r = 9$

$$u_r|_{r=9} = 1. \tag{4.3}$$

Using separation of variables find solution as a series.

Solution. Separating variables $u(r,\theta) = R(r)\Theta(\theta)$ we get

$$\frac{r^2R'' + rR'}{R} + \frac{\Theta''}{\Theta} = 0$$

and therefore both terms are constant:

$$\Theta'' + \lambda\Theta = 0, \tag{4.4}$$
$$\Theta(0) = \Theta(2\pi) = 0 \tag{4.5}$$

and therefore $\lambda_n = \frac{n^2}{4}$, $\Theta_n = \sin(\frac{n\theta}{2})$, $n = 1, 2, \ldots$. Then

$$r^2R'' + rR' - \frac{n^2}{4}R = 0 \tag{4.6}$$

and $R = Ar^{n/2} + Br^{-n/2}$ where we drop the last term as it is singular at $r = 0$. So $u_n = A_n r^{n/2}\sin(\frac{n\theta}{2})$ and

$$u = \sum_{n=1}^{\infty} A_n r^{n/2}\sin(\frac{n\theta}{2}). \tag{4.7}$$

Plugging into (4.3) we get

$$\sum_{n=1}^{\infty} A_n \frac{n}{2} 3^{n-2}\sin(\frac{n\theta}{2}) = 1; \tag{4.8}$$

then

$$A_n = \frac{2}{n} 3^{2-n} \times \frac{1}{\pi} \int_0^{2\pi} \sin(\frac{n\theta}{2})\, d\theta = \begin{cases} 0 & n = 2m, \\ \dfrac{8 \cdot 3^{1-2m}}{(2m+1)^2 \pi} & n = 2m+1 \end{cases} \tag{4.9}$$

and

$$u = \sum_{m=0}^{\infty} \frac{8 \cdot 3^{1-2m}}{(2m+1)^2 \pi} r^{(2m+1)/2} \sin(\frac{(2m+1)\theta}{2}). \tag{4.10}$$

□

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Problem 5 (4pt)**.** Find Fourier transforms of the function

$$f(x) = \cos^2(x)e^{-|x|} \tag{5.1}$$

and write this function as a Fourier integral.

Solution. The simplest:

$$\begin{aligned}
\hat{f}(k) =& \frac{1}{\pi}\operatorname{Re}\Bigl(\int_0^\infty \cos^2(x)e^{-|x|-ikx}\,dx\Bigr) = \\
& \frac{1}{2\pi}\operatorname{Re}\Bigl(\int_0^\infty \bigl(1+\cos(2x)\bigr)e^{-x-ikx}\,dx\Bigr) = \\
& \frac{1}{4\pi}\operatorname{Re}\Bigl(\int_0^\infty \bigl(2+e^{2ix}+e^{-2ix}\bigr)e^{-x-ikx}\,dx\Bigr) = \\
& \frac{1}{4\pi}\operatorname{Re}\Bigl(\int_{-\infty}^0 \bigl(2e^{-x(1+ik)}+e^{-x(1+ki-2i)}+e^{-x(1+ki+2i)}\bigr)\,dx\Bigr) = \\
& \frac{1}{4\pi}\operatorname{Re}\Bigl(\frac{2}{1+ik}+\frac{1}{1+ki-2i}+\frac{1}{1+ki+2i}\Bigr) = \\
& \frac{1}{2\pi}\Bigl(\frac{2}{1+k^2}+\frac{1}{1+(k-2)^2}+\frac{1}{1+(k+2)^2}\Bigr).
\end{aligned}$$

□