Problem 1. Solve by the method of characteristics the BVP for a wave equation

$$u_{tt} - 4u_{xx} = 0, \qquad 0 < x < \infty,\ t > 0 \tag{1.1}$$
$$u(x,0) = f(x) \tag{1.2}$$
$$u_t(x,0) = g(x), \tag{1.3}$$
$$u_x(0,t) = h(t) \tag{1.4}$$

with $f(x) = 0, \quad g(x) = \begin{cases} 1 & 0 < x < 1, \\ 0 & 1 < x < \infty, \end{cases} \quad h(t) = \begin{cases} 1 & 0 < t < 1, \\ 0 & 1 < t < \infty. \end{cases}$

Solution. From

$$u(x,t) = \varphi(x+2t) + \psi(x-2t) \tag{1.5}$$

and (1.2), (1.3) we see that for $x > 0$

$$\varphi(x) + \psi(x) = 0, \qquad 2\varphi'(x) - 2\psi'(x) = \begin{cases} 1 & 0 < x < 1, \\ 0 & 1 < x < \infty, \end{cases}$$

and then

$$\varphi(x) = \begin{cases} \frac{1}{4}x & x < 1, \\ \frac{1}{4} & x > 1 \end{cases} \tag{1.6}$$

(we set $\varphi(0) = 0$ as it can be set additionally without affecting solution), $\psi(x) = -\varphi(x)$ and then

$$u(x,t) = \begin{cases} 0 & x > 2t+1, \\ \frac{1}{4}(1+2t-x) & 2t < x < 2t+1. \end{cases} \tag{1.7}$$

which could be obtained by D'Alembert formula as well.

From (1.4) we see $\varphi'(2t) + \psi'(-2t) = \begin{cases} 1 & 0 < t < 1, \\ 0 & 1 < t < \infty \end{cases}$

$$\iff \varphi'(-x) + \psi'(x) = \begin{cases} 1 & -2 < x < 0, \\ 0 & x < -2 \end{cases}$$

$$\iff \psi(x) = \varphi(-x) + \begin{cases} x & -2 < x < 0, \\ -2 & x < -2 \end{cases}$$

$$\iff \psi(x) = \begin{cases} \frac{3}{4}x & -1 < x < 0, \\ \frac{1}{4} + x & -2 < x < -1, \\ -\frac{7}{4} & x < -2 \end{cases}$$

and finally as $0 < x < 2t$

$$u(x,t) = \begin{cases} \frac{3}{4}(x-2t) & 2t-1 < x < 2t, \\ \frac{1}{4} + x - 2t & 2t-2 < x < 2t-1, \\ -\frac{7}{4} & x < 2t-2 \end{cases} \quad + \quad \begin{cases} \frac{1}{4}x + \frac{1}{2}t & x < 1-2t, \\ \frac{1}{4} & x > 1-2t \end{cases}$$

□

Solution 2. One can solve first problem with $h(t)=0$:

$$v(x,t)=\frac{1}{2}\Big[F(x+2t)+F(x-2t)\Big]+\frac{1}{4}\int_{x-2t}^{x+2t} G(x')\,dx' \qquad x>0$$

where $F(x')=0$, $G(x')=\begin{cases}1 & |x'|<1,\\ 0 & |x'|>1\end{cases}$ and we applied method of continuation; then integral is taken over $\max(x-2t,-1)<x'<\min(x+2t,1)$ which is empty as $x>2t+1$ and otherwise it is $\min(x+2t,1)-\max(x-2t,-1)=\min(x+2t,1)+\min(-x+2t,1)=\min(4t,-x+2t+1,2)$. So

$$v(x,t)=\begin{cases}0 & x>2t+1\\ \frac{1}{4}\min(4t,-x+2t+1,2) & 0<x<2t+1\end{cases}=$$
$$\begin{cases}0 & x>2t+1\\ \frac{1}{4}(2t+1-x) & \max(1-2t,0)<x<1+2t,\\ t & 0<x<1-2t.\end{cases}$$

Next solve problem with $g(t)=0$. It is $w(x,t)=\psi(x-2t)$ with $\psi(0)=0$ and $\psi'(-2t)=h(t)$; then $\psi'(x)=\begin{cases}1 & 0>x>-2,\\ 0 & x<-2\end{cases}$ and

$\psi(x)=\begin{cases}x & 0>x>-2,\\ -2 & x<-2\end{cases}$ and

$$w(x,t)=\begin{cases}0 & x>2t,\\ x-2t & 2t-2<x<2t,\\ -2 & x<2t-2.\end{cases}$$

Finally,

$$\begin{aligned}u(x,t)&=v(x,t)+w(x,t)\\ &=\begin{cases}0 & x>2t+1\\ \frac{1}{4}(2t+1-x) & \max(1-2t,0)<x<1+2t,\\ t & 0<x<1-2t\end{cases}\\ &+\begin{cases}0 & x>2t,\\ x-2t & 2t-2<x<2t,\\ -2 & x<2t-2.\end{cases}\end{aligned}$$

□

Problem 2. Solve IBVP for the heat equation

$$\begin{cases}u_t-3u_{xx}=0, & 0<x<\infty,\ t>0,\\ u|_{t=0}=f(x), & \\ u_x|_{x=0}=0, & \end{cases} \tag{2.1}$$

with $f(x) = \begin{cases} 1 & 0 < x < 1, \\ 0 & 1 < x < \infty, \end{cases}$

Solution should be expressed through $\operatorname{erf}(z) = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_0^z e^{-z^2/2}\,dz$.

Solution. Method of continuation $u(x,t)$ should be solved as Cauchy problem with initial condition $u(x,0) = F(x)$ with $F(x) = \begin{cases} 1 & |x| < 1, \\ 0 & |x| > 1, \end{cases}$ and therefore

$$u(x,t) = \frac{1}{\sqrt{12\pi t}} \int_{-1}^{1} e^{-\frac{(x-y)^2}{12t}}\,dy = \frac{1}{\sqrt{2\pi}} \int_{\frac{x-1}{\sqrt{2t}}}^{\frac{x+1}{\sqrt{2t}}} e^{-\frac{z^2}{2}}\,dy$$

where we set $y = x + z\sqrt{6t}$ and finally

$$u(x,t) = \frac{1}{2}\Big[\operatorname{erf}\Big(\frac{(x+1)}{\sqrt{6t}}\Big) - \operatorname{erf}\Big(\frac{(x-1)}{\sqrt{6t}}\Big)\Big]$$

where we used that $\sqrt{\frac{2}{\pi}} \int_a^b e^{-z^2/2}\,dz = \operatorname{erf}(b) - \operatorname{erf}(a)$. □

Problem 3. Solve by the method of separation of variables

$$u_{tt} - 4u_{xx} = 0, \qquad 0 < x < 1,\ t > 0, \tag{3.1}$$

$$u_x(0,t) = u_x(1,t) = 0, \tag{3.2}$$

$$u(x,0) = f(x), \tag{3.3}$$

$$u_t(x,0) = g(x) \tag{3.4}$$

with $f(x) = x(1-x)$, $g(x) = 0$. Write the answer in terms of Fourier series.

Solution. Separation of variables results in $X'' + \lambda X = 0$, $X'(0) = X'(1) = 0$ and thus $\lambda_0 = 0$, $X_0 = \frac{1}{2}$ and $\lambda_n = \pi^2 n^2$, $X_n = \cos(\pi n x)$ with $n = 1, 2, \ldots$; also $T'' + 4\pi^2 T = 0$ and thus $T_0 = A_0 + B_0 t$, $T_n = A_n \cos(2\pi n t) + B_n \sin(2\pi n t)$, and

$$u = \frac{1}{2}(A_0 + B_0 t) + \sum_{n=1}^{\infty} \Big(A_n \cos(2\pi n t) + B_n \sin(2\pi n t)\Big) \cos(\pi n x). \tag{3.5}$$

The initial conditions result in

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\pi n x) = x(1-x), \qquad \frac{1}{2}B_0 + \sum_{n=1}^{\infty} 2\pi n B_n \cos(\pi n x) = 0$$

and $B_n = 0$ $(n = 0, 1, 2, \ldots)$ and

$$A_n = 2\int_0^1 x(1-x)\cos(\pi n x)\,dx = -\frac{2}{\pi n} \int_0^1 (1-2x)\sin(\pi n x) =$$

$$-\frac{2}{\pi^2 n^2}(1-2x)\cos(\pi n x)\Big|_{x=0}^{x=1} + \frac{4}{\pi^2 n^2} \int_0^1 \cos(\pi n x)\,dx = \begin{cases} -\dfrac{1}{\pi^2 m^2} & n = 2m, \\ 0 & n = 2m+1 \end{cases}$$

$m = 1, 2, \ldots$. Meanwhile $A_0 = \frac{1}{3}$. Then

$$u = \frac{1}{12} - \sum_{m=1}^{\infty} \frac{1}{\pi^2 m^2} A_n \cos(4\pi mt)\cos(2\pi mx).$$

□

Problem 4. Consider the Laplace equation in the third of the disk

$$u_{xx} + u_{yy} = 0 \qquad \text{in } \ r = \sqrt{x^2+y^2} < a^2, \ x > |y|/\sqrt{3} \tag{4.1}$$

with the boundary conditions

$$u = y \qquad \text{for } \ r = a, \ x > 0, \tag{4.2}$$

$$u = 0 \qquad \text{for } \ y > 0, \ x = y/\sqrt{3} \tag{4.3}$$

$$u = 0 \qquad \text{for } \ y < 0, \ x = -y/\sqrt{3}. \tag{4.4}$$

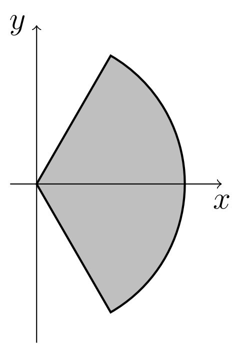
(a) Look for solutions u in the form of $u(r,\theta) = R(r)P(\theta)$ (in polar coordinates) and derive a set of ordinary differential equations for R and P. Write the correct boundary conditions for P.

(b) Solve the eigenvalue problem for P and find all eigenvalues.

(c) Solve the differential equation for R.

(d) Find the solution u of (4.1)–(4.4). Write the answer in terms of Fourier series.

Solution. Domain is a sector $-\frac{\pi}{3} < \theta < \frac{\pi}{3}$:



Then

$$P'' + \lambda P = 0, \qquad P(-\frac{\pi}{3}) = P(\frac{\pi}{3}) = 0.$$

Since our problem is symmetric with respect to $y = 0$ and $u(a,\theta) = a\sin(\theta)$ is an odd function we can replace this problem by

$$P'' + \lambda P = 0, \qquad P(0) = P(\frac{\pi}{3}) = 0$$

and $\lambda_n = 9n^2$, $P_n = \sin(3n\theta)$ with $n = 1, 2, \ldots$

Then $r^2R_n'' + rR_n' + 9n^2R_n = 0$ and $R_n = A_n r^{3n} + Br_n^{-3n}$ and we need to take $B_n = 0$.
So,

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{3n}\sin(3n\theta).$$

We need to satisfy

$$u(a,\theta) = \sum_{n=1}^{\infty} A_n a^{3n}\sin(3n\theta) = a\sin(\theta)$$

and therefore

$$A_n a^{3n} = \frac{6a}{\pi}\int_0^{\frac{\pi}{3}} \sin(3n\theta)\sin(\theta)\,d\theta = \frac{3a}{\pi}\int_0^{\frac{\pi}{3}} \Bigl[\cos((3n-1)\theta) - \cos((3n+1)\theta)\Bigr]\,d\theta =$$

$$\frac{3a}{\pi}\Bigl[\frac{\sin((3n-1)\theta)}{3n-1} - \frac{\sin((3n+1)\theta)}{3n+1}\Bigr]_{\theta=0}^{\theta=\pi/3} = \frac{3a}{\pi}\Bigl[\frac{\sin(\pi n - \frac{\pi}{3})}{3n-1} - \frac{\sin(\pi n + \frac{\pi}{3})}{3n+1}\Bigr] =$$

$$\frac{3a}{2\pi}(-1)^{n+1}\Bigl[\frac{1}{3n-1} + \frac{1}{3n+1}\Bigr] = (-1)^{n+1}\frac{9na}{(9n^2-1)\pi}$$

and

$$u(r,\theta) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{9n}{(9n^2-1)\pi} a^{1-3n} r^{3n}\sin(3n\theta).$$

□

Problem 5. Consider Laplace equation in the half-strip

$$u_{xx} + u_{yy} = 0 \qquad x > 0,\ 0 < y < \pi$$

with the boundary conditions

$$u(x,0) = 0, \qquad u(x,\pi) = 0, \qquad u_x(0,y) = g(y)$$

with $g(y) = 1$ and condition $\max|u| < \infty$.

(a) Write the associated eigenvalue problem.

(b) Find all eigenvalues and corresponding eigenfunctions.

(c) Write the solution in the form of a series expansion.

Solution. Looking at $u(x,y) = X(x)Y(y)$ we get $Y'' + \lambda Y = 0$, $Y(0) = Y(\pi) = 0$ and therefore $\lambda_n = n^2$, $Y_n = \sin(ny)$, $n = 1, 2, \ldots$, $X_n'' - n^2X_n = 0$, $X_n = A_n e^{-nx} + B_n e^{nx}$ with $B_n = 0$ (or solution would be unbounded), and

$$u(x,y) = \sum_{n=1}^{\infty} A_n e^{-nx}\sin(ny);$$

then

$$u_x(0,y) = \sum_{n=1}^{\infty} -nA_n\sin(ny) = 1$$

and

$$nA_n = -\frac{2}{\pi}\int_0^\pi \sin(ny)\,dy = \frac{2}{n\pi}\cos(ny)\Big|_0^\pi = \begin{cases} 0 & n = 2m, \\ -\dfrac{4}{n\pi} & n = 2m+1 \end{cases}$$

and

$$u(x,y) = -\sum_{m=0}^{\infty} \frac{4}{(2m+1)^2\pi} e^{-(2m+1)x}\sin((2m+1)y); \tag{5.1}$$

□

Problem 6. Consider BVP for Laplace equation on half-plane

$$u_{xx} + u_{yy} = 0 \qquad -\infty < x < \infty,\ y > 0 \tag{6.1}$$

with the Dirichlet boundary condition

$$u(x,0) = h(x) \tag{6.2}$$

with $h(x) = \begin{cases} \sin(x) & |x| < \pi, \\ 0 & |x| > \pi \end{cases}$ and condition $\max|u| < \infty$.

(a) Using Fourier transform with respect to x reduce to BVP for ODE;

(b) Solve this BVP for ODE;

(c) Write a solution of (6.1)–(6.2) in the form of Fourier integral.

Solution. Making Fourier transform by $x \to k$ we have

$$\hat{u}_{yy} - k^2\hat{u} = 0, \qquad \hat{u}(0,k) = \hat{h}(k)$$

with

$$\begin{aligned}\hat{h}(k) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(x)e^{-ikx}\,dx = \frac{1}{i\pi}\int_0^\pi \sin(x)\sin(kx)\,dx \\ &= \frac{1}{2i\pi}\int_0^\pi \Big(\cos((k-1)x) - \cos((k+1)x)\Big)\,dx \\ &= \frac{1}{2i\pi}\Big(\frac{\sin((k-1)\pi)}{k-1} - \frac{\sin((k+1)\pi)}{k+1}\Big) = \frac{1}{i\pi}\frac{\sin(k\pi)}{k^2-1}.\end{aligned}$$

Then $\hat{u}(k,y) = A(k)e^{-|k|y} + B(k)e^{|k|y}$ and we need to take $B(k) = 0$, $A(k) = \hat{h}(k)$. So

$$\hat{u}(k,y) = \frac{1}{i\pi}\frac{\sin(k\pi)}{k^2-1}e^{-|k|y}$$

and

$$u(x,y) = \frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{\sin(k\pi)}{k^2-1}e^{-|k|y+ikx}\,dk = \frac{4}{\pi}\int_0^\infty \frac{\sin(k\pi)}{k^2-1}e^{-ky}\sin(kx)\,dk.$$

□

Problem 7. Solve IVP

$$u_t - \frac{1}{4}u_{xx} = 0, \qquad -\infty < x < \infty,\ t > 0, \tag{7.1}$$

$$u|_{t=0} = g(x) \tag{7.2}$$

with $g(x) = xe^{-x^2/2}$.
HINT. Use Fourier transform by x.

Solution. Making Fourier transform by x we hade

$$\hat{u}_t + \frac{k^2}{4}\hat{u} = 0,\ t > 0,$$
$$\hat{u}|_{t=0} = \hat{g}(k).$$

Since Fourier transform of $e^{-x^2/2}$ is $(2\pi)^{-1}e^{-k^2/2}$ we conclude that

$$\hat{g}(k) = i(e^{-k^2/2})' = -(2\pi)^{-1}ike^{-k^2/2}$$

and

$$\hat{u}(k,t) = -ike^{-k^2/2 - tk^2/4} = -ik(2\pi)^{-1}e^{-k^2a^2/2}$$

with $a = \sqrt{(t+2)/2}$; then Fourier integral of $(2\pi)^{-1}e^{-k^2a^2/2}$ is $a^{-1}e^{-x^2/2a^2} = \sqrt{2/(2+t)}e^{-x^2/(2+t)}$ and finally

$$u(x,t) = -\frac{\partial}{\partial x}\Bigl[\sqrt{2/(2+t)}e^{-x^2/(2+t)}\Bigr] = x\Bigl(\frac{2+t}{2}\Bigr)^{-\frac{3}{2}}e^{-x^2/(2+t)},$$

□

Solution 2. Using formula

$$\begin{aligned}
u(x,y) =& \frac{1}{\sqrt{\pi t}}\int_{-\infty}^{\infty} y\exp\Bigl(-\frac{1}{t}(x-y)^2 - \frac{1}{2}y^2\Bigr)\,dy = \\
& \frac{1}{\sqrt{\pi t}}\int_{-\infty}^{\infty} y\exp\Bigl(-\frac{1}{t}x^2 + \frac{2}{t}xy - (\frac{1}{2}+\frac{1}{t})y^2\Bigr)\,dy = \\
& \frac{1}{\sqrt{\pi t}}\int_{-\infty}^{\infty} y\exp\Bigl(-\frac{1}{t}x^2 + \frac{2}{t}xy - \frac{t+2}{2t}y^2\Bigr)\,dy = \\
& \frac{1}{\sqrt{\pi t}}\int_{-\infty}^{\infty} y\exp\Bigl(-\frac{1}{t+2}x^2 - \frac{t+2}{2t}\bigl[y - \frac{2}{(t+2)}\bigr]^2\Bigr)\,dy;
\end{aligned}$$

plugging $y = z + \frac{2}{(t+2)}$ we get

$$u(x,y) = \frac{1}{\sqrt{\pi t}}\int_{-\infty}^{\infty}\Bigl[z + \frac{2}{(t+2)}\Bigr]\exp\Bigl(-\frac{1}{t+2}x^2 - \frac{t+2}{2t}z^2\Bigr)\,dz;$$

the second factor is even by term z and we can skip z in the first factor (integral of odd function would be 0);

$$u(x,y) = \frac{2}{\sqrt{\pi t}(t+2)}e^{-x^2/(t+2)} \times \int_{-\infty}^{\infty}\exp\Bigl(-\frac{t+2}{2t}z^2\Bigr)\,dz;$$

now integral is $\sqrt{2\pi t/(t+2)}$ (plug $z = y\sqrt{t/(t+2)}$) and

$$u(x,t) = x\Bigl(\frac{2+t}{2}\Bigr)^{-\frac{3}{2}}e^{-x^2/(2+t)}.$$

□