University of Toronto, Faculty of Arts and Science

Final Examinations, December 13, 2016, 19:00–22:00, EX 200

APM346 — Partial Differential Equations, Section L5101

Instructor: Prof. Victor Ivrii

Duration — 3 hours

The 7 problems are independent. Total marks for this paper: 105.

The paper constitutes 40% of the final mark.

A list of useful formulas is attached in the last page. No other aids allowed.

Last name	
First name	
ID	

#	points	Mark
1	[15]	
2	[15]	
3	[15]	
4	[15]	
5	[15]	
6	[15]	
7	[15]	
Total	[105]	

Term mark	
Final Mark	

Problem 1 (15 pts)**.** Solve by the method of characteristics the BVP for a wave equation

$$u_{tt} - 9u_{xx} = 0, \qquad 0 < x < \infty,\ t > 0 \tag{1.1}$$
$$u(x,0) = f(x), \tag{1.2}$$
$$u_t(x,0) = g(x), \tag{1.3}$$
$$u_x(0,t) = h(t) \tag{1.4}$$

with $f(x) = 4\cos(x)$, $\quad g(x) = 6\sin(x)$ and $h(t) = \sin(3t)$. You need to find a continuous solution.

Solution. From (1.1):

$$u(x,t) = \phi(x+3t) + \psi(x-3t). \tag{1.5}$$

Plugging to (1.2)–(1.3) we conclude that for $x > 0$ $\phi(x) + \psi(x) = 4\cos(x)$, $3\phi'(x) - 3\psi'(x) = 6\sin(x)$; integrating the second equation we get $\phi(x) - \psi(x) = -2\cos(x)$ (since we can select constant equal to 0 here) and finally

$$\phi(x) = \cos(x), \quad \psi(x) = 3\cos(x) \qquad \text{as} \quad x > 0. \tag{1.6}$$

We need to find $\psi(x)$ as $x < 0$. Plugging (1.5) into (1.4) we conclude that $\phi'(3t) + \psi'(-3t) = \sin(3t)$ as $t > 0$ and plugging $x = -3t$ we see that $\psi'(x) = -\sin(x) - \psi'(-x) = -2\sin(x)$ and therefore $\psi(x) = 2\cos(x) + C$ as $x < 0$.

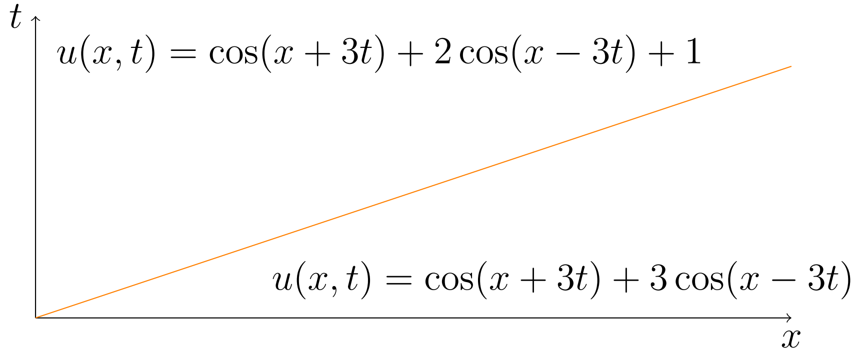
Since $\psi(+0) = 3$, $\psi(-0) = 2 + C$ we need for continuity $C = 1$. So

$$\psi(x) = 2\cos(x) + 1 \qquad \text{as} \quad x > 0. \tag{1.7}$$

Finally,

$$u(x,t) = \begin{cases} \cos(x+3t) + 3\cos(x-3t) & x > 3t > 0, \\ \cos(x+3t) + 2\cos(x-3t) + 1 & 0 < x < 3t. \end{cases} \tag{1.8}$$

□


t
$u(x,t) = \cos(x+3t) + 2\cos(x-3t) + 1$
$u(x,t) = \cos(x+3t) + 3\cos(x-3t)$
x

Problem 2 (15 pts)**.** Solve IVP for the heat equation

$$4u_t - u_{xx} = 0, \qquad 0 < x < \infty,\ t > 0, \tag{2.1}$$
$$u|_{x=0} = 0, \tag{2.2}$$
$$u|_{t=0} = f(x) \tag{2.3}$$

with $f(x) = xe^{-x^2}$.
Solution should be expressed through $\operatorname{erf}(z) = \dfrac{2}{\sqrt{\pi}} \int_0^z e^{-z^2}\,dz$.

Solution. We can safely ignore boundary condition and consider Cauchy problem. Indeed, xe^{-x^2} is an odd function. Thus

$$\begin{aligned}
u(x,t) =& \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{\infty} ye^{-y^2} e^{-\frac{1}{t}(x-y)^2}\,dy = \\
& \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{\infty} y \exp\bigl(-\frac{1}{t}x^2 + \frac{2}{t}xy - \frac{t+1}{t}y^2\bigr)\,dy = \\
& \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{\infty} y \exp\bigl(-\frac{x^2}{t+1} - \frac{t+1}{t}(y - \frac{x}{t+1})^2\bigr)\,dy = \\
& \frac{1}{\sqrt{\pi t}} \exp\bigl(-\frac{x^2}{t+1}\bigr) \int_{-\infty}^{\infty} y \exp\bigl(-\frac{t+1}{t}(y - \frac{x}{t+1})^2\bigr)\,dy =
\end{aligned}$$

plugging $y = \sqrt{t/(t+1)}z + x/(t+1)$

$$\begin{aligned}
& \frac{1}{\sqrt{\pi(t+1)}} \exp\bigl(-\frac{x^2}{t+1}\bigr) \int_{-\infty}^{\infty} \bigl(\sqrt{\frac{t}{t+1}}z + \frac{x}{t+1}\bigr) \exp\bigl(-z^2\bigr)\,dz = \\
& \frac{x}{\sqrt{\pi}(t+1)^{3/2}} \exp\bigl(-\frac{x^2}{t+1}\bigr) \int_{-\infty}^{\infty} \exp\bigl(-z^2\bigr)\,dz = \\
& \frac{x}{(t+1)^{3/2}} e^{-\frac{x^2}{t+1}}.
\end{aligned}$$

□

Solution 2. Again, we can consider Cauchy problem. Observe that $f(x) = -\frac{1}{2}g'(x)$ with $g(x) = e^{-x^2}$. Also observe that for initial function $g(x)$ solution is $v(x,t) = (t+1)^{-1/2}e^{-x^2/(t+1)}$. Indeed it is obtained is known solution of equation (2.1) by replacing t by $t+1$.
Then $u(x,t) = -\frac{1}{2}\partial_x e^{-x^2/(t+1)} = x(t+1)^{-3/2}e^{-x^2/(t+1)}$. □

Problem 3 (15 pts)**.** Solve by the method of separation of variables

$$4u_{tt} - u_{xx} = 0, \qquad 0 < x < 2,\ t > 0, \tag{3.1}$$
$$u(0,t) = u(2,t) = 0, \tag{3.2}$$
$$u(x,0) = f(x), \tag{3.3}$$
$$u_t(x,0) = g(x) \tag{3.4}$$

with $f(x) = \begin{cases} x & 0 < x < 1, \\ 2-x & 1 < x < 2, \end{cases}$ and $g(x) = 0$. Write the answer in terms of Fourier series.

Solution. Separating variables $u(x,t) = X(x)T(t)$ we get

$$X'' + \lambda X = 0, \tag{3.5}$$
$$X(0) = X(2) = 0, \tag{3.6}$$
$$4T'' + \lambda T = 0. \tag{3.7}$$

Problem (3.5)–(3.6) has solution

$$\lambda_n = \frac{\pi^2 n^2}{4}, \qquad X_n = \sin\bigl(\frac{\pi n x}{2}\bigr), \qquad n = 1, 2, \ldots \tag{3.8}$$

and therefore

$$T_n = A_n \cos\bigl(\frac{\pi n t}{4}\bigr) + B_n \sin\bigl(\frac{\pi n t}{4}\bigr), \tag{3.9}$$

and

$$u = \sum_{n=1}^{\infty} \Bigl[A_n \cos\bigl(\frac{\pi n t}{4}\bigr) + B_n \sin\bigl(\frac{\pi n t}{4}\bigr)\Bigr] \sin\bigl(\frac{\pi n x}{2}\bigr). \tag{3.10}$$

Plugging to (3.3)–(3.4) we get

$$\sum_{n=1}^{\infty} A_n \sin\bigl(\frac{\pi n x}{4}\bigr) = f(x),$$
$$\sum_{n=1}^{\infty} \frac{\pi n}{4} B_n \sin\bigl(\frac{\pi n x}{2}\bigr) = 0.$$

and $B_n = 0$,

$$A_n = \int_0^2 f(x) \sin\bigl(\frac{\pi n x}{2}\bigr)\, dx = \int_0^1 x \sin\bigl(\frac{\pi n x}{2}\bigr)\, dx + \int_1^2 (2-x) \sin\bigl(\frac{\pi n x}{2}\bigr)\, dx =$$
$$\frac{8}{\pi^2 n^2} \sin\bigl(\frac{\pi n}{2}\bigr) = \begin{cases} \dfrac{8}{\pi^2 (2m+1)^2}(-1)^m & n = 2m+1, \\ 0 & n = 2m. \end{cases}$$

Then

$$u = \sum_{m=0}^{\infty} \frac{4}{\pi^2 (2m+1)^2} \sin\bigl(\frac{\pi(2m+1)t}{4}\bigr) \sin\bigl(\frac{\pi(2m+1)x}{2}\bigr). \tag{3.11}$$

□

Problem 4 (15 pts)**.** Consider the Laplace equation in the sector

$$u_{xx} + u_{yy} = 0 \qquad \text{in } \ x^2 + y^2 < 16, x > -\sqrt{3}|y|, \tag{4.1}$$

with the boundary conditions

$$u = 1 \qquad \text{for } \ x^2 + y^2 = 16, \tag{4.2}$$

$$u = 0 \qquad \text{for } \ x = -\sqrt{3}|y|. \tag{4.3}$$

(a) Look for solutions u in the form of $u(r, \theta) = R(r)P(\theta)$ (in polar coordinates) and derive a set of ordinary differential equations for R and P. Write the correct boundary conditions for P.

(b) Solve the eigenvalue problem for P and find all eigenvalues.

(c) Solve the differential equation for R.

(d) Find the solution u of (4.1)–(4.3).

Solution. In polar coordinates $\{x = -r/2\}$ is $\theta = \pm\frac{5\pi}{6}$.
Separating variables we get domain

$$\frac{r^2R'' + rR'}{R} + \frac{P''}{P} = 0 \implies P'' + \lambda P = 0, \tag{4.4}$$

$$P(-\frac{5\pi}{6}) = P(\frac{5\pi}{6}) = 0, \tag{4.5}$$

$$r^2R'' + rR' + \lambda R = 0. \tag{4.6}$$

Since problem is symmetric with respect to $y = 0$ we conclude that u is even with respect to y (or θ) and then we consider $P_n(\theta) = \cos(\frac{3}{5}(2n + 1)\theta)$, $\lambda_n = \frac{9}{25}(2n + 1)^2$.
Then $r^2R'' + rR' + \frac{9}{25}(2n + 1)^2R = 0 \implies R_n = A_nr^{3(2n+1)/5} + B_nr^{-3(2n+1)/5}$ and $B_n = 0$ since the last term is singular as $r = 0$. Then

$$u = \sum_{n=1}^{\infty} A_nr^{3(2n+1)/5}\cos(\frac{3(2n + 1)}{5}) \tag{4.7}$$

and

$$u|_{r=16} = \sum_{n=1}^{\infty} A_n2^{3(2n+1)/5}\cos(\frac{3(2n + 1)}{5}\theta) = 1 \tag{4.8}$$

which implies

$$A_n = 2^{-12(2n+1)/5} \times \frac{12}{5\pi}\int_0^{5\pi/6} \cos(\frac{3(2n + 1)}{5}\theta)\, d\theta = \begin{cases} \dfrac{1}{2(2m + 1)\pi}2^{-6m} & n = 2m + 1, \\ 0 & n = 2m. \end{cases}$$

Finally

$$u = \sum_{m=1}^{\infty} \frac{1}{2(2m+1)\pi} 2^{-6m} r^{3(2m+1)/4} \sin(\frac{3(2m+1)}{4}(\theta + \frac{2\pi}{3})) =$$
$$\sum_{m=1}^{\infty} \frac{1}{2(2m+1)\pi} 2^{-6m} (-1)^m r^{3(2m+1)/4} \cos(\frac{3(2m+1)}{4}\theta)$$

□

Problem 5 (15 pts)**.** Consider Laplace equation in the half-strip

$$u_{xx} + u_{yy} = 0 \qquad y > 0,\ 0 < x < \pi \tag{5.1}$$

with the boundary conditions

$$u(0, y) = u(\pi, y) = 0, \tag{5.2}$$
$$u_y(x, 0) = g(x) \tag{5.3}$$

with $g(x) = \cos(x)$ and condition $\max |u| < \infty$.

(a) Write the associated eigenvalue problem.

(b) Find all eigenvalues and corresponding eigenfunctions.

(c) Write the solution in the form of a series expansion.

Solution. Separating variables $u = X(x)Y(y)$ we get

$$\frac{X''}{X} + \frac{Y''}{Y} = 0 \implies X'' + \lambda X = 0, \tag{5.4}$$
$$Y'' - \lambda Y = 0, \tag{5.5}$$
$$X(0) = X(\pi) = 0. \tag{5.6}$$

Solving (5.4), (5.6) we have

$$\lambda_n = n^2, \qquad X_n = \sin(nx) \qquad n = 1, 2, 3, \ldots, \tag{5.7}$$

and then solving (5.5) we get $Y_n = A_n e^{ny} + B_n e^{-ny}$ and the last term as growing for $y > 0$ we need to drop. So

$$Y_n = B_n e^{-ny} \tag{5.8}$$

and

$$u(x, y) = \sum_{n=1}^{\infty} B_n e^{-ny} \sin(nx). \tag{5.9}$$

Plugging to (5.3) we get

$$\sum_{n=1}^{\infty} -nB_n \sin(nx) = \cos(x). \tag{5.10}$$

and

$$B_n = -\frac{2}{n\pi} \int_0^{\pi} \cos(x)\sin(nx)\, dx = -\frac{1}{n\pi} \int_0^{\pi} \bigl(\sin((n+1)x) - \underline{\sin(n-1)x)}\bigr)\, dx =$$
$$\frac{1}{n\pi}\Bigl(\frac{\cos(n+1)x}{n+1} - \underline{\frac{\cos(n-1)x}{n-1}}\Bigr)\Bigr)\Bigr|_{x=0}^{x=\pi}.$$

with underlined term 0 as $n = 1$. Then

$$B_n = \begin{cases} 0 & n = 2m + 1, \\ -\dfrac{4}{m\pi(4m^2 - 1)} & n = 2m \end{cases}$$

and

$$u(x, y) = \sum_{m=1}^{\infty} -\frac{4}{m\pi(4m^2 - 1)} \sin(2mx)e^{-2my}$$

□

Problem 6 (15 pts)**.** Solve as $t > 0$

$$u_{tt} - \Delta u = 0, \tag{6.1}$$

with initial conditions

$$u(x,y,z,0) = \begin{cases} 1 & r := \sqrt{x^2+y^2+z^2} < 1, \\ 0 & r \ge 1, \end{cases} \qquad u_t(x,y,z,0) = 0 \tag{6.2}$$

and solve by a separation of variables.
HINT. Use spherical coordinates, observe that solution must be spherically symmetric: $u = u(r,t)$ (explain why).
Also, use equality

$$ru_{rr} + 2u_r = (ru)_{rr}. \tag{6.3}$$

Solution. Solution is spherically symmetric because the problem is. Then

$$u_{tt} - \bigl(u_{rr} + \frac{2}{r}u_r\bigr) = 0 \qquad r > 0, t > 0. \tag{6.4}$$

Multiplying by r and using (6.3) we arrive to the first equation below:

$$v_{tt} - v_{rr} = 0 \qquad r > 0, \tag{6.5}$$

$$v(0,t) = 0, \tag{6.6}$$

$$v(r,0) = g(r) = \begin{cases} r & r < 1, \\ 0 & r \ge 1, \end{cases} \qquad v_t(r,0) = 0. \tag{6.7}$$

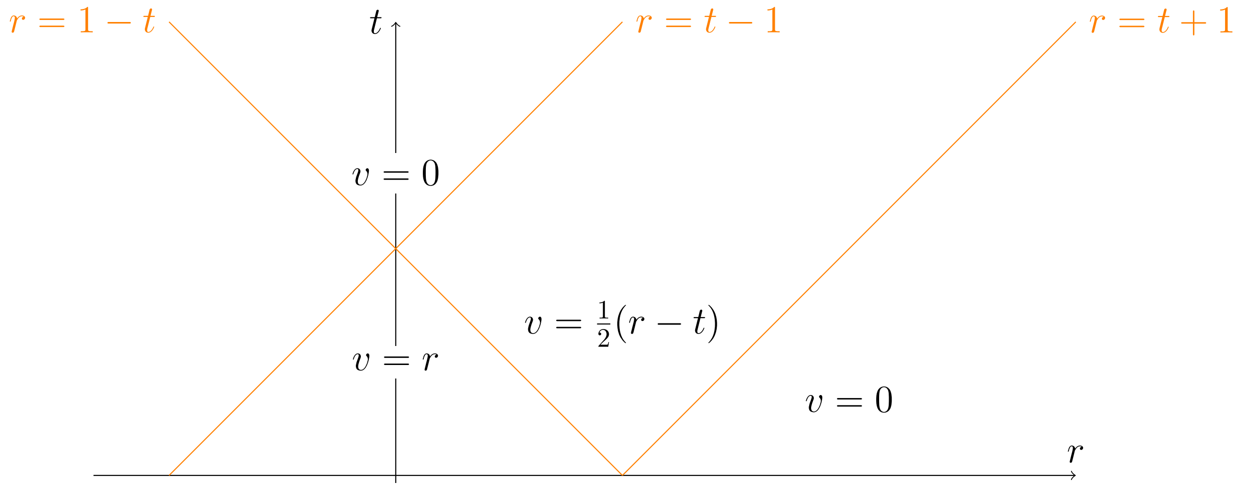
Continuing $g(r)$ as and odd function $\tilde{g}(r) = \begin{cases} r & |r| < 1, \\ 0 & |r| \ge 1, \end{cases}$ and solving Cauchy problem we get

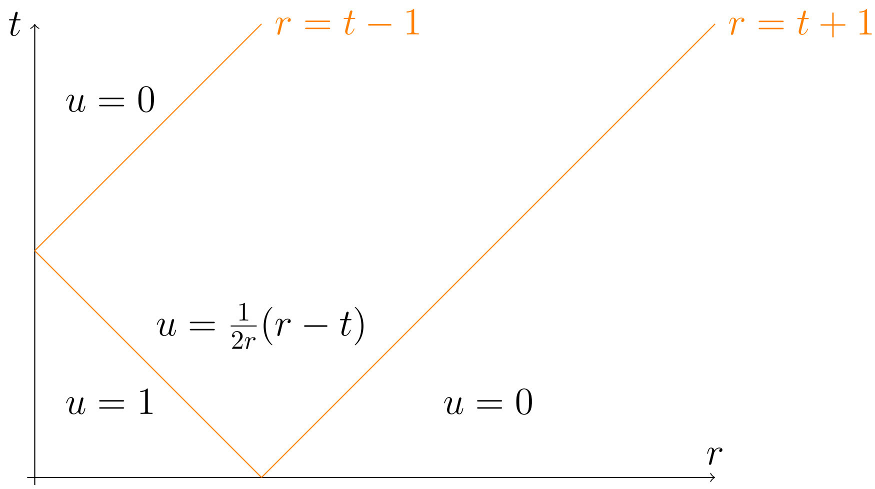
$$v(r,t) = \frac{1}{2}\bigl(\tilde{g}(r+t) + \tilde{g}(r-t)\bigr) = \begin{cases} 0 & r > t+1, \\ \frac{1}{2}(r-t) & 1-t < r < t+1, \\ r & 0 < r < 1-t, \\ 0 & 0 < r < t-1 \end{cases} \tag{6.8}$$

and finally

$$u(r,t) = r^{-1}v(r,t) == \begin{cases} 0 & r > t+1, \\ \frac{1}{2r}(r-t) & 1-t < r < t+1, \\ 1 & 0 < r < 1-t, \\ 0 & 0 < r < t-1 \end{cases} \tag{6.9}$$

□


$r = 1 - t$
t
$r = t - 1$
$r = t + 1$
$v = 0$
$v = \frac{1}{2}(r - t)$
$v = r$
$v = 0$
r


t
$r = t - 1$
$r = t + 1$
$u = 0$
$u = \frac{1}{2r}(r - t)$
$u = 1$
$u = 0$
r

Continued

Problem 7 (15 pts)**.** Solve using (partial) Fourier transform with respect to y

$$\Delta u := u_{xx} + u_{yy} = 0, \qquad x > 0, \tag{7.1}$$
$$u|_{x=0} = g(y), \tag{7.2}$$
$$\max |u| < \infty \tag{7.3}$$

with $g(y) = \frac{2}{y^2+1}$.
HINT. Fourier transform of $g(y)$ is $\hat{g} = e^{-|\eta|}$.

Solution. Making Fourier transform we get

$$\hat{u}_{xx} - \eta^2 \hat{u} = 0, \qquad x > 0, \tag{7.4}$$
$$\hat{u}|_{x=0} = \hat{g}(\eta) = e^{-|\eta|}, \tag{7.5}$$
$$\max |\hat{u}| < \infty \tag{7.6}$$

and solving (7.4) we see that $\hat{u} = A(\eta)e^{-|\eta|x} + B(\eta)e^{|\eta|x}$; (7.6) implies that $B(\eta) = 0$ and (7.5) implies then $A(\eta) = e^{-|\eta|}$. Then $\hat{u}(x,\eta) = e^{-|\eta|(1+x)}$ and

$$u(x,y) = \int_{-\infty}^{\infty} \hat{u}(x,\eta)e^{i\eta y}\, d\eta = \int_{-\infty}^{0} e^{\eta(1+x+yi)}\, d\eta + \int_{0}^{-\infty} e^{-\eta(1+x-yi)}\, d\eta = \\ \frac{1}{1+x+yi} + \frac{1}{1+x-yi} = \frac{2(1+x)}{(1+x)^2+y^2}. \tag{7.7}$$

□

Appendix: Some useful formulas.
Not exam problems.
You may detach this page

1. The two dimensional Laplacian in polar coordinates:

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}.$$

2. The Stokes theorem

$$\int_D \frac{\partial f}{\partial x_i}\,dx = \int_{\partial D} f n_i\,d\sigma$$

where n (with components n_i) is the unit normal vector pointing outside.

3. The complex Fourier series of a periodic function $f(x)$ of period $2l$, defined on the interval $(-l, l)$ is

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{\pi i n x/l}$$

with the coefficients c_n given by the formula

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-\pi i n x/l} dx$$

4. The Fourier transform of a function $f(x)$ is defined by

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} f(x) dx.$$

The inverse Fourier transform is

$$f(x) = \int_{-\infty}^{\infty} e^{ikx}\hat{f}(k) dk.$$

Here some of its properties:

(a) if $g(x) = f(ax)$, then $\hat{g}(k) = \frac{1}{|a|}\hat{f}(\frac{k}{a})$;

(b) $\widehat{f'}(k) = ik\hat{f}(k)$;

(c) if $g(x) = xf(x)$ then $\hat{g}(k) = -i\hat{f}\,'(k)$;

(d) if $g(x) = f(x-a)$, then $\hat{g}(k) = e^{-iak}\hat{f}(k)$;

(e) if $h = f * g$, then $\hat{h}(k) = 2\pi\hat{f}(k)\hat{g}(k)$;

(f) if $f(x) = e^{-x^2/2}$, then $\hat{f}(k) = \frac{1}{\sqrt{2\pi}}e^{-k^2/2}$.