Problem 1 (4 pts)**.** Consider the first order equation:

$$u_t + 3(t^2 - 1)u_x = 6t^2. \tag{1}$$

(a) Find the characteristic curves and sketch them in the (x, t) plane.

(b) Write the general solution.

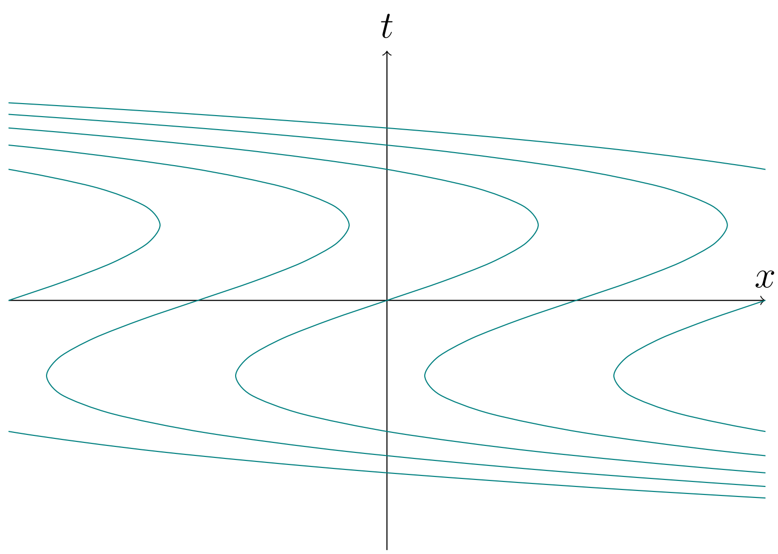
(c) Solve equation (1) with the initial condition $u(x, 0) = x$. Explain why the solution is fully determined by the initial condition.

Solution. (a)–(b) Equation of characteristics

$$\frac{dt}{1} = \frac{dx}{3t^2 - 3} = \frac{du}{6t^2} \implies x - t^3 + 3t = C_1,\ C_2 = u - 2t^3 \implies$$

$$u = 2t^3 + f(x - t^3 + 3t) \tag{2}$$

with arbitrary function f.



(c) From initial condition we conclude that $f(x) = x$ and

$$u(x, t) = x + t^3 + 3t. \tag{3}$$

□

Problem 2 (4 pts)**.** Find solution $u(x,t)$ to

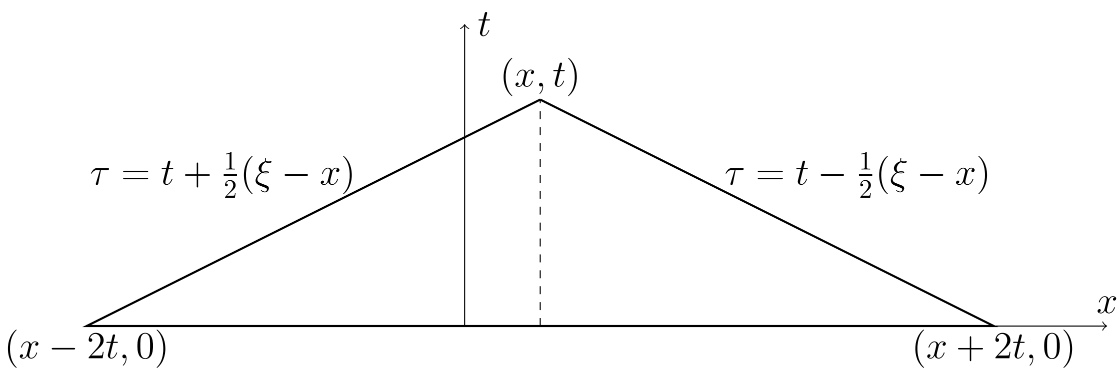
$$u_{tt} - 4u_{xx} = 8/(x^2+1), \tag{4}$$
$$u|_{t=0} = 0, \quad u_t|_{t=0} = 0. \tag{5}$$

HINT: Change order of integration over characteristic triangle.

Solution. By D'Alembert formula

$$u(x,t) = \frac{1}{4}\iint_{\Delta(x,t)} \frac{8}{\xi^2+1} d\xi d\tau, \tag{6}$$

where $\Delta(x,t)$ is bounded by $\tau = 0$, $x-\xi-2(t-\tau)=0$, $x-\xi+2(t-\tau)=0$.



Then the double integral becomes

$$2\int_{x-2t}^{x}\Bigl(\int_0^{t+\frac{1}{2}(\xi-x)} d\tau\Bigr)\frac{d\xi}{\xi^2+1} + 2\int_x^{x+2t}\Bigl(\int_0^{t-\frac{1}{2}(\xi-x)} d\tau\Bigr)\frac{d\xi}{\xi^2+1} =$$
$$\int_{x-2t}^{x} \frac{(2t-x+\xi)\,d\xi}{\xi^2+1} + \int_x^{x+2t} \frac{(2t+x-\xi)\,d\xi}{\xi^2+1} =$$
$$(2t-x)\Bigl(\arctan(x) - \arctan(x-2t)\Bigr) + \frac{1}{2}\Bigl(\ln(x^2+1) - \ln((x-2t)^2+1)\Bigr) +$$
$$(2t+x)\Bigl(\arctan(x+2t) - \arctan(x)\Bigr) + \frac{1}{2}\Bigl(\ln(x^2+1) - \ln((x+2t)^2+1)\Bigr).$$

□

Problem 3 (4 pts)**.** Find continuous solution to

$$u_{tt} - 4u_{xx} = 0, \qquad t > 0, x > -t, \tag{7}$$

$$u|_{t=0} = 0, \qquad x > 0, \tag{8}$$

$$u_t|_{t=0} = 0, \qquad x > 0, \tag{9}$$

$$u_x|_{x=-t} = \sin(t), \qquad t > 0. \tag{10}$$

Solution. (a) (2 pts) Solution to (7) is

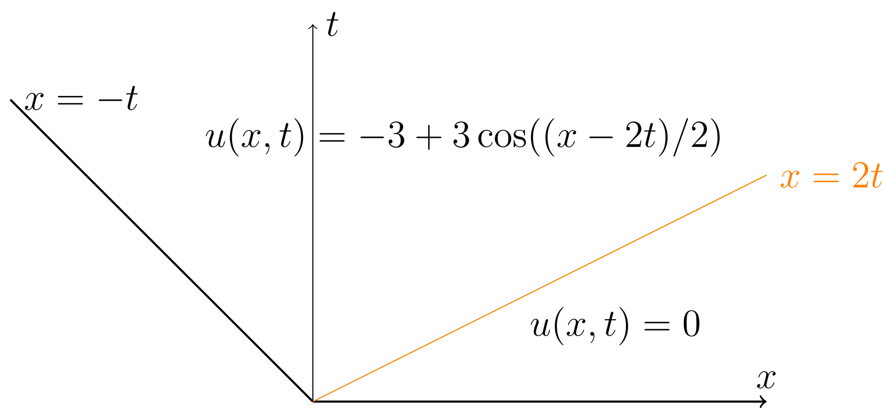
$$u(x,t) = \phi(x+2t) + \psi(x-2t) \tag{11}$$

with unknown functions ϕ and ψ. Plugging into (8)–(9) we get

$$\phi(x) + \psi(x) = 0, \quad 2\phi'(x) - 2\psi'(x) = 0 \implies \phi(x) = \psi(x) = 0 \qquad x > 0$$

and $u(x,t) = \sin(x+3t)$ as $x > 2t$.

(b) (2 pts) Plugging into (10) we get $\psi'(-3t) = \sin(t)$ as $t > 0$ or $\psi'(x) = -\sin(x/3)$ and then $\psi(x) = 3\cos(x/2) + C$ as $x < 0$ and $u(x,t) = 3\cos(x/3) + C$ as $-t < x < 2t$.
Continuity at 0 implies $C = -3$.



□

Problem 4 (4 pts)**.** Consider the PDE with boundary conditions:

$$u_{tt} - c^2 u_{xx} + au = 0, \qquad 0 < x < L, \tag{12}$$
$$(u_x - \alpha u_{tt})|_{x=0} = 0, \tag{13}$$
$$(u_x + \beta u_{tt})|_{x=L} = 0 \tag{14}$$

where $c > 0$, $\alpha > 0$, $\beta > 0$ and a are constant. Prove that the energy $E(t)$ defined as

$$E(t) = \frac{1}{2}\int_0^L \bigl(u_t^2 + c^2 u_x^2 + au^2\bigr)\,dx + \frac{\alpha c^2}{2} u_t(0,t)^2 + \frac{\beta c^2}{2} u_t(L,t)^2 \tag{15}$$

does not depend on t.

Solution.

$$\partial_t E(t) = \int_0^L \bigl(u_t u_{tt} + c^2 u_{xt} u_x + a u_t u\bigr)\,dx + \alpha u_t u_{tt}\bigr|_{x=0} + \beta u_t u_{tt}\bigr|_{x=L}$$
$$= \int_0^L c^2\bigl(u_t u_{xx} + u_x u_{xt}\bigr)\,dx + \alpha u_t u_{tt}|_{x=0} + \beta u_t u_{tt}\bigr|_{x=L} = c^2 u_t u_x\bigr|_{x=0}^{x=L} + \alpha u_t u_{tt}\bigr|_{x=0}$$
$$= c^2 u_t(-u_x + \alpha u_{tt})\bigr|_{x=0} + c^2 u_t(u_x + \beta u_{tt})\bigr|_{x=L} = 0$$

due to boundary conditions. □

Problem 5 (4 pts)**.** Find the solution $u(x,t)$ to

$$u_t = u_{xx} \qquad -\infty < x < \infty,\ t > 0, \tag{16}$$

$$u|_{t=0} = e^{-|x|} \tag{17}$$

$$\max|u| < \infty. \tag{18}$$

Calculate the integral.

Hint: For $u_t = ku_{xx}$ use $G(x,y,t) = \frac{1}{\sqrt{4\pi kt}}\exp(-(x-y)^2/4kt)$. To calculate integral make change of variables and use $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}\,dz$.

Solution. Due to hint

$$\begin{aligned}
u(x,t) =& \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{1}{4t}(x-y)^2-|y|}\,dy = \\
& \frac{1}{\sqrt{4\pi t}}\Bigl(\int_{-\infty}^{0} e^{-\frac{1}{4t}(x-y)^2+y}\,dy + \int_{0}^{\infty} e^{-\frac{1}{4t}(x-y)^2-y}\,dy\Bigr) = \\
& \frac{1}{\sqrt{4\pi t}}\Bigl(\int_{-\infty}^{0} e^{-\frac{1}{4t}(y-x-2t)^2+x+t}\,dy + \int_{0}^{\infty} e^{-\frac{1}{4t}(y-x+2t)^2-x+t}\,dy\Bigr) = \\
& \frac{1}{\sqrt{4\pi t}}e^{x+t}\int_{-\infty}^{-(x+2t)} e^{-\frac{s^2}{4t}}\,ds + \ldots = \\
& \frac{1}{\sqrt{\pi}}e^{x+t}\int_{-\infty}^{-(x+2t)/\sqrt{4t}} e^{-z^2}\,dz + \ldots = \\
& e^{x+t}\bigl(1-\operatorname{erf}((2t+x)/\sqrt{4t})\bigr) + e^{-x+t}\bigl(1-e^{-x+t}\operatorname{erf}((2t-x)/\sqrt{4t})\bigr),
\end{aligned}$$

where … denoted the same term as the first one but with $x \mapsto -x$. □