Problem 1 (4pt)**.** Solve by Fourier method

$$u_{tt} - u_{xx} = 0 \qquad 0 < x < \pi, \tag{1.1}$$
$$u|_{x=0} = -u|_{x=\pi}, \qquad u_x|_{x=0} = -u_x|_{x=\pi}, \tag{1.2}$$
$$u|_{t=0} = 1, \qquad u_t|_{t=0} = 0. \tag{1.3}$$

HINT: $\lambda_n \ge 0$. Also remember how solution looks like in the case of double eigenvalues.

Solution. Separation of variables leads to

$$X'' + \lambda X = 0, \tag{1.4}$$
$$X(0) = -X(\pi), \qquad X'(0) = -X'(\pi), \tag{1.5}$$
$$T'' + \lambda T = 0 \tag{1.6}$$

We know that for this BVP $\lambda \ge 0$, obviously $\lambda = 0 \implies X = A + Bx$ and boundary conditions imply $A = 0, B = 0$.
Consider $\lambda = k^2 > 0$; then $X = A\cos(kx) + B\sin(kx)$. Then (1.2) implies

$$\begin{cases} A = -A\cos(k\pi) - B\sin(k\pi) \\ kB = kA\sin(k\pi) - kB\cos(k\pi) \end{cases} \implies \begin{cases} A(1+\cos(k\pi)) + B\sin(k\pi) = 0 \\ -A\sin(k\pi) + B(1+\cos(k\pi)) = 0 \end{cases}$$

with determinant $(1+\cos(k\pi))^2 + \sin^2(k\pi)$ which is 0 iff $k\pi = (2n+1)\pi$ with $n = 0, 1, \ldots$, in which case we have two independent solutions $\cos((2n+1)x)$ and $\sin((2n+1)x)$.
Meanwhile (1.6) implies $T = A\cos((2n+1)t) + B\sin((2n+1)t)$ and taking into account Hint

$$u(x,t) = \sum_{n=0}^{\infty} \Big[\big(A_n \cos((2n+1)t) + C_n \sin((2n+1)t)\big)\cos((2n+1)x) + \big(B_n \cos((2n+1)t) + D_n \sin((2n+1)t)\big)\sin((2n+1)x) \Big]$$

and initial conditions imply

$$u(0,t) = \sum_{n=0}^{\infty} \Big[A_n \cos((2n+1)x) + C_n \sin((2n+1)x)\Big] = 1$$
$$u_t(0,t) = \sum_{n=0}^{\infty} (2n+1)\Big[B_n \cos((2n+1)x) + D_n \sin((2n+1)x)\Big] = 0$$

and then $B_n = D_n = 0$,

$$A_n = \frac{2}{\pi}\int_0^{\pi} \cos((2n+1)x)\,dx = 0,$$
$$C_n = \frac{2}{\pi}\int_0^{\pi} \cos((2n+1)x)\,dx = \frac{4}{(2n+1)\pi}$$

and

$$u(x,t) = \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi} \cos((2n+1)t)\sin((2n+1)x) \tag{1.7}$$

□

Problem 2 (4pt)**.** Solve

$$u_{xx} + u_{yy} = 0 \qquad -\infty < x < \infty,\ 0 < y < \infty, \tag{2.1}$$

$$u|_{y=0} = g(x) = \begin{cases} 1 & |x| < 1, \\ 0 & |x| > 1 \end{cases} \tag{2.2}$$

$$\max |u| < \infty. \tag{2.3}$$

HINT: Use partial Fourier transform with respect to x. Write solution as a Fourier integral without calculating it.

Solution. After partial Fourier transform

$$-k^2\hat{u} + \hat{u}_{yy} = 0 \qquad 0 < y < \infty, \tag{2.4}$$

$$\hat{u}|_{y=0} = \hat{g}(k). \tag{2.5}$$

One can calculate easily $\hat{g}(k) = \frac{\sin(k)}{\pi k}$.
Solving (2.5) we get $\hat{u} = A(k)e^{-|k|y} + B(k)e^{|k|y}$ and $B(k) = 0$ due to (2.3) and $A(k) = \frac{\sin(k)}{\pi k}$,

$$\hat{u} = \frac{\sin(k)}{\pi k} e^{-|k|y}. \tag{2.6}$$

and

$$u(x, y) = \int_{-\infty}^{\infty} \frac{\sin(k)}{\pi k} e^{-|k|y + ikx}\, dk = 2\int_0^{\infty} \frac{\sin(k)}{\pi k} e^{-ky} \cos(kx)\, dk. \tag{2.7}$$

□

Problem 3 (4pt)**.** Using Fourier method find eigenvalues and eigenfunctions of Laplacian in the rectangle $\{0 < x < a, 0 < y < b\}$ with Neumann boundary conditions:

$$u_{xx} + u_{yy} = -\lambda u \qquad 0 < x < a,\ 0 < y < b, \tag{3.1}$$

$$u_x|_{x=0} = u_x|_{x=a} = u_y|_{y=0} = u_y|_{y=b} = 0. \tag{3.2}$$

Solution. Separating variables $u = X(x)Y(y)$ we arrive to

$$\frac{X''}{X} + \frac{Y''}{Y} + \lambda = 0 \tag{3.3}$$

Then

$$X'' + \mu X = 0, \tag{3.4}$$

$$X'(0) = X'(a) = 0 \tag{3.5}$$

and

$$Y'' + \nu X = 0, \tag{3.6}$$

$$Y'(0) = Y'(b) = 0 \tag{3.7}$$

and

$$\lambda = \mu + \nu. \tag{3.8}$$

Next

$$\mu_m = \frac{\pi^2 m^2}{a^2}, \qquad X_m = \cos\bigl(\frac{\pi m x}{a}\bigr), \qquad m = 1, 2, \ldots, \tag{3.9}$$

$$\nu_n = \frac{\pi^2 n^2}{b^2}, \qquad Y_n = \cos\bigl(\frac{\pi n y}{b}\bigr), \qquad n = 1, 2, \ldots, \tag{3.10}$$

and finally

$$\lambda_{mn} = \pi^2\Bigl(\frac{m^2}{a^2} + \frac{n^2}{b^2}\Bigr), \quad u_{mn} = \cos\bigl(\frac{\pi m x}{a}\bigr)\cos\bigl(\frac{\pi n y}{b}\bigr), \quad m, n = 0, 1, 2 \ldots \tag{3.11}$$

□

Problem 4 (4pt)**.** Consider Laplace equation in the sector

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \qquad r < 8,\ 0 < \theta < \frac{3}{2}\pi \tag{4.1}$$

with the Dirichlet boundary conditions as $\theta = 0$ and $\theta = \frac{3}{2}\pi$

$$u|_{\theta=0} = u|_{\theta=\frac{3}{2}\pi} = 0 \tag{4.2}$$

and the Dirichlet boundary condition as $r = 8$

$$u|_{r=8} = 1. \tag{4.3}$$

Using separation of variables find solution as a series.

Solution. Separating variables $u(r,\theta) = R(r)\Theta(\theta)$ we get

$$\frac{r^2R'' + rR'}{R} + \frac{\Theta''}{\Theta} = 0$$

and therefore both terms are constant:

$$\Theta'' + \lambda\Theta = 0, \tag{4.4}$$

$$\Theta(0) = \Theta(\frac{3}{2}\pi) = 0 \tag{4.5}$$

and therefore $\lambda_n = \frac{4n^2}{9}$, $\Theta_n = \sin(\frac{2n\theta}{3})$, $n = 1, 2, \ldots$. Then

$$r^2R'' + rR' - \frac{4n^2}{9}R = 0 \tag{4.6}$$

and $R = Ar^{2n/3} + Br^{-2n/3}$ where we drop the last term as it is singular at $r = 0$. So $u_n = A_n r^{2n/3}\sin(\frac{2n}{3}\theta)$ and

$$u = \sum_{n=1}^{\infty} A_n r^{2n/3}\sin(\frac{2n\theta}{3}). \tag{4.7}$$

Plugging into (4.3) we get

$$\sum_{n=1}^{\infty} A_n 2^{2n}\sin(\frac{2n\theta}{3}) = 1; \tag{4.8}$$

then

$$A_n = 2^{-2n}\frac{4}{3\pi}\int_0^{3\pi/2}\sin(\frac{2n\theta}{3})\,d\theta = \begin{cases} 0 & n = 2m, \\ \dfrac{2^{2-2n}}{(2m+1)\pi} & n = 2m+1 \end{cases} \tag{4.9}$$

and

$$u = \sum_{m=0}^{\infty}\frac{2^{2-4m}}{(2m+1)}r^{2(2m+1)/3}\sin(\frac{2}{3}(2m+1)\theta). \tag{4.10}$$

□

Problem 5 (4pt)**.** Find Fourier transforms of the functions

$$f_\pm(x) = e^{-\varepsilon|x|}\theta(\pm x) \tag{5.1}$$

and write these function as a Fourier integrals, where θ is a Heaviside function: $\theta(t) = 1$ for $t > 0$ and $\theta(t) = 0$ for $t < 0$.

BONUS (1pt). Write Fourier transforms of the functions $g(x) = f_+(x)+f_-(x)$ and $h(x) = f_+(x) - f_-(x)$.

Solution. The simplest:

$$\begin{aligned}\hat{f}_+(k) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx = \\ & \frac{1}{2\pi}\int_0^{\infty} e^{-ikx-\varepsilon x}\,dx = \\ & \frac{1}{2\pi(\varepsilon+ik)} = \frac{1}{2\pi i(k-i\varepsilon)}.\end{aligned}$$

Then since $f_-(x) = f_+(-x)$ we conclude that $\hat{f}_-(k) = \hat{f}_+(-k) = -\frac{1}{2\pi i(k+i\varepsilon)}$. Conversely

$$f_\pm(x) = \pm\int_{-\infty}^{\infty} \frac{1}{2\pi i(k\mp i\varepsilon)}e^{ikx}\,dk.$$

□

Bonus Solution.

$$\hat{g}(k) = \hat{f}_+(k) + \hat{f}_-(k) = \frac{\varepsilon}{\pi(k^2+\varepsilon^2)}, \tag{5.2}$$

$$\hat{h}(k) = \hat{f}_+(k) - \hat{f}_-(k) = \frac{k}{i\pi(k^2+\varepsilon^2)}. \tag{5.3}$$

□