Problem 1 (15 pts)**.** Solve by the method of characteristics the BVP for a wave equation

$$u_{tt} - 16u_{xx} = 0, \qquad 0 < x < \infty,\ t > 0 \tag{1.1}$$

$$u(x,0) = f(x), \tag{1.2}$$

$$u_t(x,0) = g(x), \tag{1.3}$$

$$(u_x - u)(0,t) = h(t) \tag{1.4}$$

with $f(x) = 4e^{-2x}$, $g(x) = 16e^{-2x}$ and $h(t) = e^{-8t}$. You need to find a continuous solution.

Problem 2 (15 pts)**.** Solve IVP for the heat equation

$$2u_t - u_{xx} = 0, \qquad 0 < x < \infty,\ t > 0, \tag{2.1}$$
$$u|_{x=0} = 0, \tag{2.2}$$
$$u|_{t=0} = f(x) \tag{2.3}$$

with $f(x) = e^{-x}$.

Solution should be expressed through $\operatorname{erf}(z) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^z e^{-z^2}\,dz.$

Solution. We can consider Cauchy problem with odd function $f(x) = \sigma(x)e^{-|x|}$, $\sigma(x) = \pm 1$ for $x \gtrless 0$. Thus

$$u(x,t) = \\ \frac{1}{\sqrt{2\pi t}}\int_0^\infty e^{-y}e^{-\frac{1}{2t}(y-x)^2}\,dy \qquad - \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^0 e^{y}e^{-\frac{1}{2t}(y-x)^2}\,dy = \\ \frac{1}{\sqrt{2\pi t}}\int_0^\infty e^{-x+\frac{t}{2}}e^{-\frac{1}{2t}(y+t-x)^2}\,dy \quad - \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^0 e^{x+\frac{t}{2}}e^{-\frac{1}{2t}(y-t-x)^2} =$$

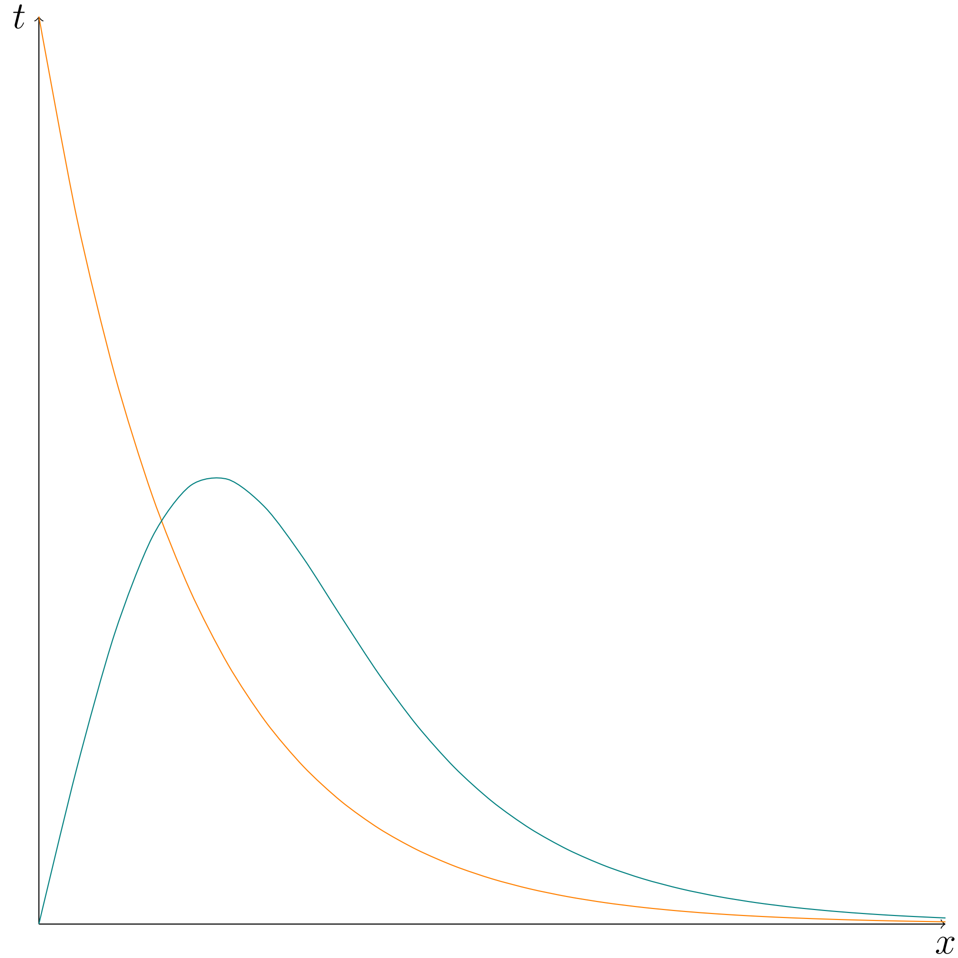
and setting $z \coloneqq \pm(y-x\pm t)/\sqrt{2t}$ in the first and second integrals respectively

$$\frac{1}{\sqrt{\pi}}e^{-x+\frac{t}{2}}\int_{\frac{t-x}{\sqrt{2t}}}^\infty e^{-z^2}\,dz \qquad - \frac{1}{\sqrt{\pi}}e^{x+\frac{t}{2}}\int_{\frac{t+x}{\sqrt{2t}}}^\infty e^{-z^2}\,dz,$$

and finally

$$u(x,t) = \frac{1}{2}\Bigl(1 - \operatorname{erf}(\frac{t-x}{\sqrt{2t}})\Bigr)e^{-x+\frac{t}{2}} - \frac{1}{2}\Bigl(1 - \operatorname{erf}(\frac{t+x}{\sqrt{2t}})\Bigr)e^{x+\frac{t}{2}}.$$

□



Continued

Problem 3 (15 pts)**.** Solve by the method of separation of variables

$$u_{tt} - u_{xx} + 4u = 0, \qquad 0 < x < \pi,\ t > 0, \tag{3.1}$$

$$u(0,t) = u(\pi,t) = 0, \tag{3.2}$$

$$u(x,0) = f(x), \tag{3.3}$$

$$u_t(x,0) = g(x) \tag{3.4}$$

with $f(x) = 0$ and $g(x) = x^2 - \pi x$. Write the answer in terms of Fourier series.

Solution. Separating variables $u(x,t) = X(x)T(t)$ we get

$$X'' + \lambda X = 0, \tag{3.5}$$

$$X(0) = X(\pi) = 0, \tag{3.6}$$

$$T'' + (\lambda + 4)T = 0. \tag{3.7}$$

Problem (3.5)–(3.6) has solution

$$\lambda_n = n^2, \qquad X_n = \sin(nx), \qquad n = 1, 2, \ldots \tag{3.8}$$

and therefore

$$T_n = A_n \cos((n^2+4)^{1/2}t) + B_n((n^2+4)^{1/2}t), \tag{3.9}$$

and

$$u = \sum_{n=1}^{\infty} \Bigl[A_n \cos((n^2+4)^{1/2}t) + B_n((n^2+4)^{1/2}t)\Bigr] \sin(nx). \tag{3.10}$$

Plugging to (3.3)–(3.4) we get

$$\sum_{n=1}^{\infty} A_n \sin(nx) = 0,$$

$$\sum_{n=1}^{\infty} (n^2+4)^{1/2} B_n \sin(nx) = x^2 - \pi x.$$

and therefore $A_n = 0$,

$$
\begin{aligned}
(n^2+4)^{1/2}B_n &= \frac{2}{\pi}\int_0^\pi (x^2-\pi x)\sin(nx)\,dx = \\
&-\frac{2}{\pi n}\int_0^\pi (x^2-\pi x)\,d\cos(nx) = \frac{2}{\pi n}\int_0^\pi (2x-\pi)\cos(nx)\,dx = \\
&\frac{2}{\pi n^2}\int_0^\pi (2x-\pi)\,d\sin(nx) = -\frac{4}{\pi n^2}\int_0^\pi \sin(nx)\,dx = \\
&\frac{4}{\pi n^3}\cos(nx)\Big|_{x=0}^{x=\pi} = \begin{cases} 0 & n=2m,\\ -\dfrac{8}{\pi(2m+1)^3} & n=2m+1.\end{cases}
\end{aligned}
$$

Then

$$
u(x,t) = -\sum_{m=0}^{\infty} \frac{8}{(2m+1)^3\pi\sqrt{(2m+1)^2+4}} \sin((2m+1)x)\sin(\sqrt{(2m+1)^2+4}t).
$$

$\square$

Problem 4 (15 pts)**.** Consider the Laplace equation in the sector

$$u_{xx} + u_{yy} = 0 \qquad \text{in } \frac{1}{4} \le x^2 + y^2 < 4, y > 0, \tag{4.1}$$

with the boundary conditions

$$u = 1 \qquad \text{for } x^2 + y^2 = 4, \tag{4.2}$$

$$u = -1 \qquad \text{for } x^2 + y^2 = \frac{1}{4}, \tag{4.3}$$

$$u = 0 \qquad \text{for } y = 0, \tag{4.4}$$

where θ is a polar angle.

(a) Look for solutions u in the form of $u(r,\theta) = R(r)P(\theta)$ (in polar coordinates) and derive a set of ordinary differential equations for R and P. Write the correct boundary conditions for P.

(b) Solve the eigenvalue problem for P and find all eigenvalues.

(c) Solve the differential equation for R.

(d) Find the solution u of (4.1)–(4.3).

Solution. In polar coordinates $\{y = 0\}$ is $\theta = 0, \pi$.

Separating variables we get

$$\frac{r^2R'' + rR'}{R} + \frac{P''}{P} = 0 \implies P'' + \lambda P = 0, \tag{4.5}$$

$$P(0) = P(\pi) = 0, \tag{4.6}$$

$$r^2R'' + rR' + \lambda R = 0. \tag{4.7}$$

This problem has solutions $\lambda_n = n^2$, $X_n = \sin(n\theta)$, $n = 1, 2, \ldots$.

Then $r^2R'' + rR' + n^2R = 0 \implies R_n = A_n r^n + B_n r^{-n}$. Then

$$u = \sum_{n=1}^{\infty} \bigl(A_n r^n + B_n r^{-n}\bigr)\sin(n\theta)$$

and using (4.2), (4.3)

$$u|_{r=2} = \sum_{n=1}^{\infty} (2^n A_n + 2^{-n} B_n)\sin(n\theta) = 1,$$

$$u|_{r=\frac{1}{2}} = \sum_{n=1}^{\infty} (2^{-n} A_n + 2^n B_n)\sin(n\theta) = -1$$

which implies

$$\sum_{n=1}^{\infty}(2^n+2^{-n})(A_n+B_n)\sin(n\theta)=0 \implies (A_n+B_n)=0$$

$$\sum_{n=1}^{\infty}(2^n-2^{-n})(A_n-B_n)\sin(n\theta)=2 \implies (2^n-2^{-n})(A_n-B_n)=$$

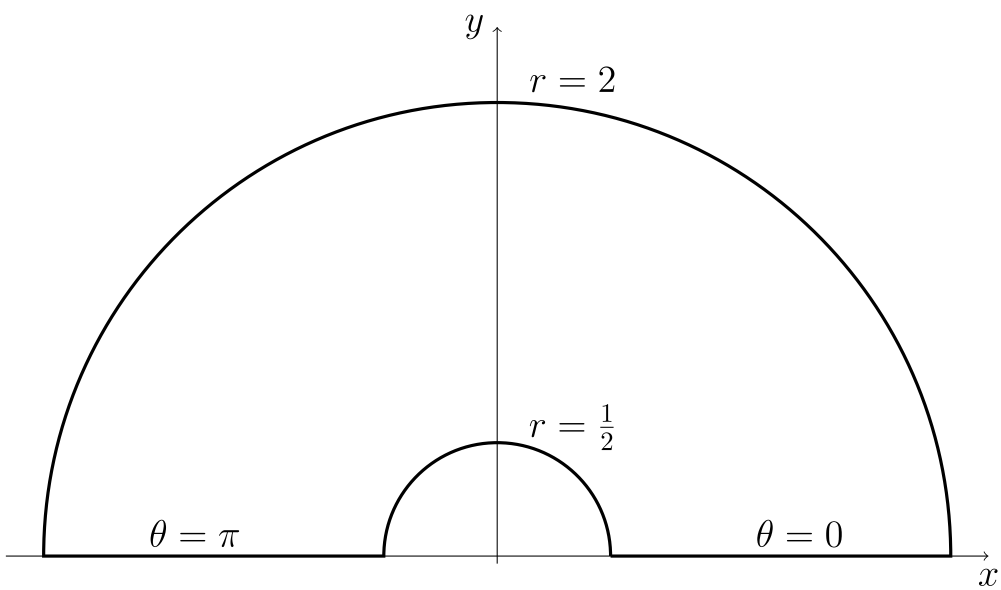
$$\frac{4}{\pi}\int_0^{\pi}\sin(n\theta)\,d\theta=\begin{cases}\dfrac{8}{\pi(2m+1)} & n=2m+1,\\ 0 & n=2m\end{cases}$$

which implies $A_{2m}=B_{2m}=0$ and

$$A_{2m+1}=-B_{2m+1}=\frac{4}{\pi(2m+1)(2^{2m+1}-2^{-2m-1})}$$

Finally

$$u=\sum_{m=0}^{\infty}\frac{4}{\pi(2m+1)(2^{2m+1}-2^{-2m-1})}(r^{2m+1}-r^{-2m-1})\sin((2m+1)\theta).$$



□

Problem 5 (15 pts)**.** Consider Laplace equation in the half-strip

$$u_{xx} + u_{yy} = 0 \qquad y > 0,\ 0 < x < \frac{\pi}{2} \tag{5.1}$$

with the boundary conditions

$$u_x(0, y) = u_x(\frac{\pi}{2}, y) = 0, \tag{5.2}$$

$$(u_y - u)(x, 0) = g(x) \tag{5.3}$$

with $g(x) = 1$ and condition $\max |u| < \infty$.

(a) Write the associated eigenvalue problem.

(b) Find all eigenvalues and corresponding eigenfunctions.

(c) Write the solution in the form of a series expansion.

Solution. Separating variables $u = X(x)Y(y)$ we get

$$\frac{X''}{X} + \frac{Y''}{Y} = 0 \implies X'' + \lambda X = 0, \tag{5.4}$$

$$Y'' - \lambda Y = 0, \tag{5.5}$$

$$X(0) = X(\frac{\pi}{2}) = 0. \tag{5.6}$$

Solving (5.4), (5.6) we have

$$\lambda_n = 4n^2, \qquad X_n = \sin(2nx) \qquad n = 1, 2, 3, \ldots,$$

and then solving (5.5) we get

$$Y_n A_n e^{-2ny} + B_n e^{2ny} \qquad n = 1, 2, 3, \ldots,$$

and the last term as growing for $y > 0$ we need to drop:

$$Y_n = A_n e^{-2ny} \qquad n = 1, 2, 3, \ldots.$$

Then

$$u(x, y) = \sum_{n=1}^{\infty} A_n e^{-2ny} \sin(2nx). \tag{5.7}$$

Plugging to (5.3) we get

$$\sum_{n=1}^{\infty} A_n(-1 - 2n) \sin(2nx) = 1$$

and

$$A_n = -\frac{4}{(2n+1)\pi}\int_0^{\frac{\pi}{2}} \sin(2nx)\,dx = \frac{2}{n(2n+1)\pi}\cos(2nx)\Big|_{x=0}^{x=\frac{\pi}{2}} =$$

$$\frac{2}{n(2n+1)\pi}\bigl(\cos(n\pi)-1\bigr) = \begin{cases} 0 & n = 2m, \\ -\dfrac{4}{(2m+1)(4m+3)\pi} & n = 2m+1 \quad m = 0, 1, \dots \end{cases}$$

Then

$$u(x,y) = -\sum_{n=1}^{\infty} \frac{4}{(2m+1)(4m+3)\pi} e^{-(4m+2)y}\cos((4m+2)x). \tag{5.8}$$

□

Problem 6 (15 pts)**.** Solve as $t > 0$

$$u_{tt} - \Delta u = 0, \tag{6.1}$$

with initial conditions

$$u(x,y,z,0) = 0, \quad u_t(x,y,z,0) = \begin{cases} r^{-1}\sin(r) & r := \sqrt{x^2+y^2+z^2} < \pi, \\ 0 & r \ge \pi, \end{cases} \tag{6.2}$$

and solve by a separation of variables.

Hint. Use spherical coordinates, observe that solution must be spherically symmetric: $u = u(r,t)$ (explain why).

Also, use equality

$$ru_{rr} + 2u_r = (ru)_{rr}. \tag{6.3}$$

Solution. Solution is spherically symmetric because the problem is. Then

$$u_{tt} - \bigl(u_{rr} + \frac{2}{r}u_r\bigr) = 0 \qquad r > 0, t > 0. \tag{6.4}$$

Multiplying by r and using (6.3) we arrive to the first equation below:

$$v_{tt} - v_{rr} = 0 \qquad r > 0, \tag{6.5}$$

$$v(0,t) = 0, \tag{6.6}$$

$$v(r,0) = 0, \qquad v_t(r,0) = h(r) = \begin{cases} \sin(r) & 0 < r < \pi, \\ 0 & r \ge \pi, \end{cases} \tag{6.7}$$

Continuing $h(r)$ as and odd function $\tilde{h}(r) = \begin{cases} \sin(r) & -\pi < r < \pi, \\ 0 & |r| \ge \pi, \end{cases}$ and solving Cauchy problem we get

$$v(r,t) = \frac{1}{2}\int_{r-t}^{r+t} \tilde{h}(z)\,dz$$

which is 0 for $r > |t| + \pi$ and for $0 < r < |t| - \pi$. Otherwise it is

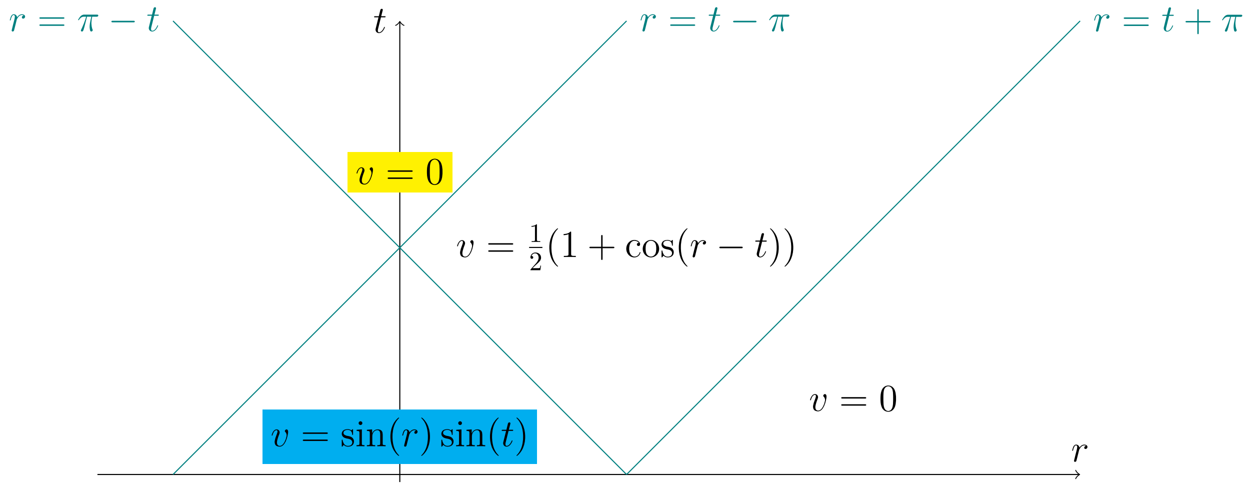
$$v(r,t) = \frac{1}{2}\int_{z_-}^{z_+} \sin(z)\,dz = \frac{1}{2}\bigl(\cos(z_-) - \cos(z_+)\bigr) = \sin(\frac{1}{2}(z_+ + z_-))\sin(\frac{1}{2}(z_+ - z_-));$$

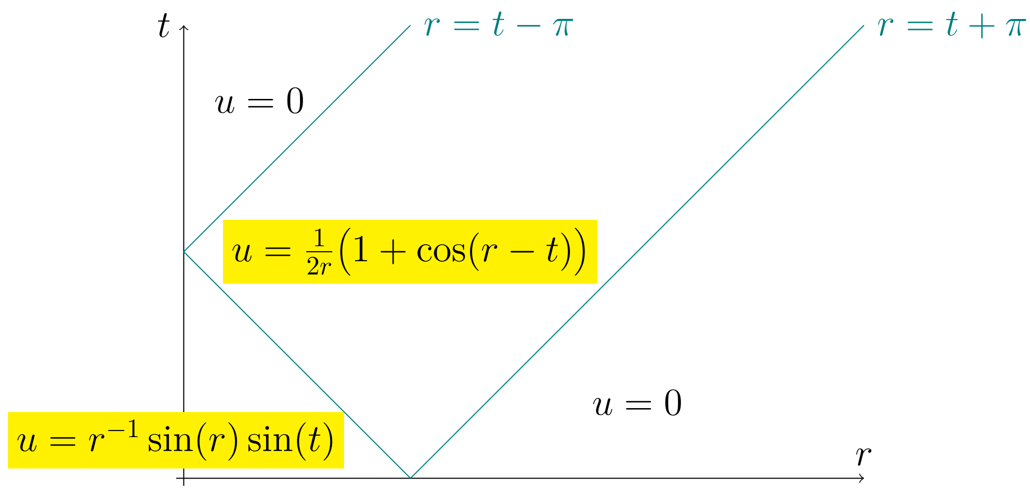
$z_- = \max(r-t, -\pi)$, $z_+ = \min(r+t, \pi)$. Then, for $r > 0$, $t > 0$, we have

$$v(r,t) = \begin{cases} 0 & r > t + \pi, \\ \frac{1}{2}\bigl(1 - \cos(r-t)\bigr) & t - \pi < r < t + \pi, r + t > \pi, \\ \sin(r)\sin(t) & r + t < \pi, \\ 0 & 0 < r < t - \pi. \end{cases}$$

and, finally,

$$u(r,t) = r^{-1}v(r,t) = \begin{cases} 0 & r > t+\pi, \\ \dfrac{1}{2}r^{-1}\bigl(1+\cos(r-t)\bigr) & t-\pi < r < t+\pi, r+t > \pi, \\ r^{-1}\sin(r)\sin(t) & r+t < \pi, \\ 0 & 0 < r < t-\pi. \end{cases}$$





□

Problem 7 (15 pts)**.** Solve using (partial) Fourier transform with respect to y

$$\begin{aligned}
&\Delta u := u_{xx} + u_{yy} = 0, && x > 0, && (7.1)\\
&u_x|_{x=0} = h(y), && && (7.2)\\
&u \to 0 && \text{as } \ x \to +\infty && (7.3)
\end{aligned}$$

with $h(y) = \frac{4y}{(y^2+1)^2}$.

HINT. Fourier transform of $g(y) = \frac{2}{y^2+1}$ is $\hat{g} = e^{-|\eta|}$ and $h(y) = -g'(y)$.

Solution. Making Fourier transform we get

$$\begin{aligned}
&\hat{u}_{xx} - \eta^2 \hat{u} = 0, && x > 0, && (7.4)\\
&\hat{u}|_{x=0} = \hat{h}(\eta), && && (7.5)\\
&\hat{u} \to 0 && \text{as } \ x \to +\infty && (7.6)
\end{aligned}$$

and solving (7.4) we see that $\hat{u} = A(\eta)e^{-|\eta|x} + B(\eta)e^{|\eta|x}$; (7.6) implies that $B(\eta) = 0$ and (7.5) implies then $A(\eta) = -|\eta|^{-1}\hat{h}(\eta)$.

Due to hint and properties of Fourier transform $\hat{h}(\eta) = -i\eta\hat{g}(\eta)$.

Then $\hat{u}(x,\eta) = i\sigma(\eta)e^{-|\eta|(1+x)}$ with $\sigma(\eta) = \pm 1$ for $\eta \gtrless 0$, and

$$\begin{aligned}
u(x,y) &= \int_{-\infty}^{\infty} \hat{u}(x,\eta)e^{i\eta y}\, d\eta = i\int_{-\infty}^{\infty} \sigma(y)e^{-|\eta|x + i\eta y}\, d\eta\\
&= -\, i\int_{-\infty}^{0} e^{\eta(1+x+yi)}\, d\eta + i\int_{0}^{-\infty} e^{-\eta(1+x-yi)}\, d\eta\\
&= -\frac{i}{1+x+yi} + \frac{i}{1+x-yi} = -\frac{2y}{(1+x)^2+y^2}.
\end{aligned}$$

□

Appendix: Some useful formulas.
Not exam problems.
You may detach this page

1. The $2D$ Laplacian in polar coordinates and $3D$ Laplacian in spherical coordinates:

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}\,,$$

$$\Delta f = \frac{\partial^2 f}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{1}{\rho^2}\cot(\phi)\frac{\partial f}{\partial \phi} + \frac{1}{\rho^2 \sin^2(\phi)}\frac{\partial^2 f}{\partial \theta^2}\,.$$

2. The n-dimensional Stokes theorem

$$\int_D \frac{\partial f}{\partial x_i}\,dx = \int_{\partial D} f\nu_i\,d\sigma\,,$$

where ν (with components ν_i) is the unit normal vector pointing outside.

3. The complex Fourier series of a periodic function $f(x)$ of period $2l$, defined on the interval $(-l,l)$ is

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{\pi i n x/l}$$

with the coefficients c_n given by the formula

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-\pi i n x/l}dx\,.$$

4. The Fourier transform of a function $f(x)$ is defined by

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx\,.$$

The inverse Fourier transform is

$$f(x) = \int_{-\infty}^{\infty} e^{ikx}\hat{f}(k)dk\,.$$

Here some of its properties:

Continued

(a) if $g(x) = f(ax)$, then $\hat{g}(k) = \frac{1}{|a|}\hat{f}(\frac{k}{a})$;

(b) $\widehat{f'}(k) = ik\hat{f}(k)$;

(c) if $g(x) = xf(x)$ then $\hat{g}(k) = -i\hat{f}\,'(k)$;

(d) if $g(x) = f(x-a)$, then $\hat{g}(k) = e^{-iak}\hat{f}(k)$;

(e) if $g(x) = f(x)e^{ixb}$, then $\hat{g}(k) = \hat{f}(k-b)$;

(f) if $h = f * g$, then $\hat{h}(k) = 2\pi\hat{f}(k)\hat{g}(k)$;

(g) if $f(x) = e^{-x^2/2}$, then $\hat{f}(k) = \frac{1}{\sqrt{2\pi}}e^{-k^2/2}$.