Problem 1 (5 pts)**.** Use the method of characteristics to solve the transport equation IVP

$$\begin{aligned} &2u_t + 5u_x = u, && -\infty < x, t < \infty, \\ &u(x,0) = \log(1 + |x|). \end{aligned}$$

Solution. Begin by solving for characteristics: if $\gamma(s) = (x(s), t(s))$ is a characteristic curve then

$$\frac{dt}{ds} = 2, \qquad \frac{dx}{ds} = 5,$$

and these can be solved by

$$t(s) = 2s, \qquad x(s) = 5s + C.$$

So the characteristic curves are given by

$$x - \frac{5}{2}t = C.$$

Set $\tilde{u}(s) := u(\gamma(s))$, then the PDE for u becomes an ODE for $\tilde{u}$,

$$\frac{d\tilde{u}}{ds} = \tilde{u},$$

which we can integrate to get

$$\tilde{u}(s) = Ae^s$$

where A is constant along γ, and so is a function of $x - \frac{5}{2}t$. Since $s = \frac{t}{2}$ along γ the general solution is

$$u(x,t) = \phi\left(x - \frac{5}{2}t\right) e^{\frac{t}{2}}.$$

To determine ϕ apply the initial condition

$$u(x,0) = \phi(x) = \log(1 + |x|)$$

So the solution is

$$u(x,t) = \log\left(1 + \left|x - \frac{5}{2}t\right|\right) e^{\frac{t}{2}}.$$

Problem 2 (5 pts). (a) Suppose that u is a solution to the 1d inhomogeneous wave IVP

$$\begin{aligned} &u_{tt} - 9u_{xx} = f(x,t), && 7 - \infty < x < \infty,\ t \geq 0, \\ &u(x,0) = 0, \\ &u_t(0,t) = 0, \end{aligned}$$

where

$$f(x,t) := \begin{cases} t(1-t)\cos(x), & 0 \leq t < 1,\ |x| \leq \frac{\pi}{2}, \\ 0, & \text{else.} \end{cases}$$

What is the value of u at $(x,t) = (15,4)$? Draw a picture to support your answer.

(b) Suppose that u is a solution to the problem

$$\begin{aligned} &u_t = u_{xx} - \pi^2 \cos(\pi x), && 0 < x < 1,\ t \geq 0, \\ &u_x(0,t) = 0, \\ &u_x(1,t) = 1, \\ &u(x,0) = 0, \end{aligned}$$

Qualitatively describe how u behaves as $t \to \infty$.

Solution. (a) Since the maximum propagation speed is $c = 3$, the domain of influence for the point $(x,t) = (15,4)$ is an isoceles triangle whose base extends only to the point $(15-4c, 0) = (15-12, 0) = (3,0)$. Therefore, since the source function f is supported on the rectangle $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]_x \times [0,1]_t$, it is zero everywhere in the domain of influence for $(15,4)$, and we conclude that $u(15,4) = 0$. Insert picture showing the disjoint domain of influence and support of f.

(b) The corresponding stationary problem does not have a solution, since the constraint $\int_0^l (-k^{-1} f(x))dx = b - a$ is not satisfied:

$$b - a = 1 - 0 = 1,$$

$$\begin{aligned} \int_0^1 \pi^2 \cos(\pi x) dx &= \pi^2 \int_0^1 \cos(\pi x) dx \\ &= -\pi \left[\sin(\pi x)\right]_0^1 = 0. \end{aligned}$$

The average discrepancy in the constraint is

$$p = 1 - 0 - 0 = 1,$$

so the solution will be of the form

$$u(x,t) = t + w(x,t),$$

where w is the solution to almost the same problem as u, the only change being in the source term. w has a stationary solution (defined up to a constant) $W(x)$, and decays exponentially to this solution as $t >> 0$.

So for large t,

$$u(x,t) \sim t + W(x),$$

i.e. u tends towards a time-independent shape $W(x)$ which is being pushed off to $+\infty$ at a linear rate ($u_t \sim +1$ for $t >> 0$).

Continue

Problem 3 (5 pts)**.** Use the method of separation of variables to solve the problem

$$
\begin{aligned}
&u_{tt} - u_{xx} + u = 0, && 0 < x < \pi,\ t > 0,\\
&u(0,t) = u(\pi,t) = 0, &&\\
&u(x,0) = x(\pi - x), &&\\
&u_t(x,0) = 0. &&
\end{aligned}
$$

Write the answer in terms of a Fourier series.

Solution. Looking for a separated solution $u(x,t) = X(x)T(t)$ results in the equation $\frac{T''}{T} + 1 - \frac{X''}{X} = 0$, and so the system of ODEs

$$
\begin{aligned}
X'' + \lambda X &= 0\\
X(0) = X(\pi) &= 0\\
T'' + (\lambda + 1)T &= 0
\end{aligned}
$$

The solution to the Dirichlet-Dirichlet eigenvalue problem for X on $[0,\pi]$ is

$$\lambda_n = n^2, \quad X_n(x) = \sin(nx), \quad n \in \mathbb{Z}_{>0}$$

So the equation for T becomes $T_n'' + (n^2 + 1)T_n = 0$, which is solved by

$$T_n(t) = A_n \cos(\sqrt{n^2 + 1}t) + B_n \sin(\sqrt{n^2 + 1}t).$$

The initial condition $u_t(x,0) = 0$ implies that in our final solution all of the B_n will vanish. So our solution is of the form

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos(\sqrt{n^2 + 1}t) \sin(nx).$$

Applying the other initial condition gives

$$\sum_{n=1}^{\infty} A_n \sin(nx) = x(\pi - x).$$

We calculate the Fourier coefficients by repeatedly integrating by parts:

$$
\begin{aligned}
A_n &= \frac{2}{\pi}\int_0^\pi x(\pi - x)\sin(nx)dx \\
&= \frac{2}{\pi}\left(\left[\frac{-1}{n}\underbrace{x(\pi-x)}_{=0}\cos(nx)\right]_0^\pi + \frac{1}{n}\int_0^\pi (\pi - 2x)\cos(nx)dx\right) \\
&= \frac{2}{n\pi}\int_0^\pi (\pi - 2x)\cos(nx)dx \\
&= \frac{2}{n\pi}\left(\left[\frac{\pi - 2x}{n}\underbrace{\sin(nx)}_{=0}\right]_0^\pi + \frac{2}{n}\int_0^\pi \sin(nx)dx\right) = \frac{4}{n^2\pi}\int_0^\pi \sin(nx)dx \\
&= -\frac{4}{n^3\pi}\left[\cos(nx)\right]_0^\pi = -\frac{4}{n^3\pi}\left((-1)^n - 1\right) \\
&= \begin{cases} 0, & n = 2m, \\ \frac{8}{\pi(2m+1)^3}, & n = 2m+1, \end{cases}
\end{aligned}
$$

where $m \in \mathbb{Z}_{\geq 0}$. So:

$$
u(x,t) = \frac{8}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^3}\cos\left(\sqrt{(2m+1)^2+1}t\right)\sin((2m+1)x)
$$

Continue

Problem 4 (10 pts)**.** Solve the Laplace equation in the sector

$$u_{xx} + u_{yy} = 0 \quad \text{in} \qquad 0 < a^2 \le x^2 + y^2 \le b^2,\ y > 0,$$

with boundary conditions

$$\begin{cases} u &= 0 & \text{on} \ \ y = 0, \\ u &= \frac{\alpha y}{a} & \text{on} \ \ x^2 + y^2 = a^2, \\ u &= \frac{2\beta xy}{b^2} & \text{on} \ \ x^2 + y^2 = b^2. \end{cases}$$

Here a, b, α, β are constants, and $a, b > 0$.
HINT: Your answer should be expressed in terms of polar coordinates, **not** cartesian coordinates.)

Solution. Using the hint, the first step is to convert the problem to polar coordinates: $x = r\cos(\theta)$, $y = r\sin(\theta)$, $0 < a \le r \le b$, $0 \le \theta \le \pi$,

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= 0 \\ u(r, 0 \text{ or } \pi) &= 0 \\ u(a,\theta) &= \alpha\sin(\theta) \\ u(b,\theta) &= \beta\sin(2\theta) \end{aligned}$$

Looking for a separated solution $u(r,\theta) = R(r)\Theta(\theta)$ gives the equation

$$0 = \underbrace{\left(\frac{r^2R'' + rR'}{R}\right)}_{=+\lambda} + \underbrace{\frac{\Theta''}{\Theta}}_{=-\lambda}.$$

The problem for Θ has Dirichlet-Dirichlet BCs

$$\begin{aligned} \Theta'' + \lambda\Theta &= 0 \\ \Theta(0) = \Theta(\pi) &= 0 \end{aligned}$$

and this eigenvalue problem has solutions

$$\lambda_n = n^2, \qquad \Theta_n(\theta) = \sin(n\theta), \qquad n \in \mathbb{Z}_{>0}.$$

Plugging these values of λ into the equation for R, we find the Euler equation

$$r^2R'' + rR' + n^2R = 0,$$

which has solutions

$$R_n(r) = A_nr^n + B_nr^{-n}.$$

The general solution looks like

$$u(r,\theta) = \sum_{n>0}(A_nr^n + B_nr^{-n})\sin(n\theta),$$

and applying the r-BCs at $r = a$ and $r = b$ gives

$$\alpha\sin(\theta) = \sum_{n>0}(a^nA_n + a^{-n}B_n)\sin(n\theta),$$
$$\beta\sin(2\theta) = \sum_{n>0}(b^nA_n + b^{-n}B_n)\sin(n\theta).$$

Since the terms on the LHS of these equations are already eigenfunctions, we can match coefficients to find:

$$a^nA_n + a^{-n}B_n = \begin{cases} \alpha, & n = 1, \\ 0, & n \neq 1, \end{cases}$$

$$b^nA_n + b^{-n}B_n = \begin{cases} \beta, & n = 2, \\ 0, & n \neq 2. \end{cases}$$

Since $a \neq b$, these equations imply that $A_n = B_n = 0$ for $n > 2$. For $n = 2$ we have

$$B_2 = -a^4A_2 \quad \Rightarrow \quad (b^2 - a^4b^{-2})A_2 = \beta$$

so that

$$A_2 = \frac{b^2}{b^4 - a^4}\beta, \quad \text{and} \quad B_2 = -a^4\frac{b^2}{b^4 - a^4}\beta.$$

Similarly for $n = 1$ we have

$$B_1 = -b^2A_1 \quad \Rightarrow \quad (a - b^2a^{-1})A_1 = \alpha,$$

so that

$$A_1 = \frac{a}{a^2 - b^2}\alpha, \quad \text{and} \quad B_1 = -\frac{ab^2}{a^2 - b^2}\alpha.$$

Therefore:

$$u(r,\theta) = \alpha\frac{a}{r}\frac{r^2 - b^2}{a^2 - b^2}\sin(\theta) + \beta\frac{b^2}{r^2}\frac{r^4 - a^4}{b^4 - a^4}\sin(2\theta)$$

Continue

Problem 5 (10 pts)**.** Consider the *telegraph equation*:

$$u_{tt} + (\alpha + \beta)u_t + \alpha\beta u = c^2 u_{xx} \tag{5.1}$$

Assume that $\alpha, \beta > 0$.

(a) Take the Fourier transform $x \to k$ of (5.1) to obtain a constant coefficient homogeneous second order ODE for the Fourier transform of u, $\hat{u}(k,t)$.

(b) Assuming that $\hat{u}(k,t)$ satisfies

$$\hat{u}(k,t) = 0 \quad \text{unless} \quad 4c^2k^2 > (\alpha - \beta)^2, \tag{5.2}$$

derive that the general solution to the ODE from (a) is

$$\hat{u}(k,t) = e^{-\gamma t}\left(\hat{F}(k)e^{-i\omega(k)t} + \hat{G}(k)e^{i\omega(k)t}\right) \tag{5.3}$$

where $\hat{F}(k)$ and $\hat{G}(k)$ are arbitrary functions of k (subject to the vanishing condition (5.2)). In particular, derive expressions for γ and $\omega(k)$.

(c) Let $F(x)$ and $G(x)$ be the inverse Fourier transforms of $\hat{F}(k)$ and $\hat{G}(k)$, and suppose that $\alpha = \beta$.

Under this assumption find the general solution $u(x,t)$ to (5.1), and give an interpretation of the individual factors in your solution.

HINT: Compare to a solution of the homogeneous wave equation.

Solution. (a) The Fourier transform takes u, u_t, u_{tt} $\to$ $\hat{u}$, $\hat{u}_t$, $\hat{u}_{tt}$, and $u_{xx} \to -k^2\hat{u}$. So the ODE for $\hat{u}$ is

$$\hat{u}_{tt} + (\alpha + \beta)\hat{u}_t + (\alpha\beta + c^2k^2)\hat{u} = 0.$$

(b) Testing the functions $\hat{u}(k,t) = e^{rt}$, the characteristic equation for the ODE is

$$r^2 + (\alpha + \beta)r + (\alpha\beta + c^2k^2) = 0.$$

By the quadratic formula this has solutions

$$r = \frac{-(\alpha+\beta) \pm \sqrt{(\alpha+\beta)^2 - 4(\alpha\beta + c^2k^2)}}{2} = \frac{-(\alpha+\beta) \pm \sqrt{(\alpha-\beta)^2 - 4c^2k^2}}{2}.$$

By assumption $4c^2k^2 > (\alpha - \beta)^2$, so

$$r = -\frac{\alpha + \beta}{2} \pm i \sqrt{\underbrace{c^2k^2 - \left(\frac{\alpha - \beta}{2}\right)^2}_{>0}} = -\gamma \pm i\omega(k)$$

where

$$\gamma = \frac{\alpha + \beta}{2} > 0, \quad \omega(k) = \sqrt{c^2k^2 - \left(\frac{\alpha - \beta}{2}\right)^2}.$$

With these values of γ and $\omega(k)$ the general solution is therefore

$$\hat{u}(k,t) = e^{-\gamma t}\left(\hat{F}(k)e^{-i\omega(k)t} + \hat{G}(k)e^{i\omega(k)t}\right).$$

(c) If $\alpha = \beta$ then $\gamma = \alpha > 0$ and $\omega(k) = ck$, so that

$$\hat{u}(k,t) = e^{-\gamma t}\left(\hat{F}(k)e^{-ickt} + \hat{G}(k)e^{ickt}\right).$$

The inverse Fourier transform of $\hat{F}(k)e^{-ickt}$ is $F(x - ct)$, and similarly for $\hat{G}$, so taking the inverse Fourier transform of $\hat{u}$ gives

$$u(x,t) = e^{-\gamma t}(F(x - ct) + G(x + ct)).$$

The terms inside the parentheses are a solution to the homogeneous wave equation with parameter c: $F(x - ct)$ is a right moving wave-packet with propagation speed c and $G(x+ct)$ is a left moving wave-packet with propagation speed c. This constant propagation speed solution to the wave equation is exponentially damped by the third factor in our solution $e^{-\gamma t}$, since $\gamma > 0$.

Upshot: The solution looks like an exponentially damped solution to the homogeneous wave equation $u_{tt} - c^2u_{xx} = 0$.

Continue

Problem 6 (5 pts). (a) Without calculating any derivatives, determine whether $\dfrac{1}{1+x_1^2+\cdots+x_n^2}$ is a harmonic function on $\mathbb{R}^n$.

(b) Suppose that u is a harmonic function on the disc $\{r<1\}$, and that

$$u|_{r=1} = \begin{cases} \sin\theta, & 0<\theta<\pi, \\ 0, & \pi<\theta<2\pi. \end{cases}$$

Without finding the solution for u, calculate the value of u at the origin.

Solution. (a) The function $\dfrac{1}{1+x_1^2+\cdots+x_n^2}$ has a global maximum at $x_1 = \cdots = x_n = 0$, and therefore does not obey the maximum principle. Hence it cannot be harmonic.

(b) By the mean value theorem for harmonic functions,

$$\begin{aligned} u(0) &= \frac{1}{2\pi}\int_0^{2\pi} u(1,\theta)d\theta \\ &= \frac{1}{2\pi}\int_0^{\pi} \sin\theta\, d\theta \\ &= \frac{1}{2\pi}\left[-\cos\theta\right]_0^{\pi} \\ &= -\frac{1}{2\pi}((-1)-1) = \frac{1}{\pi}. \end{aligned}$$

Problem 7 (5pts)**.** Let S denote the action functional on the rectangle $R = [0, T] \times [a, b]$,

$$S[u] = \int_0^T \int_a^b (u_{xx}^2 - u_t^2) dx dt. \tag{7.1}$$

Find the Euler-Lagrange equation for S with respect to variations v which satisfy $v|_{\partial R} = v_x|_{\partial R} = 0$.
HINT: Explicitly calculate the ϵ derivative of $S[u + \varepsilon v]$.

Solution. Let v be a function satisfying $v|_{\partial R} = v_x|_{\partial R} = 0$. We have

$$\begin{aligned} S[u + \epsilon v] &= \iint_R \left((u_{xx} + \epsilon v_{xx})^2 - (u_t + \epsilon v_t)^2\right)\, dx\, dt \\ &= \iint_R \left(u_{xx}^2 + 2\epsilon u_{xx} v_{xx} + \epsilon^2 v_{xx}^2 - u_t^2 - 2\epsilon u_t v_t - \epsilon^2 v_t^2\right)\, dx\, dt, \end{aligned}$$

so we may take the derivative and repeatedly integrate by parts to get

$$\begin{aligned} \frac{d}{d\epsilon} S[u + \epsilon v]|_{\epsilon=0} &= \iint_R (2u_{xx} v_{xx} - 2u_t v_t)\, dx\, dt \\ &= 2 \iint_R (-u_{xxx} v_x + u_{tt} v)\, dx\, dt \\ &= 2 \iint_R (u_{xxxx} + u_{tt}) v\, dx\, dt, \end{aligned}$$

where all boundary contributions contain factors of v or v_x, and so must vanish. Hence the Euler-Lagrange equation is:

$$u_{xxxx} + u_{tt} = 0$$

Bonus Problem 1 (2pts)**.** You want to bake a cake in your oven, which happens to be a perfect cube. The heating element is located on the bottom side of your oven, and the other sides are all perfectly insulated (even the oven door, when it is closed).
Write down an initial-boundary value problem that describes what happens when you turn your oven on to preheat. **Do not attempt to solve your IBVP.**

Solution. There is some variation possible with this problem, but the solution should look something like this:

- The oven is a perfect cube, let's suppose with side length L, so we'll take as our domain the cube $[0, L]_x \times [0, L]_y \times [0, L]_z$.
- We are interested in the temperature $T(\vec{x}, t)$ of the over, which we can model using the heat equation $\frac{\partial T}{\partial t} = k\Delta T$. k will take some particular value here, determined by the composition of the air in the oven.
- All the sides except for the bottom are perfectly insulated, so no heat can escape or enter through them, so we must have homogeneous Neumann BCs on those sides.
- The heating element is on the bottom of the oven ($z = 0$), and when we turn the oven on to preheat we are specifying a particular heat distribution $H(x, y, t)$ for this heating element, so on the bottom of the oven we have an inhomogeneous Dirichlet BC.
- Before we turn the oven on, let's assume that the air in the oven was roughly the same temperature T_0 everywhere.

So the problem is:

$$
\begin{aligned}
\frac{\partial T}{\partial t}(x, y, z, t) &= k\Delta T(x, y, z, t), & t > 0, \quad 0 \le x, y, z \le L, \\
\left.\frac{\partial T}{\partial x}\right|_{x=0,L} &= 0, & \left.\frac{\partial T}{\partial y}\right|_{y=0,L} = 0, \\
\left.\frac{\partial T}{\partial z}\right|_{z=L} &= 0, & T|_{z=0} = H(x, y, t), \\
T|_{t=0} &= T_0.
\end{aligned}
$$

Bonus Problem 2 (2pts)**.** Find the 1d Green's function $G(x, y)$ on the interval $[0, 1]$ with homogeneous Dirichlet boundary conditions by solving the following problem:

$$\frac{d^2G}{dx^2} = -\delta(x - y), \tag{9.1}$$
$$G(0, y) = G(1, y) = 0, \tag{9.2}$$
$$G \text{ is continuous.} \tag{9.3}$$

Recall that $\delta(x - y)$ is defined by the property that $\int_a^b \delta(x - y)f(x)dx = f(y)$ if $a < y < b$, and is zero otherwise.
HINT: Solve the ODE separately on the regions $x < y$ and $x > y$, then integrate (9.1) to find a condition on the derivative of G.

Solution. Using the hint, on the regions $x < y$ and $x > y$ (9.1) becomes

$$\frac{d^2G}{dx^2} = 0,$$

so on these regions we have

$$G(x, y) = \begin{cases} Ax + B, & x > y, \\ Cx + D, & x < y. \end{cases}$$

Plugging the BCs into this expression gives

$$G(0, y) = D = 0,$$
$$G(1, y) = A + B = 0,$$

so the solution becomes

$$G(x, y) = \begin{cases} A(x - 1), & x > y, \\ Cx, & x < y. \end{cases}$$

Continuity at $x = y$ imposes $A(y - 1) = Cy$, so that $A = C\frac{y}{y-1}$ and

$$G(x,y) = \begin{cases} Cy\frac{x-1}{y-1}, & x > y, \\ Cx, & x < y. \end{cases}$$

Finally, using the hint again, integrate over a small interval $[y^-, y^+] \ni y$ and take the one-sided limits to y to find

$$\begin{aligned} -1 &= \int_{y^-}^{y^+} \frac{d^2G}{dx^2} dx \\ &= \left.\frac{dG}{dx}\right|_{y^+} - \left.\frac{dG}{dx}\right|_{y^-} \\ &= C\frac{y}{y-1} - C = C\left(\frac{y}{y-1} - 1\right) = \frac{C}{y-1} \end{aligned}$$

So $C = 1 - y$ and we conclude:

$$G(x,y) = \begin{cases} (1-y)x, & 0 < x < y, \\ y(1-x), & 1 > x > y. \end{cases}$$

Continue to extra pages