Problem 1 (4 pts)**.** Consider the first order equation:

$$u_t + xtu_x = xte^{-t^2/2}. \tag{1}$$

(a) Find the characteristic curves and sketch them in the (x,t) plane.

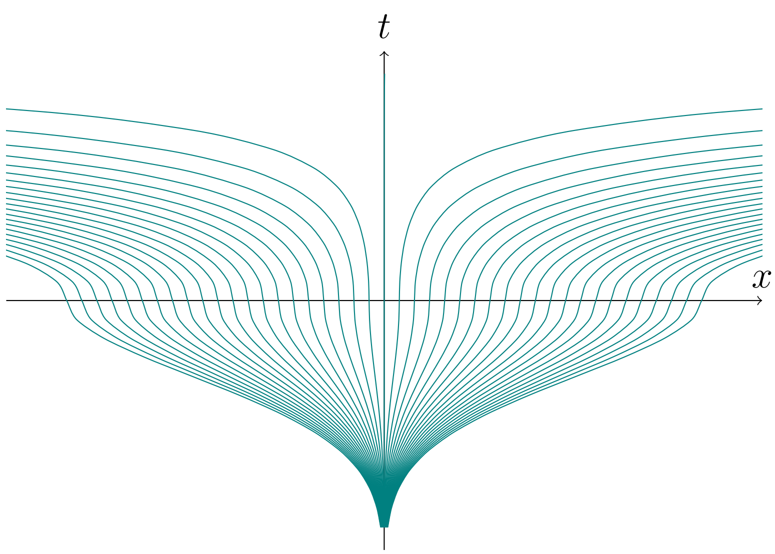
(b) Write the general solution.

(c) Solve equation (1) with the initial condition $u(x,0) = x$. Explain why the solution is fully determined by the initial condition.

Solution. (a)–(b) Equation of characteristics

$$\frac{dt}{1} = \frac{dx}{xt} = \frac{du}{xe^{-t^2/2}} \implies \frac{dx}{x} = t\,dt \implies$$
$$\ln x - t^2/2 = \ln(C_1)C_1 = xe^{-t^2/2} \implies du = xte^{-t^2/2}\,dt = C_1 t \implies$$
$$C_2 = u - C_1\frac{t^2}{2}u = C_2 + C_1\frac{t^2}{2} \implies u = \frac{1}{2}xt^2e^{-t^2/2} + f(xe^{-t^2/2}) \tag{2}$$

with arbitrary function f.



(c) From initial condition we conclude that $f(x) = 0$ and

$$u(x,t) = \frac{1}{2}xt^2e^{-t^2/2}. \tag{3}$$

□

Problem 2 (4 pts)**.** Find solution $u(x,t)$ to

$$u_{tt} - 9u_{xx} = 18e^{-x^2}, \tag{4}$$

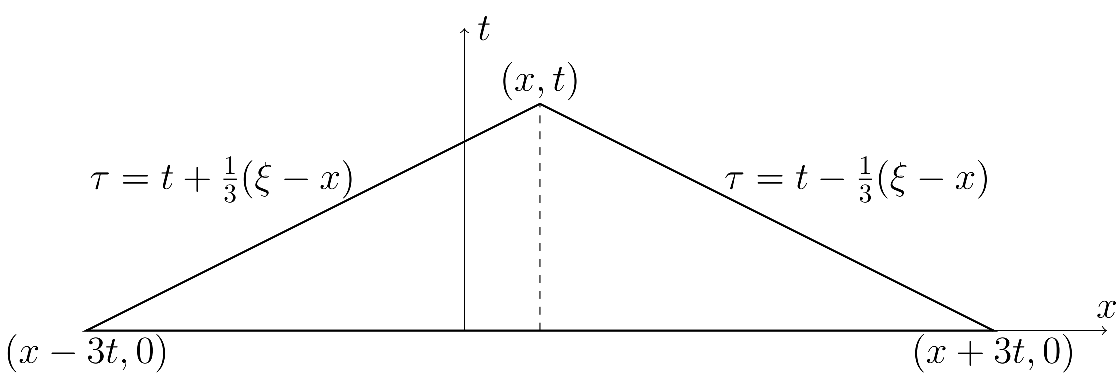
$$u|_{t=0} = 0, \quad u_t|_{t=0} = 0. \tag{5}$$

HINT: Change order of integration over characteristic triangle.
Use $\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}\,dz$.

Solution. By D'Alembert formula

$$u(x,t) = 3\iint_{\Delta(x,t)} e^{-\xi^2}\,d\xi d\tau, \tag{6}$$

where $\Delta(x,t)$ is bounded by $\tau = 0$, $x - \xi - 3(t-\tau) = 0$, $x - \xi + 3(t-\tau) = 0$.



Then the double integral becomes

$$3\int_{x-3t}^{x}\Bigl(\int_0^{t+\frac{1}{3}(\xi-x)} d\tau\Bigr)e^{-\xi^2}\,d\xi + 3\int_{x}^{x+3t}\Bigl(\int_0^{t-\frac{1}{3}(\xi-x)} d\tau\Bigr)e^{-\xi^2}\,d\xi =$$

$$\int_{x-3t}^{x}(3t - x + \xi)e^{-\xi^2}\,d\xi + \qquad \int_{x}^{x+3t}(3t + x - \xi)e^{-\xi^2}\,d\xi =$$

$$(3t-x)\int_{x-3t}^{x} e^{-\xi^2}\,d\xi - \frac{1}{2}e^{-\xi^2}\Big|_{\xi=x-3t}^{\xi=x} + (3t+x)\int_{x}^{x+3t} e^{-\xi^2}\,d\xi + \frac{1}{2}e^{-\xi^2}\Big|_{\xi=x}^{\xi=x-3t} =$$

$$(3t-x)\Bigl(\operatorname{erf}(x) - \operatorname{erf}(x-3t)\Bigr) + (3t+x)\Bigl(\operatorname{erf}(x+3t) - \operatorname{erf}(x)\Bigr) +$$

$$\frac{1}{2}\Bigl(e^{-(x-3t)^2} + e^{-(x+3t)^2} - 2e^{-x^2}\Bigr).$$

□

Problem 3 (4 pts)**.** Find solution to

$$u_{tt} - u_{xx} = 0, \qquad t > 0,\ x > 2 - 2\sqrt{t+1} \tag{7}$$
$$u|_{t=0} = 0, \qquad x > 0, \tag{8}$$
$$u_t|_{t=0} = 0, \qquad x > 0, \tag{9}$$
$$u|_{x=2-2\sqrt{t+1}} = t, \qquad t > 0. \tag{10}$$

Solution. (a) (2 pts) Solution to (7) is

$$u(x,t) = \phi(x+t) + \psi(x-t) \tag{11}$$

with unknown functions ϕ and ψ. Plugging into (8)–(9) we get

$$\phi(x) + \psi(x) = 0, \quad \phi'(x) - \psi'(x) = 0 \implies \phi(x) = \psi(x) = 0 \qquad x > 0$$

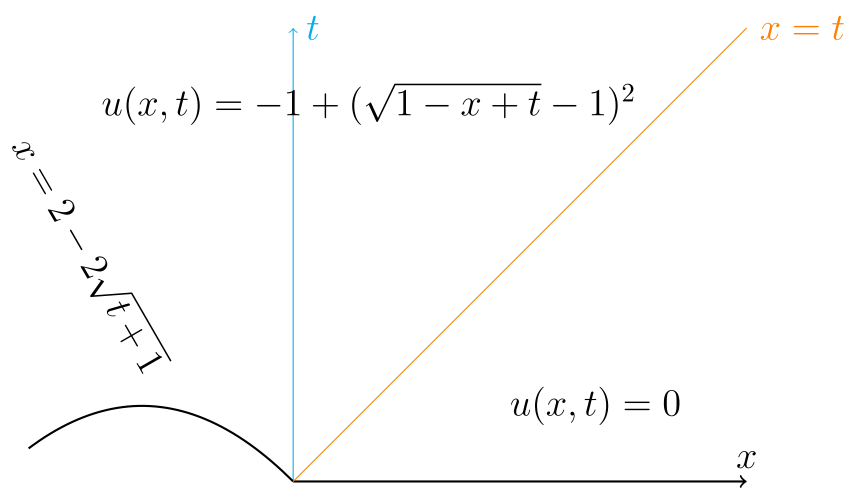
and $u(x,t) = \sin(x + 3t)$ as $x > t$.

(b) (2 pts) Plugging into (10) we get $\psi(2 - 2\sqrt{t+1} - t) = t$ as $t > 0$. For $t > 0$ $2 - 2\sqrt{t+1} - t$ is a monotone decreasing function, from 0 to $-\infty$. Solve:

$$x = -2 - 2\sqrt{t+1} - t \implies (\sqrt{t+1} + 1)^2 = -x \implies$$
$$\sqrt{t+1} = \sqrt{1-x} - 1 \implies t = -1 + (\sqrt{1-x} - 1)^2,$$

and therefore $\psi(x) = -1 + (\sqrt{1-x} - 1)^2$ for $x < 0$ and finally

$$u(x,t) = -1 + (\sqrt{1-x+t} - 1)^2 \qquad x < t.$$



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Problem 4 (4 pts)**.** Consider the PDE with boundary conditions:

$$u_{tt} + c^2 u_{xxxx} + au = 0, \qquad 0 < x < L, \tag{12}$$

$$u|_{x=0} = u_x|_{x=0} = 0, \tag{13}$$

$$u_{xx}|_{x=L} = u_{xxx}|_{x=0} = 0|_{x=L} = 0 \tag{14}$$

where $c > 0$ and a are constant. Prove that the energy $E(t)$ defined as

$$E(t) = \frac{1}{2}\int_0^L \bigl(u_t^2 + c^2 u_{xx}^2 + au^2\bigr)\,dx \tag{15}$$

does not depend on t.

Solution.

$$\partial_t E(t) = \int_0^L \bigl(u_t u_{tt} + c^2 u_{xxt} u_{xx} + a u_t u\bigr)\,dx$$

$$= \int_0^L c^2\bigl(-u_t u_{xxxx} + u_{xx} u_{xxt}\bigr)\,dx =$$

$$\int_0^L c^2\bigl(-u_t u_{xxxx} - u_{tx} u_{xxx} + u_{tx} u_{xxx} + u_{xx} u_{xxt}\bigr)\,dx =$$

$$\int_0^L c^2\Bigl(-(u_t u_{xxx})_x + (u_{tx} u_{xx})_x\Bigr)\,dx =$$

$$c^2\Bigl(-u_t u_{xxx} u_{tx} u_{xx}\Bigr)\Big|_{x=0}^{x=L} = 0$$

due to boundary conditions. ∎

Problem 5 (4 pts)**.** Find the solution $u(x,t)$ to

$$u_t = u_{xx} \qquad -\infty < x < \infty,\ t > 0, \tag{16}$$

$$u|_{t=0} = \begin{cases} x & |x| < 1, \\ 0 & |x| \geq 1 \end{cases} \tag{17}$$

$$\max |u| < \infty. \tag{18}$$

Calculate the integral.

Hint: For $u_t = ku_{xx}$ use $G(x,y,t) = \frac{1}{\sqrt{4\pi kt}} \exp(-(x-y)^2/4kt)$. To calculate integral make change of variables and use $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}\, dz$.

Solution. Due to hint

$$\begin{aligned}
u(x,t) =& \frac{1}{\sqrt{4\pi t}} \int_{-1}^{1} e^{-\frac{1}{4t}(x-y)^2} y\, dy = \\
& \frac{1}{\sqrt{4\pi t}} \int_{-1-x}^{(1-x)} e^{-\frac{s^2}{4t}} (x+s)\, ds = \\
& \frac{1}{\sqrt{\pi}} \int_{(-1-x)/\sqrt{4t}}^{(1-x)/\sqrt{4t}} e^{-z^2} (x + z\sqrt{4t})\, dz = \\
& x\Bigl(\operatorname{erf}((1-x)/\sqrt{4t}) + \operatorname{erf}((1+x)/\sqrt{4t}\Bigr)) - \\
& \frac{\sqrt{t}}{\sqrt{\pi}} \Bigl(e^{-(1+x)^2/4t)} - e^{-(1-x)^2/4t)}\Bigr).
\end{aligned}$$

□