Problem 1 (4 pts)**.** Consider the first order equation:

$$u_t + xtu_x = -u. \tag{1.1}$$

(a) Find the characteristic curves and sketch them in the (x,t) plane.

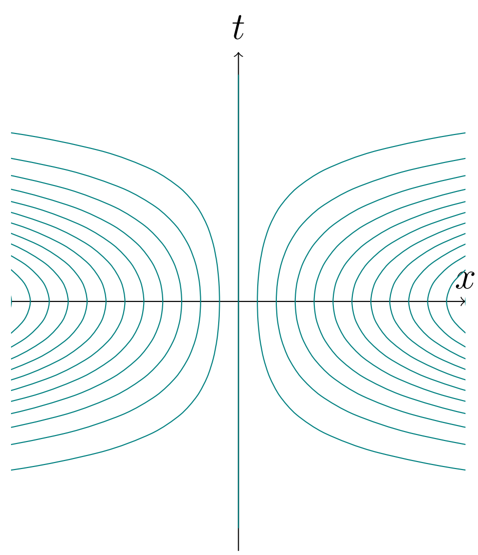
(b) Write the general solution.

(c) Solve equation (1.1) with the initial condition $u(x,0) = (x^2+1)^{-1}$. Explain why the solution is fully determined by the initial condition.

Solution. (a)–(b) Equation of characteristics

$$\frac{dt}{1} = \frac{dx}{xt} = -\frac{du}{u} \implies xe^{-t^2/2} = C_1,\ C_2 = ue^t \implies u = f(xe^{-t^2/2})e^{-t} \tag{1.2}$$

with arbitrary function f.



(c) From initial condition we conclude that $f(x) = (x^2+1)^{-1}$ and

$$u(x,t) = \frac{e^{-t}}{x^2e^{-t^2}+1}. \tag{1.3}$$

□

Problem 2 (4 pts)**.** (a) (4 pts) Find solution $u(x,t)$ to

$$u_{tt} - u_{xx} = (x^2-1)e^{-\frac{x^2}{2}}, \tag{2.1}$$

$$u|_{t=0} = -e^{-\frac{x^2}{2}}, \quad u_t|_{t=0} = 0. \tag{2.2}$$

(b) (1 pts–bonus) Find $\lim_{t\to+\infty} u(x,t)$.

Solution. (a) By D'Alembert formula

$$u(x,t) = -\frac{1}{2}e^{-(x+t)^2/2} - \frac{1}{2}e^{-(x-t)^2/2} + \underbrace{\frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} (\xi^2-1)e^{-\frac{\xi^2}{2}}\,d\xi d\tau}; \tag{2.3}$$

the inner integral is

$$-\int \xi\, de^{-\frac{\xi^2}{2}} - \int e^{-\frac{\xi^2}{2}}\,d\xi = -\xi e^{-\frac{\xi^2}{2}}$$

(we integrated by parts and cancelled integrals) and therefore the second term in the right-hand expression of (2.3) is

$$-\frac{1}{2}\int_0^t \Bigl[(x+t-\tau)e^{-\frac{1}{2}(x+t-\tau)^2} - (x-t+\tau)e^{-\frac{1}{2}(x-t+\tau)^2}d\tau\Bigr]\,d\tau$$
$$= \frac{1}{2}\Bigl[e^{-\frac{1}{2}(x+t-\tau)^2} + e^{-\frac{1}{2}(x-t+\tau)^2}\Bigr]\Bigr|_{\tau=0}^{\tau=t} = \frac{1}{2}\Bigl[e^{-\frac{1}{2}(x+t)^2} + e^{-\frac{1}{2}(x-t)^2}\Bigr] - e^{-\frac{1}{2}x^2}$$

and finally

$$u(x,t) = -e^{-\frac{1}{2}x^2}. \tag{2.4}$$

(b) As $t\to+\infty$ we have to first terms in the right-hand expression of (2.4) tending to 0 and

$$\lim_{t\to+\infty} u(x,t) = -e^{-\frac{1}{2}x^2} \tag{2.5}$$

□

Problem 3 (4 pts). Find solution to

$$u_{tt} - 9u_{xx} = 0, \qquad 0 < t < x, \tag{3.1}$$

$$u|_{t=0} = \sin(x), \qquad x > 0, \tag{3.2}$$

$$u_t|_{t=0} = 3\cos(x), \qquad x > 0, \tag{3.3}$$

$$u|_{x=t} = 0, \qquad t > 0. \tag{3.4}$$

Solution. (a) (2 pts) Solution to (3.1) is

$$u(x,t) = \phi(x+3t) + \psi(x-3t) \tag{3.5}$$

with unknown functions ϕ and ψ. Plugging into (3.2)–(3.3) we get

$$\phi(x) + \psi(x) = \sin(x), \quad 3\phi'(x) - 3\psi'(x) = 3\cos(x) \implies \phi(x) - \psi(x) = \sin(x)$$

as ϕ, ψ defined up to constants C and $-C$, and then

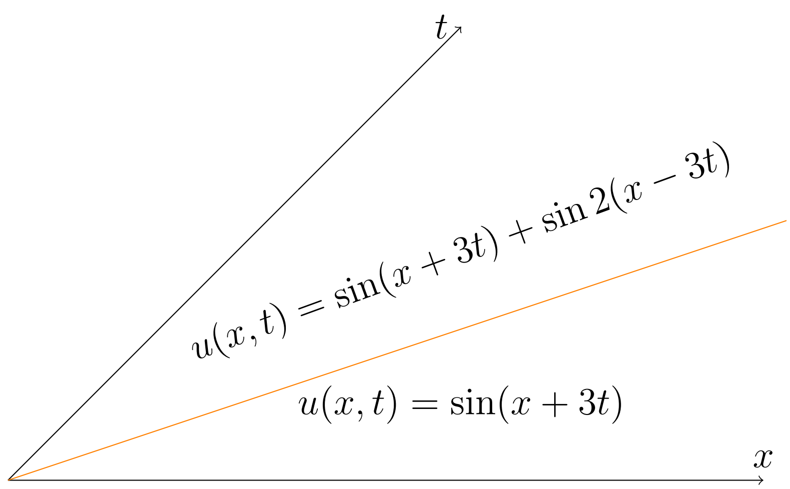
$$\phi(x) = \sin(x), \quad \psi(x) = 0 \qquad \text{as} \ \ x > 0$$

and $u(x,t) = \sin(x+3t)$ as $x > 3t$.

(b) (2 pts) Plugging into (3.4) we get $\phi(4t) + \psi(-2t) = 0$ as $t > 0$ or

$$\phi(-2x) + \psi(x) = 0 \implies \psi(x) = -\phi(-2x) = \sin(2x) \qquad x < 0$$

and $u(x,t) = \sin(x+3t) + \sin 2(x-3t)$ as $t < x < 3t$.



□

Problem 4 (4 pts)**.** Consider the PDE with boundary conditions:

$$u_{tt} - c^2 u_{xx} = 0, \qquad 0 < x < L, \tag{4.1}$$
$$(u_x - \alpha u_{tt})|_{x=0} = 0, \tag{4.2}$$
$$(u_x + \beta u_{tt})|_{x=L} = 0 \tag{4.3}$$

where $c > 0$ and $\alpha > 0$ are constant. Prove that the energy $E(t)$ defined as

$$E(t) = \frac{1}{2}\int_0^L (u_t^2 + c^2 u_x^2)\,dx + c^2\frac{\alpha}{2}u_t(0,t)^2 + c^2\frac{\beta}{2}u_t(L,t)^2 \tag{4.4}$$

does not depend on t.

Solution.

$$c^{-2}\partial_t E(t) = \int_0^L \bigl(c^{-2}u_t u_{tt} + u_x u_{xt}\bigr)\,dx + \alpha u_t u_{tt}\bigr|_{x=0} + \beta u_t u_{tt}\bigr|_{x=L}$$
$$= \int_0^L \bigl(u_t u_{xx} + u_x u_{xt}\bigr)\,dx + \alpha u_t u_{tt}|_{x=0} + \beta u_t u_{tt}\bigr|_{x=L} = u_t u_x\bigr|_{x=0}^{x=L} + \alpha u_t u_{tt}\bigr|_{x=0}$$
$$= u_t(-u_x + \alpha u_{tt})\bigr|_{x=0} + u_t(u_x + \beta u_{tt})\bigr|_{x=L} = 0$$

due to boundary conditions. □