

**Problem 1** (4 pts). Consider the first order equation:

$$u_t + 3(t^2 - 1)u_x = 6t^2. \quad (1)$$

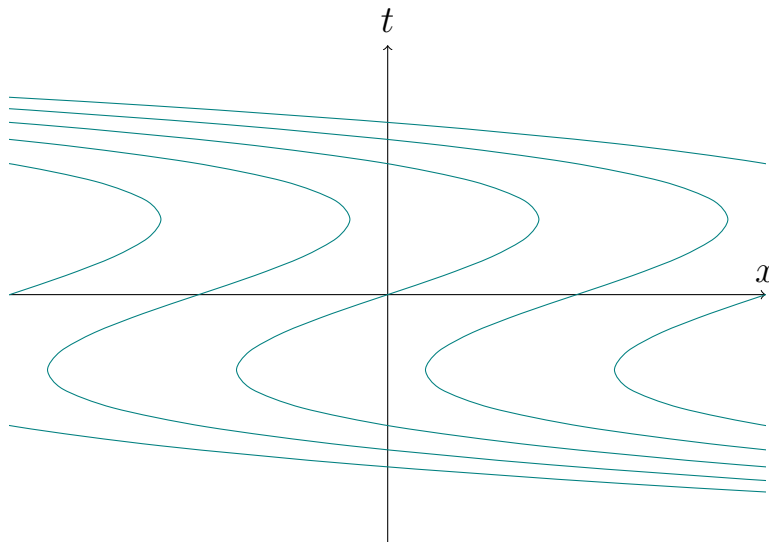
- (a) Find the characteristic curves and sketch them in the  $(x, t)$  plane.
- (b) Write the general solution.
- (c) Solve equation (1) with the initial condition  $u(x, 0) = x$ . Explain why the solution is fully determined by the initial condition.

*Solution.* (a)–(b) Equation of characteristics

$$\frac{dt}{1} = \frac{dx}{3t^2 - 3} = \frac{du}{6t^2} \implies x - t^3 + 3t = C_1, \quad C_2 = u - 2t^3 \implies$$

$$u = 2t^3 + f(x - t^3 + 3t) \quad (2)$$

with arbitrary function  $f$ .



- (c) From initial condition we conclude that  $f(x) = x$  and

$$u(x, t) = x + t^3 + 3t. \quad (3)$$

□

**Problem 2** (4 pts). Find solution  $u(x, t)$  to

$$u_{tt} - 4u_{xx} = 8/(x^2 + 1), \tag{4}$$

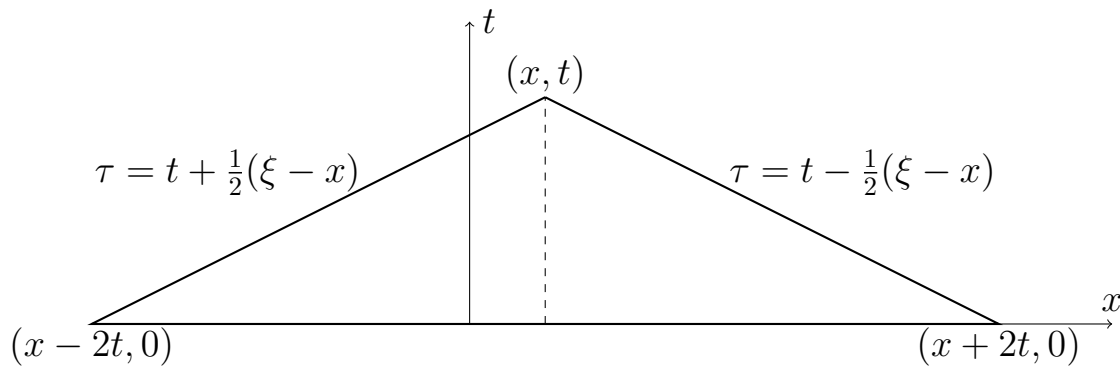
$$u|_{t=0} = 0, \quad u_t|_{t=0} = 0. \tag{5}$$

HINT: Change order of integration over characteristic triangle.

*Solution.* By D'Alembert formula

$$u(x, t) = \frac{1}{4} \iint_{\Delta(x, t)} \frac{8}{\xi^2 + 1} d\xi d\tau, \tag{6}$$

where  $\Delta(x, t)$  is bounded by  $\tau = 0$ ,  $x - \xi - 2(t - \tau) = 0$ ,  $x - \xi + 2(t - \tau) = 0$ .



Then the double integral becomes

$$\begin{aligned} & 2 \int_{x-2t}^x \left( \int_0^{t+\frac{1}{2}(\xi-x)} d\tau \right) \frac{d\xi}{\xi^2 + 1} + 2 \int_x^{x+2t} \left( \int_0^{t-\frac{1}{2}(\xi-x)} d\tau \right) \frac{d\xi}{\xi^2 + 1} = \\ & \int_{x-2t}^x \frac{(2t - x + \xi) d\xi}{\xi^2 + 1} + \int_x^{x+2t} \frac{(2t + x - \xi) d\xi}{\xi^2 + 1} = \\ & (2t - x) \left( \arctan(x) - \arctan(x - 2t) \right) + \frac{1}{2} \left( \ln(x^2 + 1) - \ln((x - 2t)^2 + 1) \right) + \\ & (2t + x) \left( \arctan(x + 2t) - \arctan(x) \right) + \frac{1}{2} \left( \ln(x^2 + 1) - \ln((x + 2t)^2 + 1) \right). \end{aligned}$$

□

**Problem 3** (4 pts). Find continuous solution to

$$u_{tt} - 4u_{xx} = 0, \quad t > 0, x > -t, \quad (7)$$

$$u|_{t=0} = 0, \quad x > 0, \quad (8)$$

$$u_t|_{t=0} = 0, \quad x > 0, \quad (9)$$

$$u_x|_{x=-t} = \sin(t), \quad t > 0. \quad (10)$$

*Solution.* (a) (2 pts) Solution to (7) is

$$u(x, t) = \phi(x + 3t) + \psi(x - 3t) \quad (11)$$

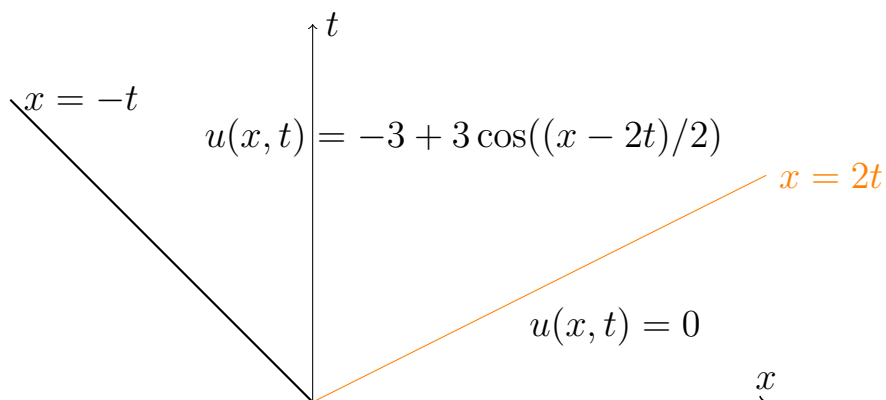
with unknown functions  $\phi$  and  $\psi$ . Plugging into (8)–(9) we get

$$\phi(x) + \psi(x) = 0, \quad 2\phi'(x) - 2\psi'(x) = 0 \implies \phi(x) = \psi(x) = 0 \quad x > 0$$

and  $u(x, t) = \sin(x + 3t)$  as  $x > 2t$ .

(b) (2 pts) Plugging into (10) we get  $\psi'(-3t) = \sin(t)$  as  $t > 0$  or  $\psi'(x) = -\sin(x/3)$  and then  $\psi(x) = 3 \cos(x/2) + C$  as  $x < 0$  and  $u(x, t) = 3 \cos(x/3) + C$  as  $-t < x < 2t$ .

Continuity at 0 implies  $C = -3$ .



□

**Problem 4** (4 pts). Consider the PDE with boundary conditions:

$$u_{tt} - c^2 u_{xx} + au = 0, \quad 0 < x < L, \quad (12)$$

$$(u_x - \alpha u_{tt})|_{x=0} = 0, \quad (13)$$

$$(u_x + \beta u_{tt})|_{x=L} = 0 \quad (14)$$

where  $c > 0$ ,  $\alpha > 0$ ,  $\beta > 0$  and  $a$  are constant. Prove that the energy  $E(t)$  defined as

$$E(t) = \frac{1}{2} \int_0^L (u_t^2 + c^2 u_x^2 + au^2) dx + \frac{\alpha c^2}{2} u_t(0, t)^2 + \frac{\beta c^2}{2} u_t(L, t)^2 \quad (15)$$

does not depend on  $t$ .

*Solution.*

$$\begin{aligned} \partial_t E(t) &= \int_0^L (u_t u_{tt} + c^2 u_{xt} u_x + a u_t u) dx + \alpha u_t u_{tt}|_{x=0} + \beta u_t u_{tt}|_{x=L} \\ &= \int_0^L c^2 (u_t u_{xx} + u_x u_{xt}) dx + \alpha u_t u_{tt}|_{x=0} + \beta u_t u_{tt}|_{x=L} = c^2 u_t u_x|_{x=0}^{x=L} + \alpha u_t u_{tt}|_{x=0} \\ &= c^2 u_t (-u_x + \alpha u_{tt})|_{x=0} + c^2 u_t (u_x + \beta u_{tt})|_{x=L} = 0 \end{aligned}$$

due to boundary conditions. □

**Problem 5** (4 pts). Find the solution  $u(x, t)$  to

$$u_t = u_{xx} \quad -\infty < x < \infty, \quad t > 0, \quad (16)$$

$$u|_{t=0} = e^{-|x|} \quad (17)$$

$$\max |u| < \infty. \quad (18)$$

Calculate the integral.

*Hint:* For  $u_t = ku_{xx}$  use  $G(x, y, t) = \frac{1}{\sqrt{4\pi kt}} \exp(-(x-y)^2/4kt)$ . To calculate integral make change of variables and use  $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz$ .

*Solution.* Due to hint

$$\begin{aligned} u(x, t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{1}{4t}(x-y)^2 - |y|} dy = \\ &= \frac{1}{\sqrt{4\pi t}} \left( \int_{-\infty}^0 e^{-\frac{1}{4t}(x-y)^2 + y} dy + \int_0^{\infty} e^{-\frac{1}{4t}(x-y)^2 - y} dy \right) = \\ &= \frac{1}{\sqrt{4\pi t}} \left( \int_{-\infty}^0 e^{-\frac{1}{4t}(y-x-2t)^2 + x+t} dy + \int_0^{\infty} e^{-\frac{1}{4t}(y-x+2t)^2 - x+t} dy \right) = \\ &= \frac{1}{\sqrt{4\pi t}} e^{x+t} \int_{-\infty}^{-(x+2t)} e^{-\frac{s^2}{4t}} ds + \dots = \\ &= \frac{1}{\sqrt{\pi}} e^{x+t} \int_{-\infty}^{-(x+2t)/\sqrt{4t}} e^{-z^2} dz + \dots = \\ &= e^{x+t} (1 - \operatorname{erf}((2t+x)/\sqrt{4t})) + e^{-x+t} (1 - e^{-x+t} \operatorname{erf}((2t-x)/\sqrt{4t})), \end{aligned}$$

where ... denoted the same term as the first one but with  $x \mapsto -x$ . □