Problem 1 (4pt)**.** Solve by Fourier method

$$u_{tt} - u_{xx} = 0 \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}, \tag{1.1}$$

$$u_x|_{x=-\pi/2} = u_x|_{x=\pi/2} = 0, \tag{1.2}$$

$$u|_{t=0} = x^2, \qquad u_t|_{t=0} = 0. \tag{1.3}$$

HINT: Since problem is symmetric with respect to $x = 0$ consider only even with respect to x eigenfunctions.

Solution. Separation of variables leads to

$$X'' + \lambda X = 0, \tag{1.4}$$

$$X'(-\frac{\pi}{2}) = X'(\frac{\pi}{2}) = 0, \tag{1.5}$$

$$T'' + \lambda T = 0 \tag{1.6}$$

We know that for this BVP $\lambda \geq 0$ and $\lambda_0 = 0$, $X_0 = \frac{1}{2}$; consider $\lambda = k^2 > 0$; then $X = C\cos(kx) + D\sin(kx)$ and according to Hint we consider only $\cos(kx)$. Then (1.2) implies $\sin(k\pi/2) = 0$ $k = 2n$, $\lambda_n = 4n^2$, $X_n = \sin(2nx)$, $n = 1, 2, \ldots$. Then $T_n = A_n\cos(2nt) + B_n\sin(2nt)$ and finally

$$u(x,t) = \frac{1}{2}(A_0 + B_0 t) + \sum_{n=1}^{\infty}\bigl(A_n\cos(2nt) + B_n\sin(2nt)\bigr)\cos(2nt). \tag{1.7}$$

Plugging to (1.3) we get

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n\cos(2nt) = x^2, \tag{1.8}$$

$$\frac{1}{2}B_0 + \sum_{n=1}^{\infty} 2nB_n\cos(2nt) = 0. \tag{1.9}$$

Then

$$\begin{aligned}A_n =& \frac{4}{\pi}\int_0^{\pi/2} x^2\cos(2nx)\,dx = \frac{2}{\pi n}\int_0^{\pi/2} x^2\,d\sin(2nx) = \\ &-\frac{4}{\pi n}\int_0^{\pi/2}\sin(2nx)x\,dx = \frac{2}{\pi n^2}\int_0^{\pi/2} x\,d\cos(2nx) = \\ &\frac{2}{\pi n^2}\Bigl(x\cos(2nx)|_{x=0}^{x=\pi/2} - \int_0^{\pi/2}\cos(2nx)\,dx\Bigr) = \frac{1}{n^2}(-1)^n\end{aligned}$$

for $n \geq 1$, $A_0 = \frac{4}{\pi}\int_0^{\pi/2} x^2\,dx = \frac{\pi^2}{6}$ while $B_n = 0$. Finally

$$u(x,t) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}(-1)^n \cos(2nt).$$

$\square$

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Problem 2 (4pt)**.** Solve

$$u_{xx} + u_{yy} = 0 \qquad -\infty < x < \infty,\ 0 < y < \infty, \tag{2.1}$$

$$u_{y=0} = \frac{1}{x^2+1}, \tag{2.2}$$

$$\max |u| < \infty. \tag{2.3}$$

HINT: Use partial Fourier transform with respect to x, and formula

$$F(x^2+a^2)^{-1} = \frac{1}{2a} e^{-|k|a} \qquad \text{as} \quad a > 0. \tag{2.4}$$

Solution. After partial Fourier transform (using (2.4))

$$-k^2\hat{u} + \hat{u}_{yy} = 0 \qquad 0 < y < \infty, \tag{2.5}$$

$$\hat{u}_{y=0} = \frac{1}{2} e^{-|k|}. \tag{2.6}$$

Solving (2.5) we get $\hat{u} = A(k)e^{-|k|y} + B(k)e^{|k|y}$ and the last term must be dropped as it grows for $y > 0$: $\hat{u} = A(k)e^{-|k|y}$. Plugging to (2.6) we find $A(k) = \frac{1}{2}e^{-|k|}$ and

$$\hat{u} = \frac{1}{2} e^{-|k|(y+1)}. \tag{2.7}$$

and using (2.4) again

$$u = \frac{y+1}{x^2+y^2+1}. \tag{2.8}$$

□

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Problem 3 (4pt)**.** Using Fourier method find eigenvalues and eigenfunctions of Laplacian in the rectangle $\{0 < x < a, < y < b\}$ with Dirichlet boundary conditions:

$$u_{xx} + u_{yy} = -\lambda u \qquad 0 < x < a,\ 0 < y < b, \tag{3.1}$$
$$u_{x=0} = u_{x=a} = u_{y=0} = u_{y=b} = 0. \tag{3.2}$$

Solution. Separating variables $u = X(x)Y(y)$ we arrive to

$$\frac{X''}{X} + \frac{Y''}{Y} + \lambda = 0 \tag{3.3}$$

Then

$$X'' + \mu X = 0, \tag{3.4}$$
$$X(0) = X(a) = 0 \tag{3.5}$$

and

$$Y'' + \nu X = 0, \tag{3.6}$$
$$Y(0) = Y(b) = 0 \tag{3.7}$$

and

$$\lambda = \mu + \nu. \tag{3.8}$$

Next

$$\mu_m = \frac{\pi^2 m^2}{a^2}, \qquad X_m = \sin\bigl(\frac{\pi m}{a}\bigr), \qquad m = 1, 2, \ldots, \tag{3.9}$$
$$\nu_n = \frac{\pi^2 n^2}{b^2}, \qquad Y_n = \sin\bigl(\frac{\pi n}{b}\bigr), \qquad n = 1, 2, \ldots, \tag{3.10}$$

and finally

$$\lambda_{mn} = \pi^2\Bigl(\frac{m^2}{a^2} + \frac{n^2}{b^2}\Bigr), \quad u_{mn} = \sin\bigl(\frac{\pi m}{a}\bigr)\sin\bigl(\frac{\pi n}{b}\bigr), \quad m, n = 1, 2 \ldots \tag{3.11}$$

□

Problem 4 (4pt)**.** Consider Laplace equation in the half-disk

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \qquad r < 1, 0 < \theta < \pi \tag{4.1}$$

with the Dirichlet boundary conditions as $\theta = 0$ and $\theta = \pi$

$$u|_{\theta=0} = u|_{\theta=\pi} = 0 \tag{4.2}$$

and the Robin boundary condition as $r = 1$

$$(u_r + u)|_{r=1} = 1. \tag{4.3}$$

Using separation of variables find solution as a series.

Solution. Separating variables $u(r,\theta) = R(r)\Theta(\theta)$ we get

$$\frac{r^2R'' + rR'}{R} + \frac{\Theta''}{\Theta} = 0$$

and therefore both terms are constant:

$$\Theta'' + \lambda\Theta = 0, \tag{4.4}$$
$$\Theta(0) = \Theta(\pi) = 0 \tag{4.5}$$

and therefore $\lambda_n = n^2$, $\Theta_n = \sin(n\theta)$, $n = 1, 2, \ldots$. Then

$$r^2R'' + rR' - n^2R = 0 \tag{4.6}$$

and $R = Ar^n + Br^{-n}$ where we drop the last term as it is singular at $r = 0$. So $u_n = A_nr^n\sin(n\theta)$ and

$$u = \sum_{n=1}^{\infty} A_nr^n\sin(n\theta). \tag{4.7}$$

Plugging into (4.3) we get

$$\sum_{n=1}^{\infty} A_n(n+1)r^n\sin(n\theta) = 1; \tag{4.8}$$

then

$$A_n(n+1)r^n = \frac{2}{\pi}\int_0^{\pi}\sin(n\theta)\,d\theta = \begin{cases} 0 & n = 2m, \\ \dfrac{2}{2m+1} & n = 2m+1 \end{cases} \tag{4.9}$$

and

$$u = \sum_{m=0}^{\infty} \frac{2}{(2m+1)(2m+2)} r^{2m+1} \sin((2m+1)\theta). \tag{4.10}$$

□

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Problem 5 (4pt)**.** Find Fourier transforms of the function

$$f(x) = \begin{cases} \cos(x) & |x| < \dfrac{\pi}{2}, \\ 0 & |x| > \dfrac{\pi}{2} \end{cases}$$

and write this function as a Fourier integral.

Solution. The simplest:

$$\begin{aligned}\hat{f}(k) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx =\\ & \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \cos(x)e^{-ikx}\,dx =\\ & \frac{1}{4\pi}\int_{-\pi/2}^{\pi/2} \bigl(e^{ix}+e^{-ix}\bigr)e^{-ikx}\,dx =\\ & \frac{1}{4\pi}\int_{-\pi/2}^{\pi/2} \bigl(e^{i(1-k)x}+e^{-i(1+k)x}\bigr)\,dx =\\ & \frac{1}{4\pi i}\Bigl((1-k)^{-1}e^{i(1-k)x}-(1+k)^{-1}e^{-i(1+k)x}\Bigr)\Bigr|_{-\pi/2}^{\pi/2} =\end{aligned}$$

since $e^{\pm i\pi/2} = \pm i$ we get

$$\begin{aligned}&\frac{1}{4\pi}\Bigl((1-k)^{-1}\bigl(e^{-ik\pi/2}+e^{ik\pi/2}\bigr)+(1+k)^{-1}\bigl(e^{ik\pi/2}+ie^{-ik\pi/2}\bigr)\Bigr) =\\ &\frac{1}{2\pi}\cos(k\pi/2)\Bigl((1-k)^{-1}+(1+k)^{-1}\Bigr) =\\ &\frac{\cos(k\pi/2)}{\pi(1+k^2)}.\end{aligned}$$

Conversely

$$f(x) = \int_{-\infty}^{\infty}\frac{\cos(k\pi/2)}{\pi(1+k^2)}e^{ikx}\,dk = \int_0^{\infty}\frac{2\cos(k\pi/2)}{\pi(1+k^2)}\cos(kx)\,dk$$

which is cos-Fourier integral (as $f(x)$ is even function.
Note: no punishment for not writing the last equality. □