

**Problem 1** (4 pts). Consider the first order equation:

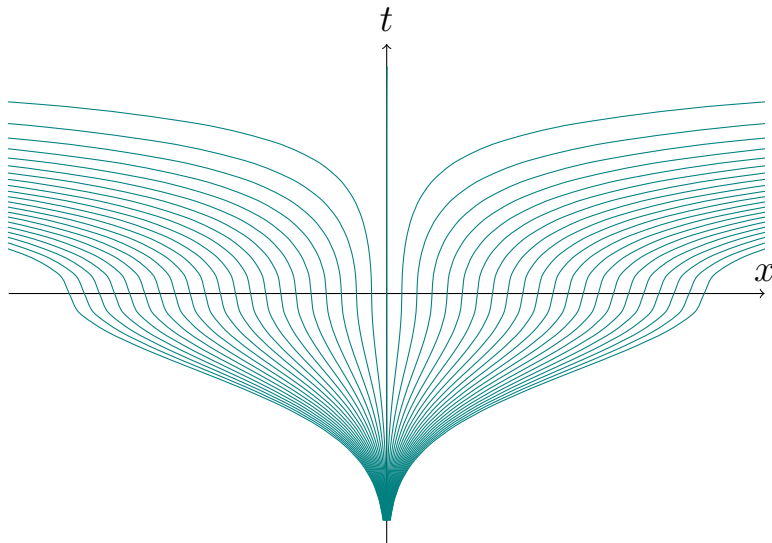
$$u_t + xt u_x = xte^{-t^2/2}. \quad (1)$$

- (a) Find the characteristic curves and sketch them in the  $(x, t)$  plane.
- (b) Write the general solution.
- (c) Solve equation (1) with the initial condition  $u(x, 0) = x$ . Explain why the solution is fully determined by the initial condition.

*Solution.* (a)–(b) Equation of characteristics

$$\begin{aligned} \frac{dt}{1} &= \frac{dx}{xt} = \frac{du}{xe^{-t^2/2}} \implies \frac{dx}{x} = t dt \implies \\ \ln x - t^2/2 &= \ln(C_1) C_1 = xe^{-t^2/2} \implies du = xte^{-t^2/2} dt = C_1 t \implies \\ C_2 &= u - C_1 \frac{t^2}{2} u = C_2 + C_1 \frac{t^2}{2} \implies u = \frac{1}{2} xt^2 e^{-t^2/2} + f(xe^{-t^2/2}) \quad (2) \end{aligned}$$

with arbitrary function  $f$ .



- (c) From initial condition we conclude that  $f(x) = 0$  and

$$u(x, t) = \frac{1}{2} xt^2 e^{-t^2/2}. \quad (3)$$

□

**Problem 2** (4 pts). Find solution  $u(x, t)$  to

$$u_{tt} - 9u_{xx} = 18e^{-x^2}, \quad (4)$$

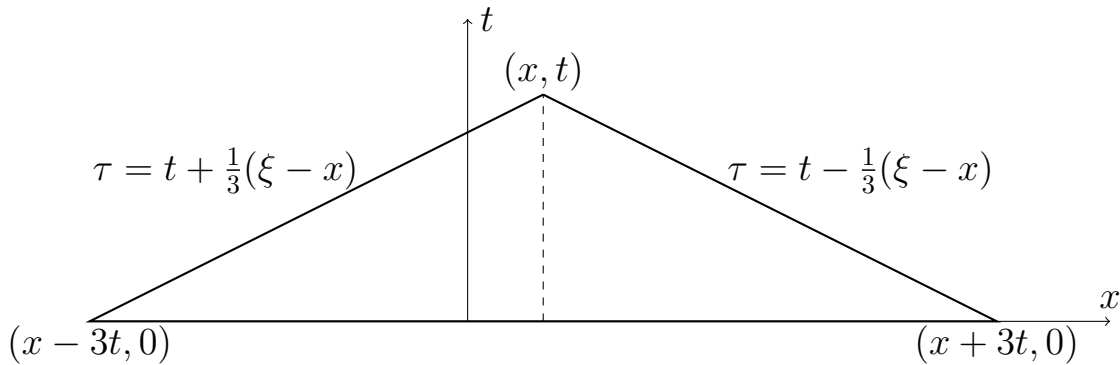
$$u|_{t=0} = 0, \quad u_t|_{t=0} = 0. \quad (5)$$

HINT: Change order of integration over characteristic triangle.  
Use  $\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz$ .

*Solution.* By D'Alembert formula

$$u(x, t) = 3 \iint_{\Delta(x, t)} e^{-\xi^2} d\xi d\tau, \quad (6)$$

where  $\Delta(x, t)$  is bounded by  $\tau = 0$ ,  $x - \xi - 3(t - \tau) = 0$ ,  $x - \xi + 3(t - \tau) = 0$ .



Then the double integral becomes

$$\begin{aligned} & 3 \int_{x-3t}^x \left( \int_0^{t+\frac{1}{3}(\xi-x)} d\tau \right) e^{-\xi^2} d\xi + 3 \int_x^{x+3t} \left( \int_0^{t-\frac{1}{3}(\xi-x)} d\tau \right) e^{-\xi^2} d\xi = \\ & \int_{x-3t}^x (3t - x + \xi) e^{-\xi^2} d\xi + \int_x^{x+3t} (3t + x - \xi) e^{-\xi^2} d\xi = \\ & (3t - x) \int_{x-3t}^x e^{-\xi^2} d\xi - \frac{1}{2} e^{-\xi^2} \Big|_{\xi=x-3t}^{\xi=x} + (3t + x) \int_x^{x+3t} e^{-\xi^2} d\xi + \frac{1}{2} e^{-\xi^2} \Big|_{\xi=x}^{\xi=x+3t} = \\ & (3t - x) \left( \operatorname{erf}(x) - \operatorname{erf}(x - 3t) \right) + (3t + x) \left( \operatorname{erf}(x + 3t) - \operatorname{erf}(x) \right) + \\ & \quad \frac{1}{2} \left( e^{-(x-3t)^2} + e^{-(x+3t)^2} - 2e^{-x^2} \right). \end{aligned}$$

□

**Problem 3** (4 pts). Find solution to

$$u_{tt} - u_{xx} = 0, \quad t > 0, \quad x > 2 - 2\sqrt{t+1} \quad (7)$$

$$u|_{t=0} = 0, \quad x > 0, \quad (8)$$

$$u_t|_{t=0} = 0, \quad x > 0, \quad (9)$$

$$u|_{x=2-2\sqrt{t+1}} = t, \quad t > 0. \quad (10)$$

*Solution.* (a) (2 pts) Solution to (7) is

$$u(x, t) = \phi(x + 3t) + \psi(x - 3t) \quad (11)$$

with unknown functions  $\phi$  and  $\psi$ . Plugging into (8)–(9) we get

$$\phi(x) + \psi(x) = 0, \quad 2\phi'(x) - 2\psi'(x) = 0 \implies \phi(x) = \psi(x) = 0 \quad x > 0$$

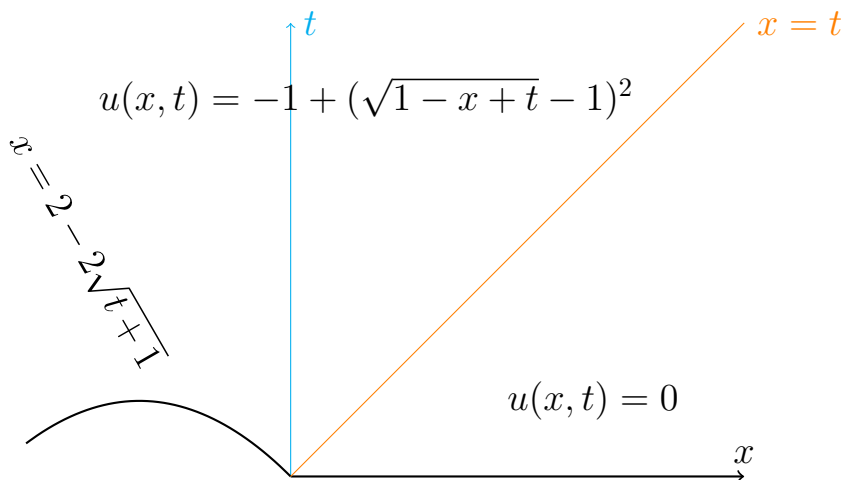
and  $u(x, t) = \sin(x + 3t)$  as  $x > t$ .

(b) (2 pts) Plugging into (10) we get  $\psi(2 - 2\sqrt{t+1} - t) = t$  as  $t > 0$ . For  $t > 0$   $2 - 2\sqrt{t+1} - t$  is a monotone decreasing function, from 0 to  $-\infty$ . Solve:

$$\begin{aligned} x = -2 - 2\sqrt{t+1} - t &\implies (\sqrt{t+1} + 1)^2 = -x \implies \\ \sqrt{t+1} &= \sqrt{1-x} - 1 \implies t = -1 + (\sqrt{1-x} - 1)^2, \end{aligned}$$

and therefore  $\psi(x) = -1 + (\sqrt{1-x} - 1)^2$  for  $x < 0$  and finally

$$u(x, t) = -1 + (\sqrt{1-x+t} - 1)^2 \quad x < t.$$



□

**Problem 4** (4 pts). Consider the PDE with boundary conditions:

$$u_{tt} + c^2 u_{xxxx} + au = 0, \quad 0 < x < L, \quad (12)$$

$$u|_{x=0} = u_x|_{x=0} = 0, \quad (13)$$

$$u_{xx}|_{x=L} = u_{xxx}|_{x=0} = 0|_{x=L} = 0 \quad (14)$$

where  $c > 0$  and  $a$  are constant. Prove that the energy  $E(t)$  defined as

$$E(t) = \frac{1}{2} \int_0^L (u_t^2 + c^2 u_{xx}^2 + au^2) dx \quad (15)$$

does not depend on  $t$ .

*Solution.*

$$\begin{aligned} \partial_t E(t) &= \int_0^L (u_t u_{tt} + c^2 u_{xxt} u_{xx} + a u_t u) dx \\ &= \int_0^L c^2 (-u_t u_{xxxx} + u_{xx} u_{xxt}) dx = \\ &= \int_0^L c^2 (-u_t u_{xxxx} - u_{tx} u_{xxx} + u_{tx} u_{xxx} + u_{xx} u_{xxt}) dx = \\ &= \int_0^L c^2 \left( -(u_t u_{xxx})_x + (u_{tx} u_{xx})_x \right) dx = \\ &= c^2 \left( -u_t u_{xxx} u_{tx} u_{xx} \right) \Big|_{x=0}^{x=L} = 0 \end{aligned}$$

due to boundary conditions. □

**Problem 5** (4 pts). Find the solution  $u(x, t)$  to

$$u_t = u_{xx} \quad -\infty < x < \infty, \quad t > 0, \quad (16)$$

$$u|_{t=0} = \begin{cases} x & |x| < 1, \\ 0 & |x| \geq 1 \end{cases} \quad (17)$$

$$\max |u| < \infty. \quad (18)$$

Calculate the integral.

*Hint:* For  $u_t = ku_{xx}$  use  $G(x, y, t) = \frac{1}{\sqrt{4\pi kt}} \exp(-(x - y)^2/4kt)$ . To calculate integral make change of variables and use  $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz$ .

*Solution.* Due to hint

$$\begin{aligned} u(x, t) &= \frac{1}{\sqrt{4\pi t}} \int_{-1}^1 e^{-\frac{1}{4t}(x-y)^2} y \, dy = \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-1-x}^{(1-x)} e^{-\frac{s^2}{4t}} (x+s) \, ds = \\ &= \frac{1}{\sqrt{\pi}} \int_{(-1-x)/\sqrt{4t}}^{(1-x)/\sqrt{4t}} e^{-z^2} (x+z\sqrt{4t}) \, dz = \\ &= x \left( \operatorname{erf}\left(\frac{1-x}{\sqrt{4t}}\right) + \operatorname{erf}\left(\frac{1+x}{\sqrt{4t}}\right) \right) - \\ &= \frac{\sqrt{t}}{\sqrt{\pi}} \left( e^{-(1+x)^2/4t} - e^{-(1-x)^2/4t} \right). \end{aligned}$$

□