Contents

0.1 Preface

This online Textbook based on half-year course APM346 at Department of Mathematics, University of Toronto but contains many additions.

This Textbook is *open* which means that anyone can use it without any permission. and *open-source*. Source (in the form of Markdown) could be downloaded. Also could be downloaded Textbook in pdf format and TeX Source (*When those are ready*).

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0.2 What one needs to know?

0.2.1 What one needs to know?

1. Multivariable Calculus
2. Ordinary Differential Equations

Assets: (useful but not required)

1. Complex Variables,
2. Elements of (Real) Analysis,
3. Any courses in Physics, Chemistry etc using PDEs (taken previously or now).

Multivariable Calculus

Differential calculus

1. Partial Derivatives (first, higher order), differential, gradient, chain rule;
2. Taylor formula;
3. Extremums, stationary points, classification of stationart points using second derivatives; *Asset:* Extremums with constrains.

Integral calculus

1. Multidimensional integral, calculations in Cartesian coordinates;
2. Change of variables, Jacobian, calculation in polar, cylindrical, spherical coordinates;
3. Path, Line, Surface integrals, calculations;
4. Green, Gauss, Stokes formulae;
5. ∇u, $\nabla \times A$, $\nabla \cdot A$, Δu where u is a scalar field and A is a vector field.

Ordinary Differential Equations

First order equations

1. Definition, Cauchy problem, existence and uniqueness;
2. Equations with separating variables, integrable, linear.

Higher order equations

1. Definition, Cauchy problem, existence and uniqueness;

Linear equations of order ≥ 2

1. General theory, Cauchy problem, existence and uniqueness;
2. Linear homogeneous equations, fundamental system of solutions, Wronskian;
3. Method of variations of constant parameters.

Linear equations of order ≥ 2 with constant coefficients

1. Fundamental system of solutions: simple, multiple, complex roots;
2. Solutions for equations with quasipolynomial right-hand expressions; method of undetermined coefficients.

Systems

1. General systems, Cauchy problem, existence and uniqueness;
2. Linear systems, linear homogeneous systems, fundamental system of solutions, Wronskian;
3. Method of variations of constant parameters;
4. Linear systems with constant coefficients.

Chapter 1

Introduction

1.1 PDE Motivations and Context

The aim of this is to introduce and motivate partial differential equations (PDE). The section also places the scope of studies in APM346 within the vast universe of mathematics.

1.1.1 What is a PDE?

A *partial differential equation (PDE)* is an *equation* involving *partial derivatives*. This is not so informative so let's break it down a bit.

1.1.1.1 What is a differential equation?

An *ordinary differential equation (ODE)* is an equation for a function which depends on one independent variable which involves the independent variable, the function, and derivatives of the function:

$$F(t, u(t), u(t), u^{(2)}(t), u^{(3)}(t), \ldots, u^{(m)}(t)) = 0.$$

This is an example of an ODE of *degree* m where m is a highest order of the derivative in the equation. Solving an equation like this on an interval $t \in [0, T]$ would mean finding a functoin $t \mapsto u(t) \in \mathbb{R}$ with the property that u and its derivatives intertwine in such a way that this equation is true for all values of $t \in [0, T]$. The problem can be enlarged by replacing the real-valued u by a vector-valued one $\mathbf{u}(t) = (u_1(t), u_2(t), \ldots, u_N(t))$. In this case we usually talk about *system of ODEs*.

Even in this situation, the challenge is to find functions depending upon exactly one variable which, together with their derivatives, satisfy the equation.

1.1.1.2 What is a partial derivative?

When you have function that depends upon several variables, you can differentiate with respect to either variable while holding the other variable constant. This spawns the idea of *partial derivatives*. As an example, consider a function depending upon two real variables taking values in the reals:

$$u : \mathbb{R}^n \to \mathbb{R}.$$

As $n = 2$ we sometimes visualize a function like this by considering its *graph* viewed as a surface in $\mathbb{R}^3$ given by the collection of points

$$\{(x, y, z) \in \mathbb{R}^3 : z = u(x, y)\}.$$

We can calculate the derivative with respect to x while holding y fixed. This leads to u_x, also expressed as $\partial_x u$, $\frac{\partial u}{\partial x}$, and $\frac{\partial}{\partial x}$. Similary, we can hold x fixed and differentiate with respect to y.

A *partial differential equation* is an equation for a function which depends on more than one independent variable which involves the independent variables, the function, and partial derivatives of the function:

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y), u_{xx}(x, y), u_{xy}(x, y), u_{yx}(x, y), u_{yy}(x, y)) = 0.$$

This is an example of a PDE of degree 2. Solving an equation like this would mean finding a function $(x, y) \to u(x, y)$ with the property that u and is partial derivatives intertwine to satisfy the equation.

Similarly to ODE case this problem can be enlarged by replacing the real-valued u by a vector-valued one $\mathbf{u}(t) = (u_1(t), u_2(t), \ldots, u_N(t))$. In this case we usually talk about *system of PDEs*.

1.1.2 Where PDEs are coming from?

Where PDEs are coming from?

PDEs are often referred as *Equations of Mathematical Physics* (or *Mathematical Physics* but it is incorrect as Mathematical Physics is now a separate field of mathematics) because many of PDEs are coming from different

domains of physics (acoustics, optics, elasticity, hydro and aerodynamics, electromagnetism, quantum mechanics, seismology etc).

However PDEs appear in other field of science as well (like quantum chemistry, chemical kinetics); some PDEs are coming from economics and financial mathematics, or computer science.

Many PDEs are originated in other fields of mathematics.

1.1.3 Examples of PDEs

(Some are actually systems)

- Simplest First Order Equation

$$u_x = 0;$$

- Transport Equation

$$u_t + cu_x = 0;$$

- Laplace's Equation (in 2D)

$$\Delta u := u_{xx} + u_{yy} = 0$$

or similarly in the higher dimensions;

- Heat Equation

$$u_t = k\Delta u;$$

(The expression Δ is called the *Laplacian* and is defined as $\partial_x^2 + \partial_y^2 + \partial_z^2$ on $\mathbb{R}^3$.)

- Schrödinger Equation (quantum mechanics)

$$iu_t + \Delta u = 0;$$

- Wave Equation

$$u_{tt} - c^2\Delta u = 0;$$

- Equation of oscillating rod (with one spatial variable) or plate (with

$$u_{tt} + K\Delta^2 u = 0;$$

- Maxwell Equation (electromagnetism)

$$\mathbf{E}_t - c\nabla \times \mathbf{H} = 0, \quad \mathbf{H}_t + c\nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{H} = 0;$$

- Dirac Equations (quantum mechanics);
- Elasticity Equation

$$\mathbf{u}_{tt} = \lambda \Delta \mathbf{u} + \mu \nabla (\nabla \cdot \mathbf{u});$$

- Navier-Stokes Equation (hydrodynamics for incompressible liquid)

$$\rho \mathbf{v}_t + (\mathbf{v} \cdot \nabla)\rho \mathbf{v} - \nu \Delta \mathbf{v} = -\nabla p, \quad \nabla \cdot \mathbf{v} = 0$$

where $\mathbf{v}$ is a velocity and p is the pressure; when viscosity $\nu = 0$ we get Euler equation;

- Yang-Mills Equation (elementary particles theory);
- Einstein Equation for General Relativity; and so on...

Remark 1.1.1. (a) Some of these examples are actually not single PDEs but the systems of PDEs.

(b) In all this examples there are *spatial variables* x, y, z and often *time variable* t but it is not necessarily so in all PDEs.

(c) Equations could be of different order with respect to different variables and it is important. However if not specified the order of equation is the highest order of the derivatives invoked.

(d) In the Textbook we will deal mainly with the wave equation, heat equation and Laplace equation in their simplest forms.

1.2 Initial and Boundary Value Problems

1.2.1 Problems for PDEs

We know that solutions of ODEs typically depend on one or several constants. For PDEs situation is more complicated. Consider simplest equations

$$u_x = 0, \tag{1.2.1}$$

$$v_{xy} = 0 \tag{1.2.2}$$

with $u = u(x, y)$ and $v = v(x, y)$. Equation (1.2.1) could be treaded as an ODE with respect to x and its solution is a constant but this is not a genuine constant as *it is constant only with respect to x and can depend on other variables*; so $u(x, y) = \phi(y)$.

Then for solution of (1.2.2) we have $v_y = \phi(y)$ where ϕ is an arbitrary function of one variable and it could be considered as ODE with respect to y; then $(v - g(y))_y = 0$ where $g(y) = \int \phi(y)\, dy$, and therefore $v - g(y) = f(x) \implies v(x, y) = f(x) + g(y)$ where f, g are arbitrary functions of one variable.

Considering these equations again but assuming that $u = u(x, y, z)$, $v = v(x, y, z)$ we arrive to $u = \phi(y, z)$ and $v = f(x, z) + g(y, z)$ where f, g are arbitrary functions of two variables.

Solutions to PDEs typically depend not on several arbitrary constants but on one or several arbitrary functions of $n - 1$ variables. For more complicated equations this dependance could be much more complicated and implicit. To select a right solutions we need to use some extra conditions.

The sets of such conditions are called *Problems*. Typical problems are

• IVP (initial value problem): one of variables is interpreted as *time* t and conditions are imposed at some moment; f.e. $u|_{t=t_0} = u_0$;

• BVP (boundary value problem) conditions are imposed on the boundary of the spatial domain Ω: f.e. $u|_{\partial\Omega} = \phi$ where $\partial\Omega$ is a boundary of *Omega*;

• IVBP (initial-boundary value problems aka mixed problems): one of variables is interpreted as *time* t and some conditions are imposed at some moment but other conditions are imposed on the boundary of the spatial domain.

Remark 1.2.1. In the course of ODEs students usually consider IVP only. F.e. for the second-order equation like

$$u_{xx} + a_1 u_x + a_2 u = f(x)$$

such problem is $u|_{x=x_0} = u_0$, $u_x|_{x=x_0} = u_1$. However one could consider BVPs like

$$(\alpha_1 u_x + \beta_1 u)|_{x=x_1} = \phi_1,$$
$$(\alpha_2 u_x + \beta_2 u)|_{x=x_2} = \phi_2$$

where solutions are sought on the interval $[x_1, x_2]$. Such are covered in advanced chapters of some of ODE textbooks (but not covered by a typical ODE class). We will need to cover such problems later in this Textbook.

1.2.2 Notion of "well-posedness"

We want that our PDE (or the system of PDEs) together with all these conditions satisfied the following requirements:

- Solutions must exist for all right-hand expressions (in equations and conditions);
- Solution must be unique;
- Solution must depend on this right-hand expressions continuously.

Such problems are called *well-posed*. PDEs are usually studied together with the problems which are well-posed for these PDEs. Different types of PDEs "admit" different problems.

Sometimes however one nedds to consider *ill-posed* problems.

1.3 Classification of equations

1.3.1 Linear and non-linear equations

Equations of the form

$$Lu = f(\mathbf{x}) \tag{1.3.1}$$

where Lu is a partial differential expression linear with respect to unknown function u is called *linear equation* (or *linear system*). This equation is *linear homogeneous equation* if $f = 0$ and *linear inhomogeneous equation* otherwise. For example,

$$Lu := a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + au = f(\mathbf{x}) \tag{1.3.2}$$

is linear; if all coefficients a_{jk}, a_j, a are constant, we call it *linear equation with constant coefficients*; otherwise we talk about *variable coefficients*.

Otherwise equation is called *non-linear*. However there is a more subtle classification of such equations. Equations of the type (1.3.1) where the right-hand expression f depend on the lower-order derivatives are called

semilinear, equations where both coefficients and right-hand expression depend on the lower-order derivatives are called *quasilinear*. For example

$$Lu := a_{11}(x,y)u_{xx} + 2a_{12}(x,y)u_{xy} + a_{22}(x,y)u_{yy} = f(x,y,u,u_x,u_y) \quad (1.3.3)$$

is semilinear, and

$$Lu := a_{11}(x,y,u,u_x,u_y)u_{xx} + 2a_{12}(x,y,u,u_x,u_y)u_{xy} + a_{22}(x,y,u,u_x,u_y)u_{yy} = f(x,y,u,u_x,u_y) \quad (1.3.4)$$

is quasilinear, while

$$F(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yx}) = 0 \quad (1.3.5)$$

is general *nonlinear*.

1.3.2 Elliptic, hyperbolic and parabolic equations

1.3.2.1 General

Consider second order equation (1.3.2):

$$Lu := \sum_{1\le i,j\le n} a_{ij}u_{x_ix_j} + \text{l.o.t.} = f(\mathbf{x}) \quad (1.3.6)$$

where l.o.t. means *lower order terms* with $a_{ij} = a_{ji}$. Let us change variables $\mathbf{x} = \mathbf{x}(\mathbf{x}')$. Then the *matrix of principal coefficients* $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$ in the new coordinate system becomes $A' = Q^*AQ$ where $Q = {T^*}^{-1}$ and $T = \left(\frac{\partial x_i}{\partial x'_j}\right)_{i,j=1,\dots,n}$ is a Jacobi matrix. The proof easily follows from the *chain rule* (Calculus II).

Therefore if the principal coefficients are real and constant, by a linear change of variables matrix of the principal coefficients could be reduced to the diagonal form, where diagonal elements could be either 1, or -1 or 0. Multiplying equation by -1 if needed we can assume that there are at least as many 1 as -1. In particular as $n = 2$ the principal part becomes either $u_{xx} + u_{yy}$, or $u_{xx} - u_{yy}$, or u_{xx} and such equations are called *elliptic*,

hyperbolic, and *parabolic* respectively (there will be always second derivative as otherwise it would be the first order equation). [1]

Let us consider equations in different dimensions:

1.3.3 2D

If we consider only 2-nd order equations with constant real coefficients then in appropriate coordinates they will look like either

$$u_{xx} + u_{yy} + \text{l.o.t} = f \tag{1.3.7}$$

or

$$u_{xx} - u_{yy} + \text{l.o.t.} = f. \tag{1.3.8}$$

and we call such equations *elliptic* and *hyperbolic* respectively.

What to do if one of the 2-nd derivatives is missing? We get *parabolic equations*

$$u_{xx} - cu_y + \text{l.o.t.} = f. \tag{1.3.9}$$

with $c \neq 0$ (we do not consider cu_y as a lower order term here) and IVP $u|_{y=0} = g$ is well-posed in the direction of $y > 0$ if $c > 0$ and in direction $y < 0$ if $c < 0$. We can dismiss $c = 0$ as not-interesting.

However this classification leaves out very important Schrödinger equation

$$u_{xx} + icu_y = 0 \tag{1.3.10}$$

with real $c \neq 0$. For it IVP $u|_{y=0} = g$ is well-posed in both directions $y > 0$ and $y < 0$ but it lacks many properties of parabolic equations (like maximum principle or mollification; still it has interesting properties on its own).

1.3.3.1 3D

Again, if we consider only 2-nd order equations with constant real coefficients then in appropriate coordinates they will look like either

$$u_{xx} + u_{yy} + u_{zz} + \text{l.o.t} = f \tag{1.3.11}$$

[1] This terminology comes from the curves of the second order (aka *conical sections*: if $a_{11}a_{22} - a_{12}^2 > 0$ equation $a_{11}\xi^2 + 2a_{12}\xi\eta + a_{22}\eta^2 + a_1\xi + a_2\eta = c$ generically defines an ellipse, if $a_{11}a_{22} - a_{12}^2 < 0$ this equation generically defines a hyperbole and if $a_{11}a_{22} - a_{12}^2 = 0$ it defines a parabole.

or

$$u_{xx} + u_{yy} - u_{zz} + \text{l.o.t.} = f. \tag{1.3.12}$$

and we call such equations *elliptic* and *hyperbolic* respectively.

Also we get *parabolic equations* like

$$u_{xx} + u_{yy} - cu_z + \text{l.o.t.} = f. \tag{1.3.13}$$

What about

$$u_{xx} - u_{yy} - cu_z + \text{l.o.t.} = f? \tag{1.3.14}$$

Algebraist-formalist would call it parabolic-hyperbolic but since this equation exhibits no interesting analytic properties (unless one considers lack of such properties interesting; in particular, IVP is ill-posed in both directions) it would be a perversion.

Yes, there will be Schrödinger equation

$$u_{xx} + u_{yy} + icu_z = 0 \tag{1.3.15}$$

with real $c \neq 0$ but $u_{xx} - u_{yy} + icu_z = 0$ would also have IVP $u|_{z=0} = g$ well-posed in both directions.

1.3.3.2 4D

Here we would get also *elliptic*

$$u_{xx} + u_{yy} + u_{zz} + u_{tt} + \text{l.o.t.} = f, \tag{1.3.16}$$

hyperbolic

$$u_{xx} + u_{yy} + u_{zz} - u_{tt} + \text{l.o.t.} = f, \tag{1.3.17}$$

but also *ultrahyperbolic*

$$u_{xx} + u_{yy} - u_{zz} - u_{tt} + \text{l.o.t.} = f \tag{1.3.18}$$

which exhibits some interesting analytic properties but these equations are way less important than elliptic, hyperbolic or parabolic.

Parabolic and Schrödinger will be here as well.

Remark 1.3.1. The notions of elliptic, hyperbolic or parabolic equations are generalized to higher dimensions (trivially) and to higher-order equations but most of the randomly written equations do not belong to any of these types and there is no reason to classify them.

There is no complete classifications of PDEs and cannot be because any reasonable classification should not be based on how equation looks like but on the reasonable analytic properties it exhibits (which IVP or BVP are well-posed etc).

1.3.3.3 Equations of the variable type

To make things even more complicated there are equations changing types from point to point, f.e. Tricomi equation

$$u_{xx} + xu_{yy} = 0 \tag{1.3.19}$$

which is elliptic as $x > 0$ and hyperbolic as $x < 0$ and at $x = 0$ has a "parabolic degeneration". It is a toy-model describing stationary transsonic flow of gas. These equations are called *equations of the variable type* (aka *mixed equations*).

Our purpose was not to give exact definitions but to explain a situation.

1.3.4 Scope of this Textbook

- We mostly consider *linear* PDE problems.
- We mostly consider *well-posed problems*
- We mostly consider problems with *constant coefficients.*
- We do not consider *numerical methods.*

1.4 Origin of some equations

1.4.1 Wave equation

Example 1.4.1. Consider a string as a curve $y = u(x,t)$ with a tension T and with a linear density ρ. We assume that $|u_x| \ll 1$.

Observe that at point x the part of the string to the left from x pulls it up with a force $-F(x) := -Tu_x$. Indeed, the force T is directed along the curve and the slope of angle θ between the tangent to the curve and horizontal line is u_x; so $\sin(\theta) = u_x/\sqrt{1+u_x^2}$ which under our assumption we can replace by u_x. On the other hand at point x the part of the string

to the right from x pulls it up with a force $F(x) := -Tu_x$. Therefore the total y-component of force applied to the segment of the string between $J = [x_1, x_2]$ equals

$$F(x_2) - F(x_1) = \int_J \partial F(x)\, dx = \int_J Tu_{xx}\, dx.$$

According to Newton law it must be equal to $\int_J \rho u_{tt}\, dx$ where ρdx is the mass and u_{tt} is the acceleration of the infinitesimal segment $[x, x + dx]$:

$$\int_J \rho u_{tt}\, dx = \int_J Tu_{xx}\, dx.$$

Since this equality holds for any segment J, the integrands coincide:

$$\rho u_{tt} = Tu_{xx} \tag{1.4.1}$$

Example 1.4.2. Consider a membrane as a surface $z = u(x, y, t)$ with a tension T and with a surface density ρ. We assume that $|u_x|, |u_{yy}| \ll 1$.

Consider a domain D on the plane, its boundary L and a small segment of the length ds of this boundary. Then the outer domain pulls this segment up with the force $-T\mathbf{n} \cdot \nabla u\, ds$ where $\mathbf{n}$ is the inner unit normal to this segment. Indeed, the total force is Tds but it pulls along the surface and the slope of the surface in the direction of $\mathbf{n}$ is $\approx \mathbf{n} \cdot \nabla u$.

Therefore the total z-component of force applied to D between $x = x_1$ and equals due to (A1.1.1)

$$-\int_L T\mathbf{n} \cdot \nabla u\, ds = \iint \nabla \cdot (T\nabla u)\, dxdy$$

According to Newton law it must be equal to $\iint_D \rho u_{tt}\, dxdy$ where $\rho dxdy$ is the mass and u_{tt} is the acceleration of the element of the area:

$$\iint_D \rho u_{tt}\, dxdy = \iint_{[x_1,x_2]} T\Delta u\, dx$$

as $\nabla \cdot (T\nabla u) = T\Delta u$. Since this equality holds for any segment, the integrands coincide:

$$\rho u_{tt} = T\Delta u. \tag{1.4.2}$$

Example 1.4.3. Consider a gas and let $\mathbf{v}$ be its velocity and ρ its density. Then

$$\rho \mathbf{v}_t + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p, \tag{1.4.3}$$
$$\rho_t + \nabla\cdot(\rho\mathbf{v}) = 0 \tag{1.4.4}$$

where p is the pressure. Indeed, in (1.4.3) the left-hand expression is $\rho\frac{d}{dt}\mathbf{v}$) (the mass per unit of the volume multiplied by acceleration) and the right hand expression is the force of the pressure; no other forces are considered. Further (1.4.4) is *continuity equation* which means the mass conservation since the flow of the mass through the surface element dS in the direction of the normal $\mathbf{n}$ for time dt equals $\rho\mathbf{n}\cdot\mathbf{v}$.

We need to add $p = p(\rho)$. Assuming that $\mathbf{v}$, $\rho - \rho_0$ and their first derivatives are small ($\rho_0 = \text{const}$) we arrive instead to

$$\rho_0 \mathbf{v}_t = -p'(\rho_0)\nabla\rho, \tag{1.4.5}$$
$$\rho_t + \rho_0\nabla\cdot\mathbf{v} = 0 \tag{1.4.6}$$

and then applying $\nabla\cdot$ to (1.4.5) and ∂_t to (1.4.6) we arrive to

$$\rho_{tt} = c^2\Delta\rho \tag{1.4.7}$$

with $c = \sqrt{p'(\rho_0)}$.

1.4.2 Diffusion equation

Example 1.4.4. Let u be a concentration of parfume in the still air. Consider some volume V, then the quantity of the parfume in V at time t equals $\iiint_V u\,dxdydz$ and its increment for time dt equals

$$\iiint_V u_t\,dxdydz \times dt.$$

On the other hand, the law of diffusion states that the flow of parfume through the surface element dS in the direction of the normal $\mathbf{n}$ for time dt equals $-k\nabla u dSdt$ where k is a *diffusion coefficient* and therefore the flow of the parfume into V from outside for time dt equals

$$\iint_S (-k\nabla u)\,dS \times dt = \iiint_V \nabla\cdot(k\nabla u)\,dxdydz \times dt$$

due to (A1.1.2). Therefore if there are neither sources nor sinks (negative sources) in V these two expression must be equal

$$\iiint_V u_t\,dxdydz = \iiint_V \nabla\cdot(k\nabla u)\,dxdydz$$

where we divided by dt. Since these equalities must hold for any volume the integrands must coincide and we arrive to *continuity equation*:

$$u_t = \nabla\cdot(k\nabla u). \tag{1.4.8}$$

If k is constant we get

$$u_t = k\Delta u. \tag{1.4.9}$$

Example 1.4.5. Consider heat propagation. Let T be a temperature. Then the heat energy contained in the volume V equals $\iiint_V Q(T)\,dxdydz$ and the heat flow (the flow of the heat energy) through the surface element dS in the direction of the normal $\mathbf{n}$ for time dt equals $-k\nabla T dSdt$ where k is a *thermoconductivity coefficient*. Applying the same arguments as above we arrive to

$$Q_t = \nabla\cdot(k\nabla T). \tag{1.4.10}$$

which we rewrite as

$$cT_t = \nabla\cdot(k\nabla T). \tag{1.4.11}$$

where $c = \frac{\partial Q}{\partial T}$ is a *thermocapacity coefficient*.

If both c and k are constant we get

$$cT_t = k\Delta T. \tag{1.4.12}$$

In the real life c and k depend on T. Further, $Q(T)$ has jumps at *phase transition temperature*. For example to melt an ice to a water (both at 0°) requires a lot of heat and to boil the water to a vapour (both at 100°) also requires a lot of heat.

1.4.3 Laplace equation

Example 1.4.6. Considering all examples above and assuming that unknown function does not depend on t (and thus replacing corresponding derivatives by 0) we arrive to the corresponding *stationary equations* the simplest of which is

$$\Delta u = 0. \tag{1.4.13}$$

Example 1.4.7. In the theory of complex variables one studies holomorphic function $f(z)$ satisfying a Cauchy-Riemann equation $\partial_{\bar z} f = 0$. Here $z = x+iy$, $f = u(x,y)+iv(x,y)$ and $\partial_{\bar z} = \frac{1}{2}(\partial_x + i\partial_y)$; then this equation could be rewritten as

$$\partial_x u - \partial_y v = 0, \tag{1.4.14}$$
$$\partial_x v + \partial_y u = 0, \tag{1.4.15}$$

which imply that both u, v satisfy (1.4.13).

1.5 Problems to Chapter 1

Problem 1.5.1. Consider first order equations and determine if they are linear homogeneous, linear inhomogeneous or non-linear (u is an unknown function):

$$u_t + xu_x = 0, \tag{1.5.1}$$
$$u_t + uu_x = 0, \tag{1.5.2}$$
$$u_t + xu_x - u = 0, \tag{1.5.3}$$
$$u_t + uu_x + x = 0, \tag{1.5.4}$$
$$u_t + u_x - u^2 = 0, \tag{1.5.5}$$
$$u_t^2 - u_x^2 - 1 = 0, \tag{1.5.6}$$
$$u_x^2 + u_y^2 - 1 = 0, \tag{1.5.7}$$
$$xu_x + yu_y + zu_z = 0, \tag{1.5.8}$$
$$u_x^2 + u_y^2 + u_z^2 - 1 = 0, \tag{1.5.9}$$
$$u_t + u_x^2 + u_y^2 = 0. \tag{1.5.10}$$

For non-linear equations determine if they are *quasilinear* (quasilinear= linear with respect to first-order derivatives (u_x, u_y), but not to derivatives and function itself (u_x, u_y, u).

Problem 1.5.2. Consider equations and determine their order; determine if they are linear homogeneous, linear inhomogeneous or non-linear (u is an

unknown function):

$$u_t + (1+x^2)u_{xx} = 0, \tag{1.5.11}$$
$$u_t - (1+u^2)u_{xx} = 0, \tag{1.5.12}$$
$$u_t + u_{xxx} = 0, \tag{1.5.13}$$
$$u_t + uu_x + u_{xxx} = 0, \tag{1.5.14}$$
$$u_{tt} + u_{xxxx} = 0, \tag{1.5.15}$$
$$u_{tt} + u_{xxxx} + u = 0, \tag{1.5.16}$$
$$u_{tt} + u_{xxxx} + \sin(x) = 0, \tag{1.5.17}$$
$$u_{tt} + u_{xxxx} + \sin(x)\sin(u) = 0. \tag{1.5.18}$$

Problem 1.5.3. Find the general solutions to the following equations

$$u_{xy} = 0, \tag{1.5.19}$$
$$u_{xy} = 2u_x, \tag{1.5.20}$$
$$u_{xy} = e^{x+y}, \tag{1.5.21}$$
$$u_{xy} = 2u_x + e^{x+y}. \tag{1.5.22}$$

Hint: Introduce $v = u_x$ and find it first.

Problem 1.5.4. Find the general solutions to the following equations

$$uu_{xy} = u_x u_y, \tag{1.5.23}$$
$$uu_{xy} = 2u_x u_y, \tag{1.5.24}$$
$$u_{xy} = u_x u_y \tag{1.5.25}$$

Hint: Divide two first equations by uu_x and observe that both the right and left-hand expressions are derivative with respect to y of $\ln(u_x)$ and $\ln(u)$ respectively. Divide the last equation by u_x.

Problem 1.5.5. Find the general solutions to the following equations

$$u_{xxyy} = 0, \tag{1.5.26}$$
$$u_{xyz} = 0, \tag{1.5.27}$$
$$u_{xxyy} = \sin(x)\sin(y), \tag{1.5.28}$$
$$u_{xyz} = \sin(x)\sin(y)\sin(z), \tag{1.5.29}$$
$$u_{xyz} = \sin(x) + \sin(y) + \sin(z). \tag{1.5.30}$$

Chapter 2

1-dimensional waves

In this Chapter we first consider first order PDE and then move to 1-dimensional wave equation which we analyze by the *method of characteristics*.

2.1 First order PDEs

2.1.1 Introduction

Consider PDE

$$au_t + bu_x = 0. \tag{2.1.1}$$

Note that the left-hand expression is a *directional derivative* of u in the direction $\ell = (a, b)$. Consider an *integral lines* of this vector field:

$$\frac{dt}{a} = \frac{dx}{b}. \tag{2.1.2}$$

Remark 2.1.1. Recall from ODE cours that an *integral line* of the vector field is a line, tangent to it in each point.

2.1.2 Constant coefficients

If a and b are constant then integral curves are just straight lines $t/a - x/b = C$ where C is a constant along integral curves and it labels them (at least as long as we consider the whole plane (x, t)). Therefore u depends only on C:

$$u = \phi\big(\frac{t}{a} - \frac{x}{b}\big) \tag{2.1.3}$$

where ϕ is an arbitrary function.

This is a *general solution* of our equation.

Consider initial value condition $u|_t = 0 = f(x)$. It allows us define ϕ: $\phi(x) = f(x)$. Plugging in u we get

$$u = f\big(\frac{t}{a} - \frac{x}{b}\big). \tag{2.1.4}$$

It is a solution of IVP

$$\begin{cases} au_t + bu_x = 0, \\ u(x,0) = f(x). \end{cases} \tag{2.1.5}$$

Obviously we need to assume that $a \neq 0$.

If $a = 1$ we can rewrite general solution in the form $u(x,t) = \phi_1(x - bt)$ where $\phi_1(x) = \phi(-x/b)$ is another arbitrary function.

Definition 2.1.1. Solutions $u = \chi(x - ct)$ are *running waves* where c is a *propagation speed*.

2.1.3 Variable coefficients

If a and/or b are not constant these integral lines are curves.

Example 2.1.1. Consider equation $u_t + tu_x = 0$. Then equation of the integral curve is $\frac{dt}{1} = \frac{dx}{t}$ or equivalently $tdt - dx = 0$ which solves as $x - \frac{1}{2}t^2 = C$ and therefore $u = \phi(x - \frac{1}{2}t^2)$ is a general solution to this equation.

One can see easily that $u = f(x - \frac{1}{2}t^2)$ is a solution of IVP.

Example 2.1.2. Consider the same equation but let us consider IVP as $x = 0$: $u(0,t) = g(t)$. However it is not a good problem: first, some integral curves intersect line $x = 0$ more than once and if in different points of intersection of the same curve initial values are different we get a contradiction (therefore problem is not solvable for g which are not even functions).

On the other hand, if we consider even function g (or equivalently impose initial condition only for $t > 0$) then u is not defined on the curves which are not intersecting $x = 0$ (which means that u is not defined for $x > \frac{1}{2}t^2$.)

In this example both *solvability* and *unicity* are broken.

2.1.4 Right-hand expression

Consider the same equation albeit with the right-hand expression

$$au_t + bu_x = f. \tag{2.1.6}$$

Then as $\frac{dt}{a} = \frac{dx}{b}$ we have $du = u_t dt + u_x dx = (au_t + bu_x)\frac{dt}{a} = f\frac{dt}{a}$ and therefore we expand our ordinary equation (2.1.2) to

$$\frac{dt}{a} = \frac{dx}{b} = \frac{du}{f}. \tag{2.1.7}$$

Example 2.1.3. Consider problem $u_t + u_x = x$. Then $\frac{dx}{1} = \frac{dt}{1} = \frac{du}{x}$. Then $x - t = C$ and $u - \frac{1}{2}x^2 = D$ and we get $u - \frac{1}{2}x^2 = \phi(x - t)$ as relation between C and D both of which are constants along integral curves. Here ϕ is an arbitrary function. So $u = \frac{1}{2}x^2 + \phi(x - t)$ is a general solution. Imposing Imposing initial condition $u|_{t=0} = 0$ (sure, we could impose another condition) we have $\phi(x) = -\frac{1}{2}x^2$ and plugging into u we get $u(x,t) = \frac{1}{2}x^2 - \frac{1}{2}(x-t)^2 = xt - \frac{1}{2}t^2$.

Example 2.1.4. Consider $u_t + xu_x = xt$. Then $\frac{dt}{1} = \frac{dx}{x} = \frac{du}{xt}$. Solving the first equation $t - \ln x = -\ln C \implies x = Ce^t$ we get integral curves. Now we have

$$\frac{du}{xt} = dt \implies du = xtdt = Cte^t dt \implies u = C(t-1)e^t + D = x(t-1) + D$$

where D must be constant along integral curves and therefore $D = \phi(xe^{-t})$ with an arbitrary function ϕ. So $u = x(t-1) + \phi(xe^{-t})$ is a general solution of this equation.

Imposing initial condition $u|_{t=0} = 0$ (sure, we could impose another condition) we have $\phi(x) = x$ and then $u = x(t - 1 + e^{-t})$.

2.1.5 Linear and semilinear equations

Definition 2.1.2. If $a = a(x,t)$ and $b = b(x,t)$ equation is *semilinear*.

In this case we first define integral curves which do not depend on u and then find u as a solution of ODE along these curves.

Definition 2.1.3. Furthermore if f is a linear function of u: $f = c(x,t)u + g(x,t)$ original equation is *linear*.

In this case the last ODE is also linear.

Example 2.1.5. Consider $u_t + xu_x = u$. Then $\frac{dt}{1} = \frac{dx}{x} = \frac{du}{u}$. Solving the first equation $t - \ln x = -\ln C \implies x = Ce^t$ we get integral curves. Now we have

$$\frac{du}{u} = dt \implies \ln u = t + \ln D \implies u = De^t = \phi(xe^{-t})e^t$$

which is a general solution of this equation.

Imposing initial condition $u|_{t=0} = x^2$ (sure, we could impose another condition) we have $\phi(x) = x^2$ and then $u = x^2e^{-t}$.

Example 2.1.6. Consider $u_t + xu_x = -u^2$. Then $\frac{dt}{1} = \frac{dx}{x} = -\frac{du}{u^2}$. Solving the first equation $x = Ce^t$ we get integral curves. Now we have

$$-\frac{du}{u^2} = dt \implies u^{-1} = t + D \implies u = (t + \phi(xe^{-t}))^{-1}.$$

which is a general solution of this equation.

2.1.6 Quasilinear equations

Definition 2.1.4. If a and/or b depend on u this is *quasininear* equation.

For such equations integral curves depend on the solution which can lead to breaking of solution.

Example 2.1.7. Consider Burgers equation $u_t + uu_x = 0$ (which is an extremely simplified model of gas dynamics.) We have $\frac{dt}{1} = \frac{dx}{u} = \frac{du}{0}$ and therefore $u = \text{const}$ along integral curves and therefore integral curves are $x - ut = C$.

Consider initial problem $u(x, 0) = f(x)$. We take initial point $(y, 0)$, find here $u = f(y)$, then $x - f(y)t = y$ (think why?) and we get $u = f(y)$ where $y = y(x, t)$ is a solution of equation $x = f(y)t + y$.

The trouble is that we can define y for all x only if $\frac{\partial}{\partial y}\big(f(y)t + y\big)$ does not vanish. So, $f'(y)t + 1 \neq 0$.

This is possible for all $t > 0$ if and only if $f'(y) \geq 0$ i.e. f is a monotone non-decreasing function.

So, classical solution breaks if f is not a monotone non-decreasing function. A proper understanding of the *global solution* for such equation goes well beyond our course.

Example 2.1.8. Traffic flow is considered in Appendix 2.A.

2.1.7 IBVP

Consider IBVP (initial-boundary value problem) for constant coefficient equation

$$\begin{cases} u_t + cu_x = 0, & x > 0,\ t > 0, \\ u|_{t=0} = f(x) & x > 0. \end{cases} \tag{2.1.8}$$

The general solution is $u = \phi(x - ct)$ and plugging into initial data we get $\phi(x) = f(x)$ (as $x > 0$).

So, $u(x,t) = f(x - ct)$. Done!–Not so fast. f is defined only for $x > 0$ so u is defined for $x - ct > 0$ (or $x > ct$). It covers the whole quadrant if $c \le 0$ (so waves run to the left) and only in this case we are done.

If $c > 0$ (waves run to the right) u is not defined as $x < ct$ and to define it here we need a *boundary condition* at $x = 0$. So we get IBVP (initial-boundary value problem)

$$\begin{cases} u_t + cu_x = 0, & x > 0, t > 0, \\ u|_{t=0} = f(x) & x > 0, \\ u|_{x=0} = g(t) & t > 0. \end{cases} \tag{2.1.9}$$

Then we get $\phi(-ct) = g(t)$ as $t > 0$ which implies $\phi(x) = g(-\frac{1}{c}x)$ as $x < 0$ and then $u(x,t) = g(-\frac{1}{c}(x - ct)) = g(t - \frac{1}{c}x)$ as $x < ct$.

So solution is

$$u = \begin{cases} f(x - ct) & x > ct, \\ g(t - \dfrac{1}{c}x) & x < ct. \end{cases} \tag{2.1.10}$$

2.A Derivation of a PDE describing traffic flow

The purpose of this discussion is to derive a toy-model PDE that describes a congested one-dimensional highway (in one direction). Let

- $\rho(x,t)$ denote the traffic density: the number of cars per kilometer at time t located at position x;
- $q(x,t)$ denote the traffic flow: the number of cars per hour passing a fixed place x at time t;

- $N(t, a, b)$ denote the number of cars between position $x = a$ and $x = b$ at time t.

It is directly implied by definition of $\rho(x, t)$ is

$$N(t, a, b) = \int_a^b \rho(t, x)dx. \tag{2.A.1}$$

By definition of q and conservation of cars we have:

$$\begin{aligned}\frac{\partial N}{\partial t}(t, a, b) &= \lim_{h\to 0} \frac{N(t + h, a, b) - N(t, a, b)}{h} \\ &= \lim_{h\to 0} \frac{h(q(t, a) - q(t, b))}{h} = q(t, a) - q(t, b)\end{aligned} \tag{2.A.2}$$

Differentiating (2.A.1) with respect to t

$$\frac{\partial N}{\partial t} = \int_a^b \rho_t(t, x)dx$$

making it equal to (2.A.2) we get the integral form of "conservation of cars":

$$\int_a^b \rho_t(t, x)dx = q(t, a) - q(t, b).$$

Since a and b are arbitrary, it implies that $\rho_t = -q_x$. The PDE

$$\rho_t + q_x = 0 \tag{2.A.3}$$

is conservation of cars equation.

After equation (2.A.3) or more general equation

$$\rho_t + \rho_x = f(x, t) \tag{2.A.4}$$

(where $f = f_{in} - f_{out}$, $f_{in}dxdt$ and $f_{out}dxdt$ are numbers of cars entering/exiting highway for time dt at the segment of the length dx) has been derived we need to connect ρ and q.

The simplest is $q = c\rho$ with a constant c: all cars are moving with the same speed c. Then (2.A.3) becomes

$$\rho_t + c\rho_x = 0. \tag{2.A.5}$$

However more realistic would be $c = c(\rho)$ being monotone decreasing function of ρ with $c(0) = c_0$ (speed on empty highway) and $c(\bar\rho) = 0$ where $\bar\rho$ is a density where movement is impossible. Assume that $q(\rho) = c(\rho)\rho$ has a single maximum at ρ^*. Then

$$\rho_t + v\rho_x = 0. \tag{2.A.6}$$

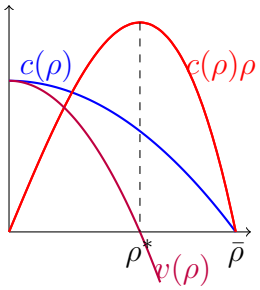
with

$$v = v(\rho) = q'(\rho) = c(\rho) + c'(\rho)\rho \tag{2.A.7}$$

where $'$ is a derivative with respect to ρ. Therefore ρ remains constant along *integral line* $x - v(\rho)t = \text{const}$.

$v = v(\rho)$ is the *group speed* namely the speed with which point where density equals given density ρ is moving. Here $v(\rho) < c(\rho)$ (because $c' < 0$), so group speed is less than the speed of the cars (simply cars may join the group from behind and leave it from its front). Further $v > 0$ as $\rho < \rho^*$ and $v < 0$ as $\rho > \rho^*$; in the latter case the group moves backward: the jam grows faster than it moves.

Also the integral lines may intersect (loose and faster moving group catches up with dense and slower group). When it happens ρ becomes discontinuous, (2.A.3) still holds but (2.A.6) fails (it is no more equivalent to (2.A.3)) and the theory becomes really complicated.



Remark 2.1.1. In the toy-model of gas dynamics $c(\rho) = \rho$, or more general $c'(\rho) > 0$ and $v(\rho) > c(\rho)$.

2.1.1 Problems

Problem 2.1.1. (a) Draw characteristics and find the general solution to each of the following equations

$$2u_t + 3u_x = 0; \tag{2.1.8}$$
$$u_t + tu_x = 0; \tag{2.1.9}$$
$$u_t + xu_x = 0; \tag{2.1.10}$$
$$u_t + x^2u_x = 0 \tag{2.1.11}$$
$$u_x + x^3u_x = 0. \tag{2.1.12}$$

(b) Consider IVP problem $u|_{t=0} = f(x)$ as $-\infty < x < \infty$; does solution always exists? If not, what conditions should satisfy $f(x)$?

(c) Where this solution is uniquely determined?

(d) Consider this equation in $\{t > 0, x > 0\}$ with the initial condition $u|_{t=0} = f(x)$ as $x > 0$; where this solution defined? Is it defined everywhere in $\{t > 0, x > 0\}$ or do we need to impose condition at $x = 0$? In the latter case impose condition $u|_{x=0} = g(t)$ $(t > 0)$ and solve this IVBP;

(e) Consider this equation in $\{t > 0, x < 0\}$ with the initial condition $u|_{t=0} = f(x)$ as $x < 0$; where this solution defined? Is it defined everywhere in $\{t > 0, x < 0\}$ or do we need to impose condition at $x = 0$? In the latter case impose condition $u|_{x=0} = g(t)$ $(t > 0)$ and solve this IVBP;

(f) Consider problems (d) as $t < 0$;

(g) Consider problems (e) as $t < 0$;

Problem 2.1.2. (a) Find the general solution to each of the following equations

$$xu_x + yu_y = 0, \tag{2.1.13}$$
$$xu_x - yu_y = 0 \tag{2.1.14}$$

in $\{(x, y) \neq (0, 0)\}$; when this solution is continuous at $(0, 0)$? Explain the difference between these two cases;

(b) Find the general solution to each of the following equations

$$yu_x + xu_y = 0, yu_x - xu_y = 0 \tag{2.1.15}$$

in $\{(x,y) \neq (0,0)\}$; when this solution is continuous at $(0,0)$? Explain the difference between these two cases;

Problem 2.1.3. In the same way consider equations

$$(x^2+1)yu_x + (y^2+1)xu_y = 0;$$
$$(x^2+1)yu_x - (y^2+1)xu_y = 0.$$

Problem 2.1.4. Find the solution of

$$\begin{cases} u_x + 3u_y = xy, \\ u|_{x=0} = 0. \end{cases} \tag{2.1.16}$$

Problem 2.1.5. Find the general solutions to each of

$$yu_x - xu_y = x; \tag{2.1.17}$$
$$yu_x - xu_y = x^2; \tag{2.1.18}$$
$$xu_x + xu_y = x; \tag{2.1.19}$$
$$yu_x + xu_y = x^2; \tag{2.1.20}$$

In one instance solution does not exist. Explain why.

Problem 2.1.6. Solve IVP

$$u_t + uu_x = 0, \qquad t > 0; \tag{2.1.21}$$
$$u|_{t=0} = f(x) \tag{2.1.22}$$

and describe domain in (x,t) where this solution is properly defined with one of the following initial data

$$f(x) = \quad \tanh(x); \tag{2.1.23}$$
$$f(x) = -\tanh(x); \tag{2.1.24}$$
$$f(x) = \begin{cases} -1 & x < -a, \\ x/a & -a \le x \le a, \\ 1 & x > a; \end{cases} \tag{2.1.25}$$
$$f(x) = \begin{cases} 1 & x < -a, \\ -x/a & -a \le x \le a, \\ -1 & x > a; \end{cases} \tag{2.1.26}$$

$$f(x) = \begin{cases} -1 & x < 0, \\ 1 & x > 0; \end{cases} \tag{2.1.27}$$

$$f(x) = \sin(x). \tag{2.1.28}$$

$$f(x) = \begin{cases} \sin(x) & |x| < \pi, \\ 0 & |x| > \pi, \end{cases} \tag{2.1.29}$$

$$f(x) = \begin{cases} -\sin(x) & |x| < \pi, \\ 0 & |x| > \pi, \end{cases} \tag{2.1.30}$$

Here $a > 0$ is a parameter.

2.2 First order PDEs (continued)

2.2.1 Multidimensional equations

Remark 2.2.1. Multidimensional equations (from linear to semilinear)

$$au_t + \sum_{j=1}^{n} b_j u_{x_j} = f(x_1, \dots, x_n, t, u) \tag{2.2.1}$$

and nonlinear

$$F(x_1, \dots, x_n, t, u, u_{x_1}, \dots, u_{x_n}, u_t) = 0 \tag{2.2.2}$$

could be solved by the same methods.

For example, if $a = 1$, $b_j = \text{const}$ and $f = 0$ the general solution of (2.2.1) is $u = \phi(x_1 - b_1 t, \dots, x_n - b_n t)$ where ϕ is an arbitrary function of n variables.

2.2.2 Multidimensional non-linear equations

We consider fully non-linear multidimensional equation in $\mathbb{R}^n$

$$F(x, u, \nabla u) = 0 \tag{2.2.3}$$

(we prefer such notations here) with $x = (x_1, \dots, x_n)$, $\nabla u = (u_{x_1}, \dots, u_{x_n})$ and the *initial condition*

$$u|_\Sigma = g \tag{2.2.4}$$

where Σ is a hypersurface. If $F = u_{x_1} - f(x, u, u_{x_2}, \dots, u_{x_n})$ and $\Sigma = \{x_1 = 0\}$ then such problem has a local solution and it is unique. However we consider a general form under assumption

$$\sum_{1\le j\le n} F_{p_j}(x,u,p)\big|_{p=\nabla u}\nu_j \neq 0 \tag{2.2.5}$$

where $\nu = \nu(x) = (\nu_1, \dots, \nu_n)$ is a normal to Σ at point x.

Consider $p = \nabla u$ and consider a *characteristic curve* in x-space (n-dimensional) $\frac{dx_j}{dt} = F_{p_j}$ which is exactly (2.2.8) below. Then by the chain rule

$$\frac{dp_j}{dt} = \sum_k p_{j,x_k}\frac{dx_k}{dt} = \sum_k u_{x_j x_k} F_{p_k} \tag{2.2.6}$$

$$\frac{du}{dt} = \sum_k u_{x_k}\frac{dx_k}{dt} = \sum_k p_k F_{p_k}. \tag{2.2.7}$$

The last equation is exactly (2.2.10) below. To deal with (2.2.6) we differentiate (2.2.3) by x_j; by the chain rule we get

$$0 = \partial_{x_j}\big(F(x,u,\nabla u)\big) = F_{x_j} + F_u u_{x_j} + \sum_k F_{p_k} p_{k,x_j} = F_{x_j} + F_u u_{x_j} + \sum_k F_{p_k} u_{x_k x_j}$$

and therefore the r.h.e. in (2.2.6) is equal to $-F_{\{x_j\}} - F_u u_{\{x_j\}}$ and we arrive exactly to equation (2.2.9) below.

So we have a system defining a *characteristic trajectory* which lives in $(2n+1)$-dimensional space:

$$\frac{dx_j}{dt} = F_{p_j}, \tag{2.2.8}$$

$$\frac{dp_j}{dt} = -F_{x_j} - F_u p_j, \tag{2.2.9}$$

$$\frac{du}{dt} = \sum_{1\le j\le n} F_{p_j} p_j. \tag{2.2.10}$$

Characteristic curve is n-dimensional x-projection of the *characteristic trajectory*. Condition (2.2.5) simply means that characteristic curve *is transversal* (i. e. is not tangent) to Σ.

Therefore, to solve (2.2.3)-(2.2.4) we

(a) Find $\nabla_\Sigma u = \nabla_\Sigma g$ at Σ (i.e. we find gradient of u along Σ; if $\Sigma = \{x_1 = 0\}$ then we just calculate $u_{x_2}, \dots, u_{x_n}$);

(b) From (2.2.3) we find the remaining normal component of ∇u at Σ; so we have $(n-1)$-dimensional surface $\Sigma^* = \{(x, u, \nabla u), x \in \Sigma\}$ in $(2n+1)$-dimensional space.

(c) From each point of Σ^* we issue a characteristic trajectory described by (2.2.8)-(2.2.10). These trajectories together form n-dimensional hypesurface Λ in $(2n+1)$-dimensional space.

(d) Locally (near $t = 0$) this surface Λ has one-to-one x-projection and we can restore $u = u(x)$ (and $\nabla u = p(x)$).

However this property (d) is just local.

Remark 2.2.2. We have not proved directly that this construction always gives us a solution but if we know that solution exists then our arguments imply that it is unique and could be founds this way. Existence could be proven either directly or by some other arguments.

Remark 2.2.3. (a) Important for application case is when F does not depend on u (only on $x, p = \nabla u$) and (2.2.8)-(2.2.10) become highly symmetrical with respect to (x, p):

$$\frac{dx_j}{dt} = F_{p_j}, \tag{2.2.11}$$

$$\frac{dp_j}{dt} = -F_{x_j}, \tag{2.2.12}$$

$$\frac{du}{dt} = \sum_{1 \le j \le n} p_j F_{p_j}. \tag{2.2.13}$$

This is so called *Hamiltonian system* with the *Hamiltonian* $F(x, p)$.

(b) In this case we can drop u from consideration and consider only (x, p)-projections of Σ^* and Λ.

2.2.3 Problems

Problem 2.2.1. (a) Find the general solution to each of the following equations

$$u_t + 3u_x - 2u_y = 0; \tag{2.2.14}$$
$$u_t + xu_x + yu_y = 0; \tag{2.2.15}$$
$$u_t - xu_x - yu_y = 0; \tag{2.2.16}$$
$$u_t + yu_x + xu_y = 0; \tag{2.2.17}$$
$$u_t - yu_x - xu_y = 0. \tag{2.2.18}$$

(b) Solve IVP $u(x, y, 0) = f(x, y)$.

Problem 2.2.2. (a) Find the general solution to each of the following equations

$$u_t + 3u_x - 2u_y = x; \tag{2.2.19}$$
$$u_t + xu_x + yu_y = x; \tag{2.2.20}$$
$$u_t - xu_x - yu_y = x; \tag{2.2.21}$$
$$u_t + yu_x + xu_y = x; \tag{2.2.22}$$
$$u_t - yu_x - xu_y = x. \tag{2.2.23}$$

(b) Solve IVP $u(x, y, 0) = 0$.

2.3 Homogeneous 1D Wave equation

Consider equation

$$u_{tt} - c^2 u_{xx} = 0. \tag{2.3.1}$$

2.3.1 Physical examples

Remark 2.3.1. As we mentioned in section 1.4 this equation describes a lot of things.

Example 2.3.1. Consider a string with the points deviating from the original position (along x) in the orthogonal direction (y); so the string is described by $y = u(x, t)$ at the moment t (so u is a displacement along y). In this case $c^2 = T/\rho$ where T is a *tension* of the string and ρ is a *linear density* of it.

Example 2.3.2. This equation also describes compression-rarefication waves in elastic 1-dimensional media. Then $u(x,t)$ is displacement along x.

Example 2.3.3. Consider a pipe filled by an ideal gas. Then $c^2 = p(\rho)/\rho$ where ρ is a density and $p(\rho)$ is a pressure (for an ideal gas at the given temperature such ratio is constant and due to Mendeleev-Clapeyron equation it is proportional to absolute temperature T which is assumed to be a constant). Then f.e. u may denote a density $\rho(x,t)$ at point x at time t.

Remark 2.3.2. c has a dimension of the speed. In the example above c is a speed of sound.

2.3.2 General solution

Let us rewrite formally equation (2.3.1) as

$$(\partial_t^2 - c^2\partial_x^2) = (\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0. \tag{2.3.2}$$

Denoting $v = (\partial_t + c\partial_x)u = u_t + cu_x$ and $w = (\partial_t - c\partial_x)u = u_t - cu_x$ we have

$$v_t - cv_x = 0, \tag{2.3.3}$$
$$w_t + cw_x = 0. \tag{2.3.4}$$

But from Section 2.1 we know how to solve these equations

$$v = 2c\phi'(x+ct), \tag{2.3.5}$$
$$w = -2c\psi'(x-ct) \tag{2.3.6}$$

where ϕ' and ψ' are arbitrary functions. We find convenient to have factors $2c$ and $-2c$ and to denote by ϕ and ψ their primitives (aka indefinite integrals). Recalling definitions of v and w we have

$$u_t + cu_x = 2c\phi'(x+ct),$$
$$u_t - cu_x = -2c\psi'(x-ct).$$

Then

$$c^{-1}u_t = \phi'(x+ct) - \psi'(x-ct),$$
$$u_x = \phi'(x+ct) + \psi'(x-ct).$$

The second equation implies that $u = \phi(x+ct) + \psi(x-ct) + \Phi(t)$ and plugging to the first equation we get $\Phi' = 0$, thus $\Phi = \text{const}$.

So,

$$u = \phi(x+ct) + \psi(x-ct) \tag{2.3.7}$$

is a *general* solution to (2.3.1). This solution is a superposition of two waves $u_1 = \phi(x+ct)$ and $u_2 = \psi(x-ct)$ running to the left and to the rightrespectively with the speed c. So c *is a propagation speed.*

Remark 2.3.3. Adding constant C to ϕ and $-C$ to ψ we get the same solution u. However it is the only arbitrarness.

2.3.3 Cauchy problem

Let us consider IVP (*initial–value problem*, aka *Cauchy problem*) for (2.3.1):

$$u_{tt} - c^2 u_{xx} = 0, \tag{2.3.8}$$

$$u|_{t=0} = g(x), \qquad u_t|_{t=0} = h(x). \tag{2.3.9}$$

Plugging (2.3.7) into initial conditions we have

$$\phi(x) + \psi(x) = g(x), \tag{2.3.10}$$

$$c\phi'(x) - c\psi'(x) = h(x) \implies \phi(x) - \psi(x) = \frac{1}{c}\int^x h(y)\,dy. \tag{2.3.11}$$

Then

$$\phi(x) = \frac{1}{2}g(x) + \frac{1}{2c}\int^x h(y)\,dy, \tag{2.3.12}$$

$$\psi(x) = \frac{1}{2}g(x) - \frac{1}{2c}\int^x h(y)\,dy. \tag{2.3.13}$$

Plugging into (2.3.7) and using property of an integral we get *D'Alembert formula*

$$u(x,t) = \frac{1}{2}\big[g(x+ct) + g(x-ct)\big] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\,dy. \tag{2.3.14}$$

Remark 2.3.4. Later we generalize it to the case of inhomogeneous equation (with the right-hand expression $f(x,t)$ in (2.3.8).

2.3.4 Problems

Problem 2.3.1. Find the general solutions of

$$u_{tt} - u_{xx} = 0; \tag{2.3.15}$$
$$u_{tt} - 4u_{xx} = 0; \tag{2.3.16}$$
$$u_{tt} - 9u_{xx} = 0; \tag{2.3.17}$$
$$4u_{tt} - u_{xx} = 0; \tag{2.3.18}$$
$$u_{tt} - 9u_{xx} = 0. \tag{2.3.19}$$

Problem 2.3.2. Solve IVP

$$u_{tt} - c^2 u_{xx} = 0, \tag{2.3.20}$$
$$u|_{t=0} = g(x), \quad u_t|_{t=0} = h(x) \tag{2.3.21}$$

with

$$g(x) = 0, \qquad h(x) = 1;$$
$$g(x) = \begin{cases} 0 & x < 0, \\ 1 & x \geq 0. \end{cases} \qquad h(x) = 0;$$
$$g(x) = \begin{cases} 1 & |x| < 1, \\ 0 & |x| \geq 1. \end{cases} \qquad h(x) = 0;$$
$$g(x) = \begin{cases} \cos(x) & |x| < \pi/2, \\ 0 & |x| \geq \pi/2. \end{cases} \qquad h(x) = 0;$$
$$g(x) = 0, \qquad h(x) = \begin{cases} 0 & x < 0, \\ 1 & x \geq 0. \end{cases}$$
$$g(x) = 0, \qquad h(x) = \begin{cases} 1 & |x| < 1, \\ 0 & |x| \geq 1. \end{cases}$$
$$g(x) = 0, \qquad h(x) = \begin{cases} \cos(x) & |x| < \pi/2, \\ 0 & |x| \geq \pi/2. \end{cases}$$

Problem 2.3.3. Find solution to equation

$$Au_{tt} + 2Bu_{tx} + Cu_{tt} = 0 \tag{2.3.22}$$

as

$$u = f(x - c_1 t) + g(x - c_2 t) \tag{2.3.23}$$

with arbitrary f, g and real $c_1 < c_2$.

(a) What equation should satisfy c_1 and c_2?

(b) When this equation has such roots?

Problem 2.3.4. A spherical wave is a solution of the three-dimensional wave equation of the form $u(r,t)$, where r is the distance to the origin (the spherical coordinate). The wave equation takes the form

$$u_{tt} = c^2\big(u_{rr} + \frac{2}{r}u_r\big) \qquad (\text{"spherical wave equation"}). \tag{2.3.24}$$

(a) Change variables $v = ru$ to get the equation for v: $v_{tt} = c^2 v_{rr}$.

(b) Solve for v using

$$v = f(r + ct) + g(r - ct) \tag{2.3.25}$$

and thereby solve the spherical wave equation.

(c) Use

$$v(r,t) = \frac{1}{2}\big[\phi(r + ct) + \phi(r - ct)\big] + \frac{1}{2c}\int_{r-ct}^{r+ct} \psi(s)\, ds \tag{2.3.26}$$

with $\phi(r) = v(r,0)$, $\psi(r) = v_t(r,0)$ to solve it with initial conditions $u(r,0) = \Phi(r)$, $u_t(r,0) = \Psi(r)$.

(d) Find the general form of solution u to (2.3.24) which is continuous as $r = 0$.

Problem 2.3.5. Find formula for solution of the *Goursat problem*

$$u_{tt} - c^2 u_{xx} = 0, \qquad x > c|t|, \tag{2.3.27}$$

$$u|_{x=-ct} = g(t), \qquad t < 0, \tag{2.3.28}$$

$$u|_{x=ct} = h(t), \qquad t > 0 \tag{2.3.29}$$

as $g(0) = h(0)$.

Problem 2.3.6. Often solution in the form of *travelling wave* $u = \phi(x - vt)$ is sought for more general equations. Here we are interested in the bounded solutions, especially in those with $\phi(x)$ either tending to 0 as $|x| \to \infty$ (solitons) or periodic (kinks). Plugging such solution to equation we get ODE for function ϕ, which could be either solved or at least explored. Sure we are not interested in the trivial solution which is identically equal to 0.

(a) Find such solutions for each of the following equations

$$u_{tt} - c^2 u_{xx} + m^2 u = 0; \tag{2.3.30}$$
$$u_{tt} - c^2 u_{xx} - m^2 u = 0; \tag{2.3.31}$$

the former is *Klein-Gordon equation*. Describe all possible velocities v.

(b) Find such solutions for each of the following equations

$$u_t - K u_{xxx} = 0; \tag{2.3.32}$$
$$u_t - iK u_{xx} = 0; \tag{2.3.33}$$
$$u_{tt} + K u_{xxxx} = 0; \tag{2.3.34}$$

Problem 2.3.7. Look for solutions in the form of travelling wave for sine-Gordon equation

$$u_{tt} - c^2 u_{xx} + \sin(u) = 0. \tag{2.3.35}$$

observe that resulting ODE is describing *mathematical pendulum* which could be explored. Describe all possible velocities v.

Problem 2.3.8. Look for solutions in the form of travelling wave for each of the following equations

$$u_{tt} - u_{xx} + u - 2u^3 = 0; \tag{2.3.36}$$
$$u_{tt} - u_{xx} - u + 2u^3 = 0; \tag{2.3.37}$$

(a) Describe kinks. Describe all possible velocities v.

(b) Find solitons. Describe all possible velocities v.

Problem 2.3.9. For a solution $u(x,t)$ of the wave equation $u_{tt} = c^2 u_{xx}$, the *energy density* is defined as $e = \frac{1}{2}\bigl(u_t^2 + c^2 u_x^2\bigr)$ and the *momentum density* as $p = c u_t u_x$.

(a) Show that

$$\frac{\partial e}{\partial t} = c\frac{\partial p}{\partial x} \qquad \text{and} \qquad \frac{\partial p}{\partial t} = c\frac{\partial e}{\partial x}. \tag{2.3.38}$$

(b) Show that both $e(x,t)$ and $p(x,t)$ also satisfy the same wave equation.

Problem 2.3.10. (a) Consider wave equation $u_{tt} - u_{xx} = 0$ in the rectangle $0 < x < a$, $0 < t < b$ and prove that if a and b are not *commensurable* (i.e. $a : b$ is not rational) then Dirichlet problem $u|_{t=0} = u_{t=b} = u|_{x=0} = u|_{x=a} = 0$ has only trivial solution.

(b) On the other hand, prove that if a and b are *commensurable* then there exists a nontrivial solution $u = \sin(px/a)\sin(qt/b)$.

2.4 1D Wave equation reloaded: characteristic coordinates

2.4.1 Characteristic coordinates

We realize that lines $x + ct = \text{const}$ and $x - ct = \text{const}$ play a very special role in our analysis. We call these lines *characteristics*. Let us introduce *characteristic coordinates*

$$\begin{cases} \xi = x + ct, \\ \eta = x - ct. \end{cases} \tag{2.4.1}$$

Proposition 2.4.1.

$$u_{tt} - c^2 u_{xx} = -4c^2 u_{\xi\eta}. \tag{2.4.2}$$

Proof. From (2.4.1) we see that $x = \frac{1}{2}(\xi+\eta)$ and $t = \frac{1}{2c}(\xi-\eta)$ and therefore due to chain rule $v_\xi = \frac{1}{2}v_x + \frac{1}{2c}v_t$ and $v_\eta = \frac{1}{2}v_x - \frac{1}{2c}v_t$ and therefore

$$-4c^2 u_{\xi\eta} = -\frac{1}{4}(c\partial_x + \partial_t)(c\partial_x - \partial_t)u = u_{tt} - c^2 u_{xx}.$$

□

Therefore wave equation (2.3.1) becomes in the characteristic coordinates

$$u_{\xi\eta} = 0 \tag{2.4.3}$$

which we rewrite as $(u_\xi)_\eta = 0 \implies u_\xi = \phi'(\xi)$ (really, u_ξ should not depend on η and it is convenient to denote by $\phi(\xi)$ the primitive of u_ξ). Then $(u - \phi(\xi))_\xi = 0 \implies u - \phi(\xi) = \psi(\eta)$ (due to the same arguments) and therefore

$$u = \phi(\xi) + \psi(\eta) \tag{2.4.4}$$

is the *general solution* to (2.4.3).

2.4.2 Application of characteristc coordinates

Example 2.4.1. Consider *Goursat problem* for (2.4.3):

$$\begin{aligned} u_{\xi\eta} &= 0 \qquad \text{as } \xi > 0, \eta > 0 \\ u|_{\eta=0} &= g(\xi) \qquad \text{as } \xi > 0, \\ u|_{\xi=0} &= h(\eta) \qquad \text{as } \eta > 0 \end{aligned}$$

where g and h must satisfy *compatibility condition* $g(0) = h(0)$ (really $g(0) = u(0,0) = h(0)$).

Then one can see easily that $u(\xi, \eta) = g(\xi) + h(\eta) - g(0)$ solves Goursat problem.

Plugging (2.4.1) into (2.4.4) we get for a general solution (2.3.1)

$$u = \phi(x + ct) + \psi(x - ct) \tag{2.4.5}$$

which is exactly (2.3.7).

2.4.3 D'Alembert formula

So far we achieved nothing new. Consider now IVP:

$$u_{tt} - c^2 u_{xx} = f(x,t), \tag{2.4.6}$$

$$u|_{t=0} = g(x), \tag{2.4.7}$$

$$u_t|_{t=0} = h(x). \tag{2.4.8}$$

It is convenient for us to assume that $g = h = 0$. Later we will get rid off this assumption. Rewriting (2.4.6) as

$$\tilde{u}_{\xi\eta} = -\frac{1}{4c^2}\tilde{f}(\xi, \eta)$$

(where $\tilde{u}$ etc means that we use characteristic coordinates) we get after integration

$$\tilde{u}_\xi = -\frac{1}{4c^2}\int^{\eta} \tilde{f}(\xi,\eta')\,d\eta' = -\frac{1}{4c^2}\int_{\xi}^{\eta} \tilde{f}(\xi,\eta')\,d\eta' + \phi'(\xi)$$

with an indefinite integral in the middle. Note that $t=0$ means exactly that $\xi=\eta$ but then $u_\xi = 0$ there. Really, u_ξ is a linear combination of u_t and u_x but both of them are 0 as $t=0$. Therefore $\phi'(\xi)=0$ and

$$\tilde{u}_\xi = \frac{1}{4c^2}\int_{\eta}^{\xi} \tilde{f}(\xi,\eta')\,d\eta'$$

where we flipped limits and changed sign.

Integrating with respect to ξ we arrive to

$$\tilde{u} = \frac{1}{4c^2}\int^{\xi}\Big[\int_{\xi'}^{\eta} \tilde{f}(\xi',\eta')\,d\eta'\Big]\,d\xi' = \frac{1}{4c^2}\int_{\eta}^{\xi}\Big[\int_{\eta}^{\xi'} \tilde{f}(\xi',\eta')\,d\eta'\Big]\,d\xi' + \psi(\eta)$$

and $\psi(\eta)$ also must vanish because $u=0$ as $t=0$ (i.e. $\xi=\eta$). So

$$\tilde{u}(\xi,\eta) = \frac{1}{4c^2}\int_{\eta}^{\xi}\Big[\int_{\eta}^{\xi'} \tilde{f}(\xi',\eta')\,d\eta'\Big]\,d\xi'. \tag{2.4.9}$$

We got a solution as a *double integral* but we want to write it down as *2-dimensional integral*

$$\tilde{u}(\xi,\eta) = \frac{1}{4c^2}\iint_{\tilde{\Delta}(\xi,\eta)} \tilde{f}(\xi',\eta')\,d\eta d\xi'. \tag{2.4.10}$$

But what is $\tilde{\Delta}$? Consider $\xi>\eta$. Then ξ' should run from η to ξ and for fixed ξ', $\eta<\xi'<\xi$ eta should run from η to ξ'. So, we get a triangle bounded by $\xi'=\eta'$, $\xi'=\xi$ and $\eta'=\eta$:

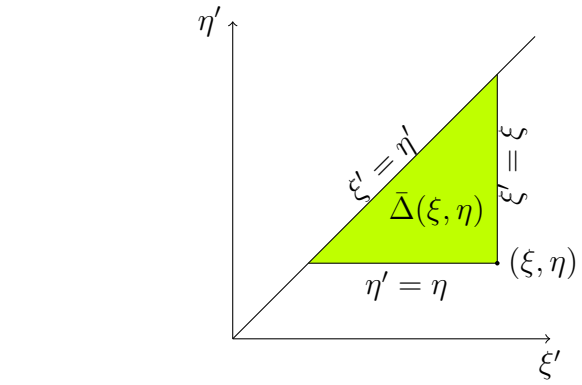
But in coordinates (x,t) this domain $\Delta(x,t)$ is bounded by $t=0$ and two characteristics:

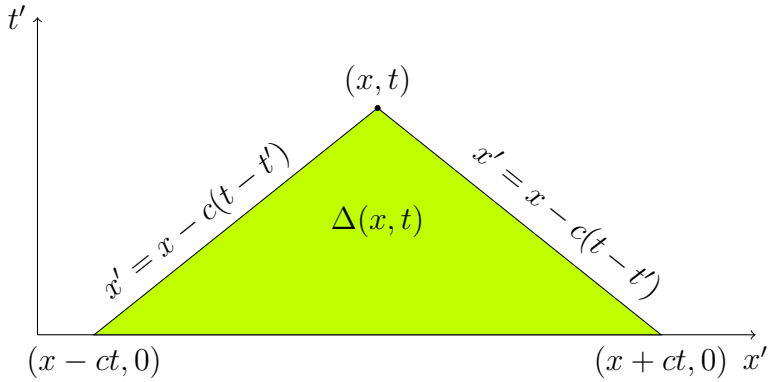
- So, we get

$$u(x,t) = \frac{1}{2c}\iint_{\Delta(x,t)} f(x',t')\,dx'dt'. \tag{2.4.11}$$

because we need to replace $d\xi' d\eta'$ by $|J|dx'dt'$ with *Jacobian* J.

Exercise 2.4.1. Calculate J and justify factor $2c$.





2.4.4 Problems

Problem 2.4.1. Solve IVP

$$u_{tt} - c^2 u_{xx} = f(x,t); \tag{2.4.12}$$
$$u|_{t=0} = g(x), \tag{2.4.13}$$
$$u_t|_{t=0} = h(x) \tag{2.4.14}$$

with

$$f(x,t) = \sin(\alpha x), \qquad g(x) = 0, \qquad h(x) = 0; \tag{2.4.15}$$
$$f(x,t) = \sin(\alpha x)\sin(\beta t), \qquad g(x) = 0; \qquad h(x) = 0, \tag{2.4.16}$$
$$f(x,t) = f(x), \qquad g(x) = 0, \qquad h(x) = 0; \tag{2.4.17}$$
$$f(x,t) = f(x)t, \qquad g(x) = 0, \qquad h(x) = 0, \tag{2.4.18}$$

in the case (2.4.17) assume that $f(x) = F''(x)$ and in the case (2.4.18) assume that $f(x) = F'''(x)$.

Problem 2.4.2. Find formula for solution of the *Goursat problem*

$$u_{tt} - c^2 u_{xx} = f(x,t), \qquad x > c|t|, \tag{2.4.19}$$
$$u|_{x=-ct} = g(t), \qquad t < 0, \tag{2.4.20}$$
$$u|_{x=ct} = h(t), \qquad t > 0 \tag{2.4.21}$$

as $g(0) = h(0)$.

Hint. Contribution of the right-hand expression will be

$$-\frac{1}{4c^2}\iint_{R(x,t)} f(x',t')\,dx'dt' \tag{2.4.22}$$

with $R(x,t) = \{(x',t') : 0 < x' - ct' < x - ct,\ 0 < x' + ct' < x + ct\}$.

Problem 2.4.3. Find the general solutions of the following equations:

$$u_{xy} = u_x u_y u^{-1}; \tag{2.4.23}$$
$$u_{xy} = u_x u_y; \tag{2.4.24}$$
$$u_{xy} = \frac{u_x u_y u}{u^2+1}; \tag{2.4.25}$$

2.5 Wave equation reloaded (continued)

We solved IVP with homogeneous (= 0) initial data for inhomogeneous wave equation. Now consider both inhomogeneous data and equation. So we consider problem (2.4.6)–(2.4.8):

$$u_{tt} - c^2 u_{xx} = f(x,t), \tag{2.5.1}$$
$$u|_{t=0} = g(x), \tag{2.5.2}$$
$$u_t|_{t=0} = h(x). \tag{2.5.3}$$

when neither f, nor g, h are necessarily equal to 0.

The good news is that our equation is linear and therefore $u = u_2 + u_1$ where u_1 satisfies problem with right-hand function $f(x,t)$ but with g and h replaced by 0 and u_2 satisfies the same problem albeit with f replaced by 0 and original g, h:

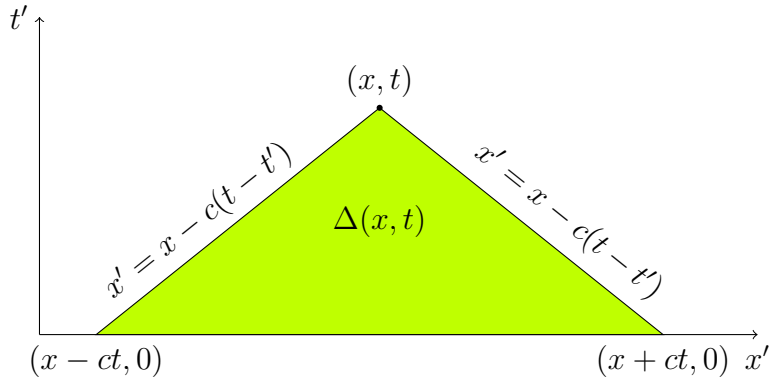
$$u_{1tt} - c^2 u_{1xx} = f(x,t), \qquad u_{2tt} - c^2 u_{2xx} = 0,$$
$$u_1|_{t=0} = 0, \qquad u_2|_{t=0} = g(x),$$
$$u_{1t}|_{t=0} = 0 \qquad u_{2t}|_{t=0} = h(x).$$

Exercise 2.5.1. Prove it.

Then u_1 is given by (2.4.11 and u_2 is given by (2.3.14) (with $(f,g) \mapsto (g,h)$)and adding them we arrive to the final

$$u(x,t) = \underbrace{\frac{1}{2}\bigl[g(x+ct)+g(x-ct)\bigr] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\,dy}_{=u_2} + \underbrace{\frac{1}{2c}\iint_{\Delta(x,t)} f(x',t')\,dx\,dt'}_{=u_1}. \quad (2.5.4)$$

where recall that $\Delta(x,t)$ is a *characteristic triangle:*



This formula is also called *D'Alembert formula.*

Remark 2.5.1. Note that integral of h in u_2 is taken over base of the characteristic triangle $\Delta(x,t)$ and g is taken in the ens of this base.

2.5.1 Wave equation reloaded (Duhamel integral)

We discuss formula (2.5.4) in details in the next lecture and now let us derive it by a completely different method. Again, in virtue of (2.3.14) we need to consider only the case when $g = h = 0$.

Let us define an auxillary function $U(x,t,\tau)$ $(0<-\tau<-t)$ as a solution of an auxillary problem

$$U_{tt}-c^2U_{xx}=0, \tag{2.5.5}$$

$$U|_{t=\tau}=0, \tag{2.5.6}$$

$$U_t|_{t=\tau}=f(x,\tau). \tag{2.5.7}$$

We claim that

Proposition 2.5.1.

$$u(x,t)=\int_0^t U(x,t,\tau)\,d\tau \tag{2.5.8}$$

is a required solution.

Proof. Note first that we can differentiate (2.5.8) by x easily:

$$u_x=\int_0^t U_x(x,t,\tau)\,d\tau, \qquad u_{xx}=\int_0^t U_{xx}(x,t,\tau)\,d\tau \tag{2.5.9}$$

and so on. Let us find u_t. Note that u depends on t through its upper limit and through integrand. We apply formula

$$\frac{d}{dt}\Big(\int_{\alpha(t)}^{\beta(t)} F(t,\tau)\,d\tau\Big)= \\ -F(t,\alpha(t))\alpha'(t)+F(t,\beta(t))\beta'(t)+\int_{\alpha(t)}^{\beta(t)}\frac{\partial F}{\partial t}(t,\tau)\,d\tau \tag{2.5.10}$$

which you should know but we will prove it anyway.

As $\alpha=0$, $\beta=t$ we have $\alpha'=0$, $\beta'=1$ and

$$u_t=U(x,t,t)+\int_0^t U_t(x,t,\tau)\,d\tau.$$

But $U(x,t,t)=0$ due to (2.5.6) and therefore

$$u_t=\int_0^t U_t(x,t,\tau)\,d\tau. \tag{2.5.11}$$

We can differentiate this with respect to x as in (2.5.9). Let us differentiate by t. Applying the same (2.5.10) we get

$$u_{tt} = U_t(x,t,t) + \int_0^t U_{tt}(x,t,\tau)\,d\tau.$$

Due to (2.5.7) $U_t(x,t,t) = f(x,t)$:

$$u_{tt} = f(x,t) + \int_0^t U_{tt}(x,t,\tau)\,d\tau.$$

Therefore

$$u_{tt} - c^2u_{xx} = f(x,t) + \int_0^t \underbrace{\bigl(U_{tt} - c^2U_{xx}\bigr)}_{=0}(x,t,\tau)\,d\tau = f(x,t)$$

where integrand vanishes due to (2.5.5). So, u really satisfies (2.5.1). Due to (2.5.8) and (2.5.11) $u|_{t=0} = u_t|_{t=0} = 0$. □

Formula (2.5.8) is *Duhamel integral formula.*

Remark 2.5.2. It is not important that it is wave equation: we need assume that equation is linear and has a form $u_{tt} + Lu = f$ where Lu is a linear combination of u and its derivatives but with no more than 1 differentiation by t.

Now we claim that

$$U(x,t,\tau) = \frac{1}{2c}\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(x',\tau)\,dx'. \tag{2.5.12}$$

Really, changing variable $t' = t - \tau$ we get the same problem albeit with t' instead of t and with initial data at $t' = 0$.

Plugging (2.5.12) into (2.5.8) we get

$$u(x,t) = \int_0^t \frac{1}{2c}\Bigl[\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(x',\tau)\,dx'\Bigr]\,d\tau. \tag{2.5.13}$$

Exercise 2.5.2. Rewrite double integral (2.5.13) as a $2D$-integral in the right-hand expression of (2.4.11).

Proof of (2.5.10). Let us plug γ into $F(x,t,\tau)$ instead of t. Then integral $I(t) = J(\alpha(t),\beta(t),\gamma(t))$ with $J(\alpha,\beta,\gamma) = \int_\alpha^\beta F(\gamma,\tau)\,d\tau$ and by chain rule $I'(t) = J_\alpha\alpha' + J_\beta\beta' + J_\gamma\gamma'$.

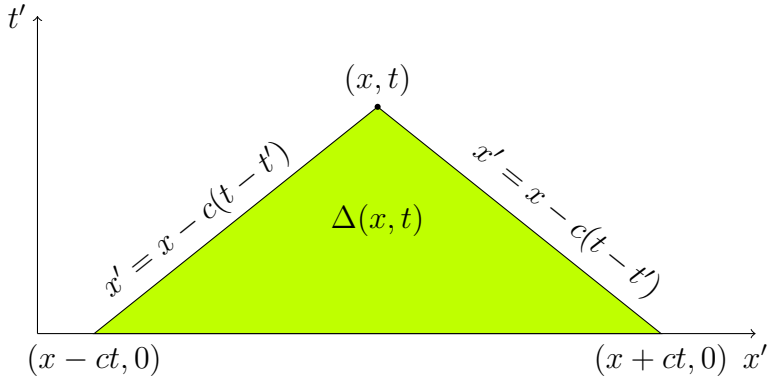
But $J_\alpha = -F(\gamma,\alpha)$, $J_\beta = F(\gamma,\beta)$ (differentiation by lower and upper limits) and $J_\gamma = \int_\alpha^\beta F_\gamma(\gamma,\tau)\,d\tau$. Plugging $\gamma = t$, $\gamma' = 1$ we arrive to (2.5.10). □

2.5.2 Domains of dependence and influence

Recall formula (2.5.4):

$$u(x,t) = \frac{1}{2}\big[g(x+ct) + g(x-ct)\big] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(x')\,dx' + \frac{1}{2c}\iint_{\Delta(x,t)} f(x',t')\,dxdt'$$

where $\Delta(x,t)$ is the same characteristic triangle as above:

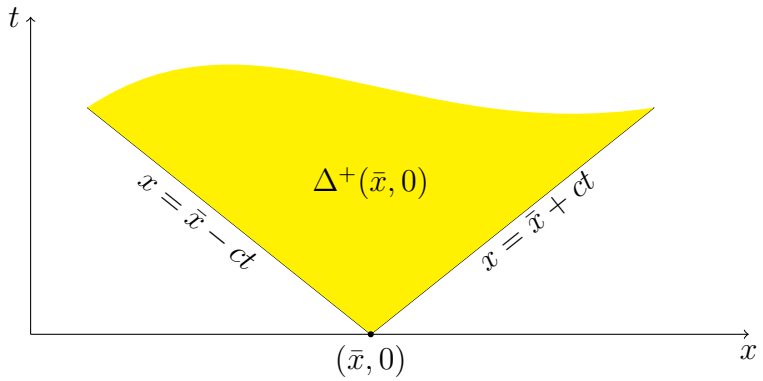


Therefore

Proposition 2.5.2. *Solution $u(x,t)$ depends only on the right hand expression f in $\Delta(x,t)$ and on the initial data g,h on the base of $\Delta(x,t)$.*

Definition 2.5.1. $\Delta(x,t)$ is a *triangle of dependence* for point (x,t).

Conversely, if we change functions g,h only near some point $(\bar x,0)$ then solution can change only at points (x,t) such that $(\bar x,0) \in \Delta(x,t)$; let $\Delta^+(\bar x,0)$ be the set of such points (x,t):



Definition 2.5.2. $\Delta^+(\bar{x},0)$ is a *triangle of influence* for point $(\bar{x},0)$.

Remark 2.5.3. (a) We can introduce *triangle of influence* for each point (not necessary at $t=0$).

(b) These notions work in much more general settings and we get *domain of dependence* and *domain of influence* which would not be triangles.

(c) F.e. for $3D$ wave equation $u_{tt}-c^2(u_{xx}+u_{yy}+u_{zz})=f$ those would be the *backward light cone* and *forward light cone* respectively and if c is variable instead of cones we get *conoids* (imagine tin coin and then deform it but the vertex should remain).

We see also that

Proposition 2.5.3. *Solution propagates with the speed not exceeding c.*

2.5.3 Examples

In examples we rewrite the last term in (2.5.4) as a double integral:

$$u(x,t)=\frac{1}{2}\big[g(x+ct)+g(x-ct)\big]+\frac{1}{2c}\int_{x-ct}^{x+ct}h(x')\,dx'+ \\ \frac{1}{2c}\int_0^t\int_{x-c(t-t')}^{x+c(t-t')}f(x',t')\,dx'dt' \tag{2.5.14}$$

Example 2.5.1.

$$\begin{aligned} &u_{tt} - 4u_{xx} = \sin(x)\cos(t),\\ &u|_{t=0} = 0,\\ &u_t|_{t=0} = 0. \end{aligned}$$

Then $c = 2$ and according to (2.5.14)

$$\begin{aligned} u(x,t) = \frac{1}{4}\int_0^t \int_{x-2(t-t')}^{x+2(t-t')} \sin(x')\cos(t')\,dx'dt' = \\ \frac{1}{4}\int_0^t \Big[\cos\big(x-2(t-t')\big) - \cos\big(x+2(t-t')\big)\Big]\cos(t')\,dt' = \\ \frac{1}{2}\int_0^t \sin(x)\sin(2(t-t'))\cos(t')\,dt' = \\ \frac{1}{4}\sin(x)\int_0^t [\sin(2(t-t')+t') + \sin(2(t-t')-t')]\,dt' = \\ \frac{1}{4}\sin(x)\int_0^t [\sin(2t-t') + \sin(2t-3t')]\,dt' = \\ \frac{1}{4}\sin(x)\big[\cos(2t-t') + \frac{1}{3}\cos(2t-3t')\big]_{t'=0}^{t'=t} = \\ \frac{1}{3}\sin(x)\big[\cos(t) - \cos(2t)\big]. \end{aligned}$$

Sure, *separation of variables* would be simpler here.

2.6 1D Wave equation: IBVP

2.6.1 1D Wave equation on half-line

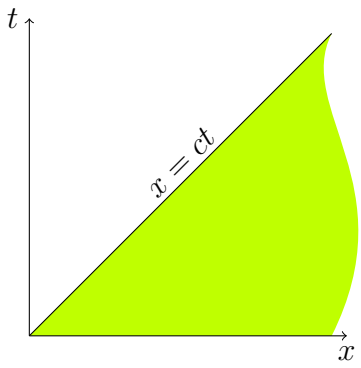
Consider wave equation in domain $\{x > 0, t > 0\}$ and initial conditions

$$u_{tt} - c^2 u_{xx} = 0 \qquad x > 0,\ t > 0 \tag{2.6.1}$$

$$u|_{t=0} = g(x) \qquad x > 0, \tag{2.6.2}$$

$$u_t|_{t=0} = h(x) \qquad x > 0. \tag{2.6.3}$$

Here we take $f(x,t) = 0$ for simplicity. Then according to the previous section solution $u(x,t)$ is defined uniquely in the domain $\{t > 0, x \geq ct\}$:



where it is given by D'Alembert formula

$$u(x,t) = \frac{1}{2}\big[g(x+ct) + g(x-ct)\big] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(x')\,dx'. \tag{2.6.4}$$

What about domain $\{0 < -x < -ct\}$? We claim that *we need one boundary condition as* $x = 0, t > 0$. Indeed, recall that the general solution of (2.6.1) is

$$u(x,t) = \phi(x+ct) + \psi(x-ct) \tag{2.6.5}$$

where initial conditions (2.6.2)-(2.6.3) imply that

$$\begin{aligned} &\phi(x) + \psi(x) = g(x),\\ &\phi(x) - \psi(x) = \frac{1}{c}\int_0^x h(x')\,dx' \end{aligned}$$

as $x > 0$ (where we integrated the second equation) and then

$$\phi(x) = \frac{1}{2}g(x) + \frac{1}{2c}\int_0^x h(x')\,dx', \qquad x > 0, \tag{2.6.6}$$

$$\psi(x) = \frac{1}{2}g(x) - \frac{1}{2c}\int_0^x h(x')\,dx' \qquad x > 0. \tag{2.6.7}$$

Therefore, $\phi(x+ct)$ and $\psi(x-ct)$ are defined respectively as $x + ct > 0$ (which is automatic as $x > 0, t > 0$) and $x - ct < 0$ (which is fulfilled only as $x > ct$).

To define $\psi(x-ct)$ as $0 < x < ct$ we need to define $\psi(x)$ as $x < 0$ and we need a boundary condition as $x = 0, t > 0$.

Example 2.6.1. Consider *Dirichlet boundary condition*

$$u|_{x=0} = p(t), \qquad t > 0. \tag{2.6.8}$$

Plugging (2.6.4) we see that $\phi(ct) + \psi(-ct) = p(t)$ as $t > 0$ or equivalently $\phi(-x) + \psi(x) = p(-x/c)$ as $x < 0$ (and $-x > 0$) where we plugged $t := -x/c$. Plugging (2.6.6) we have

$$\psi(x) = p(-x/c) - \phi(x) = p(-x/c) - \frac{1}{2}g(-x) - \frac{1}{2c}\int_0^{-x} h(x')\,dx'. \tag{2.6.9}$$

Then plugging $x := x + ct$ into (2.6.6) and $x := x - ct$ into (2.6.9) and adding we get from (2.6.4) that

$$u(x,t) = \underbrace{\frac{1}{2}g(x+ct) + \frac{1}{2c}\int_0^{x+ct} h(x')\,dx'}_{=\phi(x+ct)} + \underbrace{p(t-x/c) - \frac{1}{2}g(ct-x) - \frac{1}{2c}\int_0^{ct-x} h(x')\,dx'}_{=\psi(x-ct)}. \tag{2.6.10}$$

This formula defines $u(x,t)$ as $0 < x < ct$ solution is given by (2.6.4).

(a) As $g = h = 0$ (wave is generated by the perturbation of the end)

$$u(x,t) = \begin{cases} 0 & 0 < ct < x, \\ p(t - x/c) & 0 < x < ct. \end{cases} \tag{2.6.11}$$

(b) As $g(x) = \phi(x)$, $h(x) = c\phi'(x)$, $p = 0$ (wave initially running to the left and then reflected)

$$u(x,t) = \begin{cases} \phi(x+ct) & 0 < ct < x, \\ \phi(x+ct) - \phi(ct-x) & 0 < x < ct; \end{cases} \tag{2.6.12}$$

Example 2.6.2. Alternatively, consider *Neumann boundary condition*

$$u_x|_{x=0} = q(t), \qquad t > 0. \tag{2.6.13}$$

Plugging (2.6.4) we see that $\phi'(ct)+\psi'(-ct)=q(t)$ as $t>0$ or equivalently $\phi(-x)-\psi(x)=c\int_0^{-x/c} q(t')\,dt'$ as $x<0$ (and $-x>0$) where we integrated first and then plugged $t:=-x/c$.

Plugging (2.6.6) we have

$$\psi(x)=-c\int_0^{-x/c} q(t')\,dt'+\phi(x)= \\ -c\int_0^{-x/c} q(t')\,dt'+\frac{1}{2}g(-x)+\frac{1}{2c}\int_0^{-x} h(x')\,dx'. \tag{2.6.14}$$

Then plugging $x:=x+ct$ into (2.6.6) and $x:=x-ct$ into (2.6.14) and adding we get from (2.6.4) that

$$u(x,t)=\underbrace{\frac{1}{2}g(x+ct)+\frac{1}{2c}\int_0^{x+ct} h(x')\,dx'}_{=\phi(x+ct)}+ \\ \underbrace{-c\int_0^{t-x/c} q(t')\,dt'+\frac{1}{2}g(ct-x)+\frac{1}{2c}\int_0^{ct-x} h(x')\,dx'}_{=\psi(x-ct)}. \tag{2.6.15}$$

This formula defines $u(x,t)$ as $0<x<ct$. Recall that for $x>ct$ solution is given by (2.6.4). In particular

(a) As $g=h=0$ (wave is generated by the perturbation of the end)

$$u(x,t)=\begin{cases} 0 & 0<ct<x,\\ -c\displaystyle\int_0^{t-x/c} & 0<x<ct. \end{cases} \tag{2.6.16}$$

(b) As $g(x)=\phi(x)$, $h(x)=c\phi'(x)$, $p=0$ (wave initially running to the left and then reflected)

$$u(x,t)=\begin{cases} \phi(x+ct) & 0<ct<x,\\ \phi(x+ct)+\phi(ct-x) & 0<x<ct; \end{cases} \tag{2.6.17}$$

Example 2.6.3. Alternatively, consider boundary condition

$$(\alpha u_x+\beta u_t)|_{x=0}=q(t), \qquad t>0. \tag{2.6.18}$$

Again we get equation to $\psi(-ct)$: $(\alpha-c\beta)\psi`(-ct)+(\alpha+c\beta)\phi'(ct)=q(t)$ and everything works well as long as $\alpha\neq c\beta$.

Example 2.6.4. Alternatively, consider *Robin boundary condition*

$$(u_x + \sigma u)|_{x=0} = q(t), \qquad t > 0. \tag{2.6.19}$$

and we get $\psi`(-ct) + \sigma\psi(-ct) + \phi'(ct) + \sigma\phi(ct) = q(t)$ or equivalently

$$\psi'(x) + \sigma\psi(x) = q(-x/c) - \phi'(-x) + \sigma\phi(-x) \tag{2.6.20}$$

where the right-hand expression is known.

In this case we define $\psi(x)$ as $x < 0$ solving ODE (2.6.20) as we know $\psi(0) = \frac{1}{2}g(0)$ from (2.6.7).

2.6.2 1D Wave equation on the finite interval

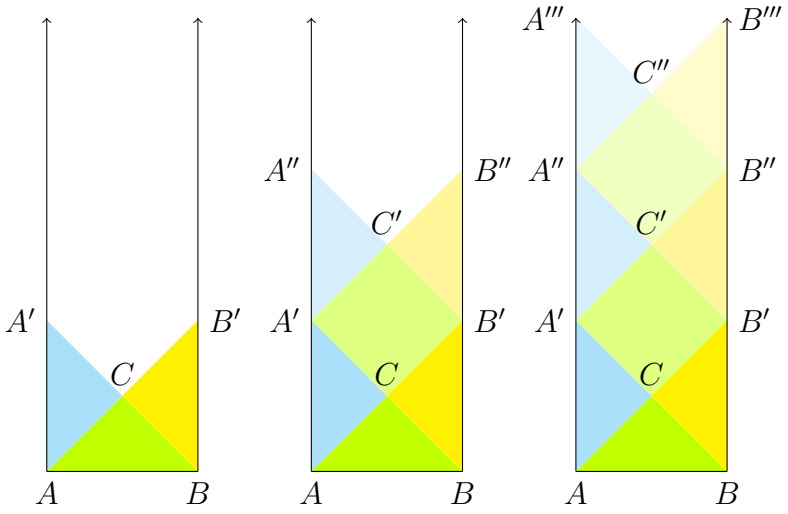
Consider wave equation in domain $\{a < x < b, t > 0\}$ and initial conditions

$$u_{tt} - c^2 u_{xx} = 0 \qquad a < x < b,\ t > 0 \tag{2.6.21}$$

$$u|_{t=0} = g(x) \qquad a < x < b, \tag{2.6.22}$$

$$u_t|_{t=0} = h(x) \qquad a < x < b. \tag{2.6.23}$$

Here we take $f(x,t) = 0$ for simplicity. Then according to the previous section solution $u(x,t)$ is defined uniquely in the characteristic triangle ABC.



(a) However $\phi(x)$, $\psi(x)$ are defined as $a < x < b$.

(b) Now the boundary condition on the left end (f. e. $u|_{x=a} = p_l(t)$, or $u_x|_{x=a} = q_l(t)$, or more general condition) allows us to find $\psi(a-ct)$ as $0 < t < (b-a)/c$ and then $\psi(x-ct)$ is defined in $ABB''; A'$. Similarly the boundary condition on the right end (f. e. $u|_{x=b} = p_r(t)$, or $u_x|_{x=a} = q_r(t)$, or more general condition) allows us to find $\phi(b+ct)$ as $0 < t < (b-a)/c$ and then $\phi(x+ct)$ is defined in $ABB'A''$. So, u is defined in the intersection of those two domains which is $AA'C'B'B$.

(c) Continuing this process in steps we define $\phi(x+ct)$, $\psi(x-ct)$ and $u(x,t)$ in expanding "up" set of domains.

2.6.3 Half-line: method of continuation

Consider wave equation in domain $\{x > 0, t > 0\}$, initial conditions, and a boundary condition

$$u_{tt} - c^2 u_{xx} = f(x,t) \qquad x > 0, \tag{2.6.24}$$
$$u|_{t=0} = g(x) \qquad x > 0, \tag{2.6.25}$$
$$u_t|_{t=0} = h(x) \qquad x > 0, \tag{2.6.26}$$
$$u|_{x=0} = 0. \tag{2.6.27}$$

Alternatively, instead of (2.6.27) we consider

$$u_x|_{x=0} = 0. \tag{2.6.27}'$$

Remark 2.6.1. It is crucial that we consider either Dirichlet or Neumann homogeneous boundary conditions. To deal with this problem consider first IVP on the whole line:

$$U_{tt} - c^2 U_{xx} = F(x,t) \qquad x > 0, \tag{2.6.28}$$
$$U|_{t=0} = G(x) \qquad x > 0, \tag{2.6.29}$$
$$U_t|_{t=0} = H(x) \qquad x > 0, \tag{2.6.30}$$

and consider $V(x,t) = \varsigma U(-x,t)$ with $\varsigma = \pm 1$.

Proposition 2.6.1. *If U satisfies (2.6.28)-(2.6.30) then V satisfies similar problem albeit with right-hand expression $\varsigma F(-x,t)$, and initial functions $\varsigma G(-x)$ and $\varsigma H(-x)$.*

Proof. Plugging V into equation we use the fact that $V_t(x,t) = \varsigma U_t(-x,t)$, $V_x(x,t) = -\varsigma U_x(-x,t)$, $V_{tt}(x,t) = \varsigma U_{tt}(-x,t)$, $V_{tx}(x,t) = -\varsigma U_{tx}(-x,t)$, $V_{xx}(x,t) = \varsigma U_{xx}(-x,t)$ etc and exploit the fact that wave equation contains only even-order derivatives with respect to x.

Note that if F, G, H are even functions with respect to x, and $\varsigma = 1$ then $V(x,t)$ satisfies the same IVP as $U(x,t)$. Similarly, if F, G, H are odd functions with respect to x, and $\varsigma = -1$ then $V(x,t)$ satisfies the same IVP as $U(x,t)$.

However we know that solution of (2.6.28)-(2.6.30) is unique and therefore $U(x,t) = V(x,t) = \varsigma U(-x,t)$. □

Therefore

Corollary 2.6.1. *(a) If $\varsigma = -1$ and F, G, H are odd functions with respect to x then $U(x,t)$ is also an odd function with respect to x.*

(b) If $\varsigma = 1$ and F, G, H are even functions with respect to x then $U(x,t)$ is also an even function with respect to x.

However we know that

(a) odd function with respect to x vanishes as $x = 0$;

(b) derivative of the even function with respect to x vanishes as $x = 0$

and we arrive to

Corollary 2.6.2. *(a) In the framework of Proposition 1a U satisfies* (2.6.27)*;*

(b) In the framework of Proposition 1b U satisfies (2.6.27)$'$.

Therefore

Corollary 2.6.3. *(a) To solve* (2.6.24)–(2.6.26), (2.6.27) *we need to take an odd continuation of f, g, h to $x < 0$ and solve the corresponding IVP;*

(b) To solve (2.6.24)-(2.6.26), (2.6.27)$'$ *we need to take an even continuation of f, g, h to $x < 0$ and solve the corresponding IVP.*

So far we have not used much that we have exactly wave equation (similar argments with minor modification work for heat equation as well etc). Now we apply D'Alembert formula (2.5.4):

$$u(x,t) = \frac{1}{2}\big(G(x+ct) + G(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} H(x')\,dx' + \frac{1}{2c}\int_0^t \int_{x-c(t-t')}^{x+c(t-t')} F(x',t')\,dx'dt'$$

and we need to take $0 < x < ct$, resulting for $f = 0 \implies F = 0$

$$u(x,t) = \frac{1}{2}\big(g(ct+x) - g(ct-x)\big) + \frac{1}{2c}\int_{ct-x}^{ct+x} h(x')\,dx', \tag{2.6.31}$$

and

$$u(x,t) = \frac{1}{2}\big(g(ct+x) + g(ct-x)\big) + \frac{1}{2c}\int_0^{ct-x} h(x')\,dx' + \frac{1}{2c}\int_0^{ct+x} h(x')\,dx' \tag{31'}$$

for boundary condition (2.6.27) and (2.6.27)′ respectively.

Example 2.6.5. Consider wave equation with $c = 1$ and let $f = 0$,

(a) $g = \sin(x)$, $h = 0$ and Dirichlet boundary condition. Obviously $G(x) = \sin(x)$ (since we take odd continuation and $\sin(x)$ is an odd function). Then

$$u(x,t) = \frac{1}{2}\big(\sin(x+t) + \sin(x-t)\big) = \sin(x)\cos(t). \tag{2.6.32}$$

(b) $g = \sin(x)$, $h = 0$ and Neumann boundary condition. Obviously $G(x) = -\sin(x)$ as $x < 0$ (since we take even continuation and $\sin(x)$ is an odd function). Then u is given by (2.6.32) for $x > t > 0$ and

$$u(x,t) = \frac{1}{2}\big(\sin(x+t) - \sin(x-t)\big) = \sin(t)\cos(x) \tag{2.6.33}$$

as $0 < x < t$.

(c) $g = \cos(x)$, $h = 0$ and Neumann boundary condition. Obviously $G(x) = \cos(x)$. Then

$$u(x,t) = \frac{1}{2}\big(\cos(x+t) + \cos(x-t)\big) = \cos(x)\cos(t). \tag{2.6.34}$$

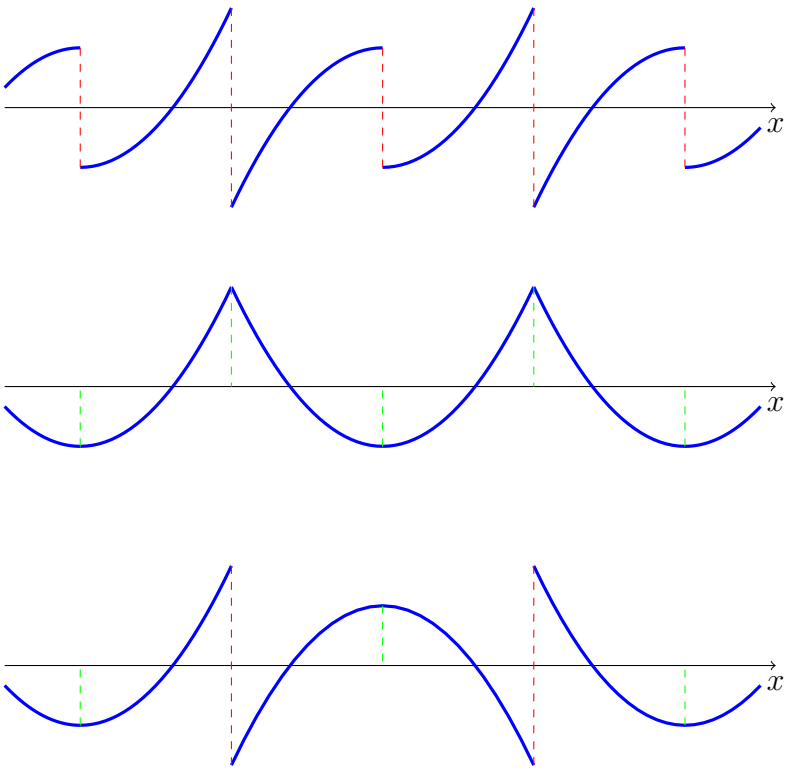
(d) $g = \cos(x)$, $h = 0$ and Dirichlet boundary condition. Obviously $G(x) = -\cos(x)$ as $x < 0$. Then u is given by (2.6.35) for $x > t > 0$ and

$$u(x,t) = \frac{1}{2}\big(\cos(x+t) - \cos(x-t)\big) = \sin(t)\sin(x) \tag{2.6.35}$$

as $0 < x < t$.

2.6.4 Finite interval: method of continuation

Consider the same problem albeit on interval $0 < x < l$ with either Dirichlet or Neumann condition on each end. Then we need to take odd continuation through "Dirichlet end" and even continuation through "Neumann end". On figures below we have respectively Dirichlet conditions on each end (indicated by red), Neumann conditions on each end (indicated by green), and Dirichlet condition on one end (indicated by red) and Neumann conditions on another end (indicated by green). Resulting continuations are $2l$, $2l$ and $4l$ periodic respectively.



2.6.5 Problems

Problem 2.6.1. (a) Find solution

$$u_{tt} - c^2 u_{xx} = 0, \qquad t > 0, x > 0, \tag{2.6.36}$$
$$u|_{t=0} = \phi(x), \qquad x > 0, \tag{2.6.37}$$
$$u_t|_{t=0} = c\phi'(x), \qquad x > 0, \tag{2.6.38}$$
$$u|_{x=0} = \chi(t), \qquad t > 0. \tag{2.6.39}$$

(separately in $x > ct$ and $0 < x < ct$).

(b) Find solution

$$u_{tt} - c^2 u_{xx} = 0, \qquad t > 0, x > 0, \tag{2.6.40}$$
$$u|_{t=0} = \phi(x), \qquad x > 0, \tag{2.6.41}$$
$$u_t|_{t=0} = c\phi'(x), \qquad x > 0, \tag{2.6.42}$$
$$u_x|_{x=0} = \chi(t), \qquad t > 0. \tag{2.6.43}$$

(separately in $x > ct$ and $0 < x < ct$).

Problem 2.6.2. (a) Find solution

$$\begin{aligned}
&u_{tt} - c_1^2 u_{xx} = 0, && && t > 0, x > 0,\\
&u_{tt} - c_2^2 u_{xx} = 0, && && t > 0, x < 0,\\
&u|_{t=0} = \phi(x), && u_t|_{t=0} = c_1\phi'(x) && x > 0,\\
&u|_{t=0} = 0, && u_t|_{t=0} = 0, && x < 0,\\
&u|_{x=+0} = \alpha u|_{x=-0}, && u_x|_{x=+0} = \beta u_x|_{x=-0} && t > 0
\end{aligned}$$

(separately in $x >_1 t$, $0 < x < c_t$, $-c_2 t < x < 0$ and $x < -c_2 t$.

(b) Discuss *reflected wave* and *refracted wave*.

Problem 2.6.3. Consider equation with the initial conditions

$$u_{tt} - c^2 u_{xx} = 0, \qquad t > 0, x > vt, \tag{2.6.44}$$
$$u|_{t=0} = f(x), \qquad x > 0, \tag{2.6.45}$$
$$u_t|_{t=0} = g(x), \qquad x > 0. \tag{2.6.46}$$

(a) Find which of these conditions (a)-(c) at $x = vt$, $t > 0$ could be added to (2.6.44)-(2.6.45) so that the resulting problem would have a unique solution and solve the problem you deemed as a good one:

(A) None,

(B) $u|_{x=vt} = 0$ $(t > 0)$,

(C) $(\alpha u_x + \beta u_t)|_{x=vt} = 0$ $(t > 0)$,

(D) $u|_{x=vt} = u_x|_{x=vt} = 0$ $(t > 0)$. Consider cases $v > c$, $-c < v < c$ and $v < -c$. In the case condition (3) find necessary restrictions to α, β.

(b) Find solution in the cases when it exists and is uniquely determined; consider separately zones $x > ct$, $-ct < -x < -ct$ and $x > ct$ (intersected with $x > vt$).

Problem 2.6.4. By method of continuation combined with D'Alembert formula solve each of the following four problems (a)–(d).

(a)

$$\begin{cases} u_{tt} - 9u_{xx} = 0, & x > 0, \\ u|_{t=0} = 0, & x > 0, \\ u_t|_{t=0} = \cos(x), & x > 0, \\ u|_{x=0} = 0, & t > 0. \end{cases} \tag{2.6.47}$$

(b)

$$\begin{cases} u_{tt} - 9u_{xx} = 0, & x > 0, \\ u|_{t=0} = 0, & x > 0, \\ u_t|_{t=0} = \cos(x), & x > 0, \\ u_x|_{x=0} = 0, & t > 0. \end{cases} \tag{2.6.48}$$

(c)

$$\begin{cases} u_{tt} - 9u_{xx} = 0, & x > 0, \\ u|_{t=0} = 0, & x > 0, \\ u_t|_{t=0} = \sin(x), & x > 0, \\ u|_{x=0} = 0, & t > 0. \end{cases} \tag{2.6.49}$$

(d)

$$\begin{cases} u_{tt} - 9u_{xx} = 0, & x > 0, \\ u|_{t=0} = 0, & x > 0, \\ u_t|_{t=0} = \sin(x), & x > 0, \\ u_x|_{x=0} = 0, & t > 0. \end{cases} \tag{2.6.50}$$

Problem 2.6.5. Solve

$$(t^2+1)u_{tt} + tu_t - u_{xx} = 0, \tag{2.6.51}$$

$$u|_{t=0} = 0, \qquad u_t|_{t=0} = 1. \tag{2.6.52}$$

Hint: Make a change of variables $x = \frac{1}{2}(\xi + \eta)$, $t = \sinh(\frac{1}{2}(\xi - \eta))$ and calculate u_ξ, u_η, $u_{\xi\eta}$.

2.7 Energy integral

2.7.1 Energy integral: wave equation

Consider multidimensional wave equation

$$u_{tt} - c^2\Delta u = 0. \tag{2.7.1}$$

Recall that $\Delta = \nabla \cdot \nabla = \partial_x^2 + \partial_y^2 + \partial_z^2$ (the number of terms depends on dimension). Multiplying by u_t we get in the left-hand expression

$$\begin{aligned} u_t u_{tt} - c^2 u_t \nabla^2 u =& \partial_t\big(\frac{1}{2}u_t^2\big) + \nabla\cdot(-c^2 u_t \nabla u) + c^2 \nabla u_t \cdot \nabla u \\ =& \partial_t\big(\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2\big) + \nabla\cdot\big(-c^2 u_t\nabla u\big). \end{aligned}$$

So we arrive to

$$\partial_t\big(\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2\big) + \nabla\cdot\big(-c^2 u_t\nabla u\big). \tag{2.7.2}$$

This is *energy conservation law in the differential form.* Here

$$e = \big(\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2\big) \tag{2.7.3}$$

is a *density of energy* and

$$\mathbf{S} = -c^2 u_t \nabla u \tag{2.7.4}$$

is a vector of *energy flow.*

Then if we fix a volume (or an area in $2D$ case, or just an interval in $1D$ case) V and introduce a full energy in V at moment t

$$E_V(t) = \iiint_V \big(\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2\big)\, dV \tag{2.7.5}$$

then

$$E_V(t_2) - E_V(t_1) + \int_{t_1}^{t_2} dt \iint_\Sigma \mathbf{S}\cdot\mathbf{n}\, d\sigma = 0 \tag{2.7.6}$$

where Σ is the surface bounding V, $d\sigma$ is an element of the surface area, and $\mathbf{n}$ is an unit exterior normal to Σ.

2.7.1.1 Energy integral: Maxwell equation

Similarly, *Maxwell equations* without charges and currents are

$$\varepsilon\mathbf{E}_t = c\nabla\times\mathbf{H}, \tag{2.7.7}$$

$$\mu\mathbf{H}_t = -c\nabla\times\mathbf{E}, \tag{2.7.8}$$

$$\nabla\cdot\varepsilon\mathbf{E} = \nabla\cdot\mu\mathbf{H} = 0. \tag{2.7.9}$$

Here $\mathbf{E}, \mathbf{H}$ are intensities of electric and magnetic field respectively, c is the speed of light in the vacuum, ε and μ are dialectric and magnetic characteristics of the media ($\varepsilon \ge 1$, $\mu \ge 1$ and $\varepsilon = \mu = 1$ in the vacuum).

Multiplying (taking an inner product) (2.7.7) by $\mathbf{E}$ and (2.7.8) by $\mathbf{H}$ and adding we arrive to

$$\partial_t\big(\frac{1}{2}\varepsilon|\mathbf{E}|^2 + \frac{1}{2}\mu|\mathbf{H}|^2\big) = c\big(\mathbf{E}\cdot\nabla\times\mathbf{H} - \mathbf{H}\cdot\nabla\times\mathbf{E}\big) = c\nabla\cdot\big(\mathbf{E}\times\mathbf{H}\big)$$

where the last equality follows from vector calculus.

Then

$$\partial_t\big(\frac{1}{2}\varepsilon|\mathbf{E}|^2 + \frac{1}{2}\mu|\mathbf{H}|^2\big) + \nabla\cdot\big(-c\mathbf{E}\times\mathbf{H}\big) = 0. \tag{2.7.10}$$

In the theory of electromagnetism

$$e = \frac{1}{2}\big(\varepsilon|\mathbf{E}|^2 + \mu|\mathbf{H}|^2\big) \tag{2.7.11}$$

is again *density of energy* and

$$\mathbf{S} = -c\mathbf{E}\times\mathbf{H} \tag{2.7.12}$$

is a vector of *energy flow* (aka *Poynting vector*).

Remark 2.7.1. $\frac{c}{\sqrt{\mu\varepsilon}}$ is the speed of light in the given media.

Remark 2.7.2. In inhomogeneous media ε and μ depend on (x, y, z); in anisotropic media (crystals) ε and μ are matrices and then

$$e = \frac{1}{2}\big(\varepsilon\mathbf{E}\cdot\mathbf{E} + \mu\mathbf{H}\cdot\mathbf{E}\big).$$

2.7.2 Elasticity equations

Elasticity equations in homogeneous isotropic media are

$$\mathbf{u}_{tt} = 2\mu\Delta\mathbf{u} + \lambda\nabla(\nabla\cdot\mathbf{u}) \tag{2.7.13}$$

where $\mathbf{u}$ is a *displacement* and $\lambda > 0$, $\mu > 0$ are *Lam parameters*.

Exercise 2.7.1. Multiplying (taking an inner product) (2.7.13) by $\mathbf{u}_t$ write conservation law in the differential form. What are e and $\mathbf{S}$?

2.7.3 Problems

Problem 2.7.1. For equation

$$u_{tt} - c^2u_{xx} + f(u) = 0, \qquad x > 0 \tag{2.7.14}$$

prove energy conservation law

$$E(t) = \frac{1}{2}\int_0^\infty \big(u_t^2 + c^2u_x^2 + F(u)\big)\,dx \tag{2.7.15}$$

under Dirichlet ($u|_{x=0} = 0$) or Neumann ($u_x|_{x=0} = 0$) boundary condition; here F is a primitive of f.

Problem 2.7.2. For *beam equation*

$$u_{tt} + Ku_{xxxx}, \qquad x > 0, \tag{2.7.16}$$

prove energy conservation law

$$E(t) = \frac{1}{2}\int_0^\infty (u_t^2 + Ku_{xx}^2)\,dx \tag{2.7.17}$$

under each of the pair of the boundary conditions:

$$u|_{x=0} = u_x|_{x=0} = 0; \tag{2.7.18}$$

$$u|_{x=0} = u_{xxx}|_{x=0} = 0; \tag{2.7.19}$$

$$u_x|_{x=0} = u_{xx}|_{x=0} = 0. \tag{2.7.20}$$

,

Problem 2.7.3. (a) For problem

$$u_{tt} - c^2 u_{xx} = 0, \qquad x > 0, \tag{2.7.21}$$
$$(\alpha_0 u_x + \alpha_1 u)|_{x=0} \tag{2.7.22}$$

find energy conservation law

$$E(t) = \frac{1}{2}\int_0^\infty (u_t^2 + c^2 u_x^2)\,dx + \frac{1}{2}au(0)^2 \tag{2.7.23}$$

(you need to calculate a).

(b) For problem

$$u_{tt} - c^2 u_{xx} = 0, \qquad 0 < x < l, \tag{2.7.24}$$
$$(\alpha_0 u_x + \alpha_1 u)|_{x=0}, \tag{2.7.25}$$
$$(\beta_0 u_x - \beta_1 u)|_{x=0} \tag{2.7.26}$$

find energy conservation law

$$E(t) = \frac{1}{2}\int_0^\infty (u_t^2 + c^2 u_x^2)\,dx + \frac{1}{2}au(0)^2 + \frac{1}{2}bu(l)^2 \tag{2.7.27}$$

(you need to calculate a, b).

Problem 2.7.4. For problem

$$u_{tt} - c^2 u_{xx} = 0, \qquad x > 0, \tag{2.7.28}$$
$$(u_x - i\alpha u)_t|_{x=0} \tag{2.7.29}$$

with real α prove energy conservation law

$$E(t) = \frac{1}{2}\int_0^\infty (|u_t|^2 + c^2|u_x|^2)\,dx \tag{2.7.30}$$

Problem 2.7.5. Consider Problem 2.6.2

$$\begin{aligned}
&u_{tt} - c_1^2 u_{xx} = 0, && && t > 0, x > 0,\\
&u_{tt} - c_2^2 u_{xx} = 0, && && t > 0, x < 0,\\
&u|_{t=0} = \phi(x), && u_t|_{t=0} = c_1\phi'(x) && x > 0,\\
&u|_{t=0} = 0, && u_t|_{t=0} = 0, && x < 0,\\
&u|_{x=+0} = \alpha u|_{x=-0}, && u_x|_{x=+0} = \beta u_x|_{x=-0} && t > 0
\end{aligned}$$

Let $E_1(t) = \frac{m_1}{2}\int_0^\infty (u_t^2 + c_1^2 u_x^2)\,dx$, $E_2(t) = \frac{m_2}{2}\int_{-\infty}^0 (u_t^2 + c_2^2 u_x^2)\,dx$.

alph*) Find $m_1 : m_2$ such that $E(t) = E_1(t) + E_2(t)$ is conserved.

alph*) In this case prove that if $\phi(x) = 0$ for $x > L$ then for $t > L/c_1$ $E_1(t) = k_1 E(0)$, $E_2(t) = k_2 E(0)$. Calculate k_1, k_2 and prove that $k_1 + k_2 = 1$.

2.8 Hyperbolic first order systems with one spatial variable

2.8.1 Definition

We consider system

$$EU_t + AU_x + BU = F \tag{2.8.1}$$

where E, A, B are $n \times n$-matrices, U is unknown n-vector (column) and F is known n-vector (column).

2.8.2 Completely separable systems

Assume that E and A are constant matrices, E is non-degenerate, $E^{-1}A$ has real eigenvalues $\lambda_1, \dots, \lambda_n$ and is diagonalisable: $E^{-1}A = Q^{-1}\Lambda Q$ with $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ (diagonal matrix with $\lambda_1, \dots, \lambda_n$ on the diagonal.

Then substituting $U = QV$ (or $V = Q^{-1}U$) we have

$$QV_t + E^{-1}AQV_x + E^{-1}BQV = E^{-1}F$$

or

$$V_t + \Lambda V_x + Q^{-1}E^{-1}BQV = Q^{-1}E^{-1}F. \tag{2.8.2}$$

In particular if $Q^{-1}E^{-1}BQ$ is also a diagonal matrix: $Q^{-1}E^{-1}BQ = \operatorname{diag}(\alpha_1, \dots, \alpha_n)$ (which is the case provided $B = 0$) we have n separate equations

$$V_{j,t} + \lambda_j V_{j,x} + \alpha_j V_j = f_j \tag{2.8.3}$$

and we can apply the theory of Section 2.1.

Definition 2.8.1. Lines $x - \lambda_j t = \text{const}$ are *characteristics*, V_j are called *Riemannian invariants*. If $\alpha_j = 0$, $f_j = 0$ these Riemannian invariants are constant along characteristics.

2.8.3 IVP (Cauchy problem)

Consider Cauchy problem: $U|_t = 0 = G(x)$, $x \in \mathbb{R}$.

Proposition 2.8.1. *(a) Let $E = I$, $A = \Lambda$ (already diagonalized) and $B = 0$. Then U_j at point P) is defined by $G_j(P_j)$ and F_j on a segment of characteristics connecting P and P_j where P_j is an intersection of $x - \lambda_j t =$* const *passing through P and $\{t = 0\}$; $(j = 1, \dots, n)$.*

(b) Let $B = 0$. Then U at point P is defined by $G(P_j)$ and by F on segments of characteristics connecting P and P_j; $(j = 1, \dots, n)$.

Proof. The first statement is obvious and the second follows from it. Note that transform by Q messes up components of U, and F, and G. □

2.8.4 IBVP

Consider now equations (2.8.3) in domain $\Omega = \{t > 0, x > \mu t\}$ assuming that the *lateral boundary* $\Gamma = \{x = \mu t, t > 0\}$ is not a characteristic, i.e. μ is not one of the numbers $\lambda_1, \dots, \lambda_n$. Then (renumbering Riemannian invariants if necessary) we have

$$\lambda_1 \le \dots \le \lambda_m < \mu < \lambda_{m+1} \le \dots \le \lambda_n. \tag{2.8.4}$$

Then with equation in Ω and initial data on $\{t = 0, x > 0\}$ we can find $V_1, \dots, V_m$ everywhere in Ω and thus on Γ; we call $V_1, \dots, V_m$ *incoming Riemannian invariants*. On the other hand, we define this way $V_{m+1}, \dots, V_n$ only as $x \ge \lambda_{m+1}, \dots, \lambda_n$ respectively and therefore not on Γ; we call them *outgoing Riemannian invariants*.

To define *outgoing Riemannian invariants* $V_{m+1}, \dots, V_n$ on Γ and thus in the rest of Ω we need boundary condition $CU|_\Gamma = H$ where C is $(n-m) \times n$-matrix and $H = H(t)$ is $(n-m)$-vector.

Indeed we need as many equations as outgoing Riemannian invariants. However it is not sufficient. We need also to assume that the following *non-degeneracy* assumption is fulfilled: $(n-m) \times (n-m)$-matrix C' obtained from C by selecting last $(n-m)$ columns (corresponding to outgoing Riemannian invariants) is non-degenerate.

2.8.5 Compatibility condition

The solution is continuous if and only if *compatibility condition* $CG(0) = H(0)$. Think why. If this condition fails U is discontinuous (has jumps) along characteristics going into Ω from the corner point $(0,0)$.

But even if solution is continuous it is not necessarily continuously differentiable (one needs more compatibility conditions for this); even more compatibility conditions for U be twice continuously differentiable etc.

Exercise 2.8.1. (a) Prove compatibility condition $CG(0) = H(0)$ for continuity of solution U;

(b) Derive a compatibility condition for continuity of the first derivatives U_t, U_x of the solution;

(c) Derive a compatibility condition for continuity of the second derivatives U_{tt}, U_{tx}, U_{xx} of the solution.

2.8.6 General case

What happens if $Q^{-1}E^{-1}BQ$ is not a diagonal matrix? More generally consider $E = E(x,t)$, $A = A(x,t)$, $B = B(x,t)$. Then assuming that $E^{-1}A$ is *smooth diagonalisable* (t.m. one can select $Q = Q(x,t)$ a smooth matrix; this is the case provided $\lambda_1, \dots, \lambda_n$ are real and distinct at every point) we have

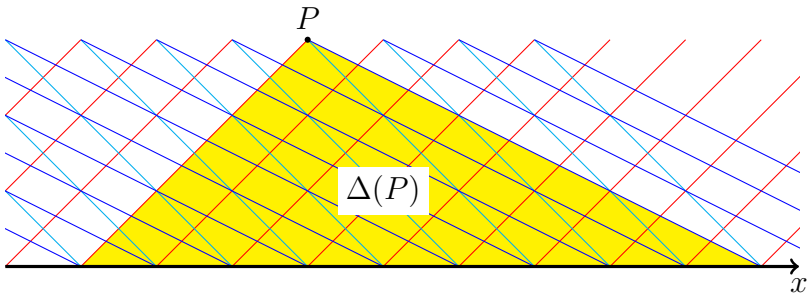
$$V_t + \Lambda V + Q^{-1}(Q_t + AQ_x + CQ)V = Q^{-1}F; \tag{2.8.5}$$

so while main part of system broke into separate equations, they are entangled through lower order terms.

Then main conclusions are the same:

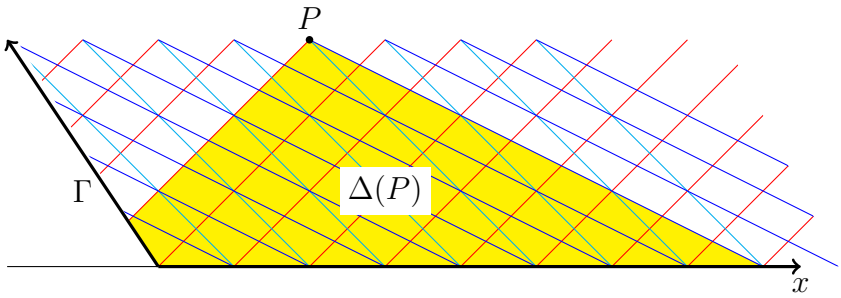
2.8.6.1 IVP

For Cauchy problem consider point P and a triangle $\Delta(P)$ formed by two characteristics: the leftmost and the rightmost going back in time and by initial line. This triangle (curvilinear if λ_j are not constant) is the domain of dependence of P.



2.8.6.2 IBVP

Again we need to assume that Γ (which could be a smooth curve rather than the straight ray) is non-characteristic at each point, then the numbers of incoming and outgoing Riemannian invariants remain constant along Γ and we need impose as many boundary condition as there are outgoing Riemannian invariants and we need to impose non-degeneracy condition in each point of Γ.



2.8.7 Problems

Problem 2.8.1. Find the characteristics and write down the general solutions to the systems $U_u + AU_x = 0$ with

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}, \tag{2.8.6}$$

$$A = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}, \tag{2.8.7}$$

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}, \tag{2.8.8}$$

$$A = \begin{pmatrix} -1 & -1 \\ 2 & -4 \end{pmatrix}, \tag{2.8.9}$$

$$A = \begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix}, \tag{2.8.10}$$

$$A = \begin{pmatrix} 3 & 0 \\ 2 & -1 \end{pmatrix} \tag{2.8.11}$$

Problem 2.8.2. For each system from Problem 1 in $\{x > 0, t > 0\}$ determine which of the following IVBPs is well-posed and find solution ($U = \begin{pmatrix} u \\ v \end{pmatrix}$):

$$u|_{t=0} = f(x), v|_{t=0} = g(x); \tag{2.8.12}$$

$$u|_{t=0} = f(x), v|_{t=0} = g(x); \qquad u|_{x=o} = \phi(t), \tag{2.8.13}$$

$$u|_{t=0} = f(x), v|_{t=0} = g(x); \qquad u|_{x=o} = \phi(t), v|_{x=0} = \psi(t) \tag{2.8.14}$$

Problem 2.8.3. Find the characteristics and write down the general solutions to the systems $U_u + AU_x = 0$ with

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.8.15}$$

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \tag{2.8.16}$$

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \tag{2.8.17}$$

$$A = \begin{pmatrix} -3 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \tag{2.8.18}$$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix}. \tag{2.8.19}$$

Problem 2.8.4. For each system from 2.8.3 in $\{x > 0, t > 0\}$ determine which of the following IVBPs is well-posed and find solution ($U = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$):

$$u|_{t=0} = f(x), v|_{t=0} = g(x), w|_{t=0} = h(x); \tag{2.8.20}$$

$$u|_{t=0} = f(x), v|_{t=0} = g(x), w|_{t=0} = h(x),$$
$$u|_{x=0} = \phi(t); \tag{2.8.21}$$

$$u|_{t=0} = f(x), v|_{t=0} = g(x), w|_{t=0} = h(x),$$
$$u|_{x=0} = \phi(t), v|_{x=0} = \psi(t); \tag{2.8.22}$$

$$u|_{t=0} = f(x), v|_{t=0} = g(x), w|_{t=0} = h(x),$$
$$u|_{x=0} = \phi(t), v|_{x=0} = \psi(t), w|_{x=0} = \chi(t). \tag{2.8.23}$$

Chapter 3

Heat equation in 1D

In this Chapter we consider 1-dimensional heat equation (also known as diffusion equation). Instead of more standard Fourier method (which we will postpone a bit) we will use the method of *self-similar solutions*.

3.1 1D Heat equation

3.1.1 Introduction

Heat equation which is in its simplest form

$$u_t = ku_{xx} \tag{3.1.1}$$

is another classical equation of mathematical physics and it is very different from wave equation. This equation describes also a diffusion, so we sometimes will refer to it as *diffusion equation*. It also describes random walks (see Subsection 3.A(Project "Random walks").

3.1.2 Self-similar solutions

We want to solve IVP for equation (3.1.1) with $t > 0$, $-\infty < x < \infty$. Let us plug

$$u_{\alpha,\beta,\gamma}(x,t) = \gamma u(\alpha x, \beta t). \tag{3.1.2}$$

Proposition 3.1.1. *If u satisfy* (3.1.1) *then $u_{\alpha,\beta,\gamma}$ also satisfies* (3.1.1) *provided $\beta = \alpha^2$.*

Proof. is just by calculation. Note that $\beta = \alpha^2$ because one derivative with respect to t is "worth" of two derivatives with respect to x. □

We impose another assumption:

Condition 3.1.1. Total heat energy

$$I(t) := \int_{-\infty}^{\infty} u(x,t)\,dx \tag{3.1.3}$$

is finite and does not depend on t.

The second part is due to the first one. Really (not rigorous) integrating (3.1.1) by x from $-\infty$ to $+\infty$ and assuming that $u_x(\pm\infty) = 0$ we see that $\partial_t I(t) = 0$.

Note that $\int_{-\infty}^{\infty} u_{\alpha,\beta,\gamma}\,dx = \gamma|\alpha|^{-1}\int_{-\infty}^{\infty} u\,dx$ and to have them equal we should take $\gamma = |\alpha|$ (actually we restrict ourselves by $\alpha > 0$). So (3.1.2) becomes

$$u_\alpha(x,t) = \alpha u(\alpha x, \alpha^2 t). \tag{3.1.4}$$

This is *transformation of similarity*. Now we are looking for a *self-similar* solution of (3.1.1) i.e. solution such that $u_\alpha(x,t) = u(x,t)$ for all $\alpha > 0, x, t > 0$. So we want

$$u(x,t) = \alpha u(\alpha x, \alpha^2 t) \qquad \forall \alpha > 0, t > 0, x. \tag{3.1.5}$$

We want to get rid off one of variables; so taking $\alpha = t^{-\frac{1}{2}}$ we get

$$u(x,t) = t^{-\frac{1}{2}} u(t^{-\frac{1}{2}}x, 1) = t^{-\frac{1}{2}}\phi(t^{-\frac{1}{2}}x) \tag{3.1.6}$$

with $\phi(\xi) := u(\xi, 1)$. Equality (3.1.6) is equivalent to (3.1.5).

Now we need to plug it into (3.1.1). Note that

$$u_t = -\frac{1}{2}t^{-\frac{3}{2}}\phi(t^{-\frac{1}{2}}x) + t^{-\frac{1}{2}}\phi'(t^{-\frac{1}{2}}x) \times \bigl(-\frac{1}{2}t^{-\frac{3}{2}}x\bigr) = \\ -\frac{1}{2}t^{-\frac{3}{2}}\bigl(\phi(t^{-\frac{1}{2}}x) + t^{-\frac{1}{2}}x\phi'(t^{-\frac{1}{2}}x)\bigr)$$

and

$$u_x = t^{-1}\phi'(t^{-\frac{1}{2}}x), \qquad u_{xx} = t^{-\frac{3}{2}}\phi''(t^{-\frac{1}{2}}x)$$

and after multiplication by $t^{\frac{3}{2}}$ and plugging $t^{-\frac{1}{2}}x = \xi$ we arrive to

$$-\frac{1}{2}\bigl(\phi(\xi) + \xi\phi'(\xi)\bigr) = k\phi''(\xi). \tag{3.1.7}$$

Good news: it is ODE. Really good news: $\phi(\xi) + \xi\phi'(\xi) = \big(\xi\phi(\xi)\big)'$. Then integrating we get

$$-\frac{1}{2}\xi\phi(\xi) = k\phi'(\xi). \tag{3.1.8}$$

Remark 3.1.1. Sure there should be $+C$ but we are looking for a solution fast decaying with its derivatives at ∞ and it implies that $C = 0$.

Separating in (3.1.8) variables and integrating we get

$$\frac{d\phi}{\phi} = -\frac{1}{2k}\xi d\xi \implies \log\phi = -\frac{1}{4k}\xi^2 + \log c \implies \phi(\xi) = ce^{-\frac{1}{4k}\xi^2}$$

and plugging into (3.1.6) we arrive to

$$u(x,t) = \frac{1}{2\sqrt{\pi}kt}e^{-\frac{x^2}{4kt}}. \tag{3.1.9}$$

Remark 3.1.2. We took $c = \frac{1}{2\sqrt{\pi}k}$ to satisfy $I(t) = 1$. Really,

$$I(t) = c\int_{-\infty}^{+\infty} t^{-\frac{1}{2}}e^{-\frac{x^2}{4kt}}\,dx = c\sqrt{2k}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}z^2}\,dz = 2c\sqrt{k\pi}$$

where we changed variable $x = z/\sqrt{2kt}$ and used the equality

$$J = \int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2}\,dx = \sqrt{2\pi}. \tag{3.1.10}$$

To prove (3.1.10) just note that

$$J^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2}\times e^{-\frac{1}{2}y^2}\,dxdy = \int_0^{2\pi} d\theta\int_0^{\infty} e^{-\frac{1}{2}r^2}r\,dr = 2\pi$$

where we used polar coordinates; since $J > 0$ we get (3.1.10).

Remark 3.1.3. Solution which we got is a very important one. However we have a problem understanding what is $u|_{t=+0}$ as $u(x,t) \to 0$ as $t \to +0$ and $x \neq 0$ but $u(x,t) \to \infty$ as $t \to +0$ and $x = 0$ and $\int_{-\infty}^{\infty} u(x,t)\,dx = 1$. In fact $u|_{t=+0} = \delta(x)$ which is *Dirac δ-function* which actually is not an ordinary function but a *distribution*.

To work around this problem we consider

$$U(x,t) = \int_{-\infty}^{x} u(x,t)\,dx. \tag{3.1.11}$$

We claim that

Proposition 3.1.2. *(a) $U(x,t)$ also satisfies* (3.1.1).

(b) $U(x,0) = \theta(x) = \begin{cases} 0 & x<0,\\ 1 & x>0. \end{cases}$

Proof. Plugging $u = U_x$ into (3.1.1) we see that $(U_t - kU_{xx})_x = 0$ and then $(U_t - kU_{xx}) = \Phi(t)$. However one can see easily that as $x \to -\infty$ U is fast decaying with all its derivatives and therefore $\Phi(t) = 0$ and (a) is proven.

Note that

$$U(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x}{\sqrt{2kt}}} e^{-\frac{1}{2}z^2}\,dz =: \frac{1}{2} + \frac{1}{2}\operatorname{erf}\Big(\frac{x}{\sqrt{2kt}}\Big) \tag{3.1.12}$$

with

$$\operatorname{erf}(z) := \sqrt{\frac{2}{\pi}} \int_0^z e^{-z^2/2}\,dz \tag{erf}$$

and that an upper limit in integral tends to $\mp\infty$ as $t \to +0$ and $x \lessgtr 0$. Then since an integrand is very fast decaying at $\mp\infty$ we using (3.1.10) arrive to (b). □

Remark 3.1.4. (a) One can construct $U(x,t)$ as a self-similar solution albeit with $\gamma = 1$.

(b) We can avoid analysis of $U(x,t)$ completely just by noting that $u(x,t)$ is a *δ-sequence* as $t \to +0$: $u(x,t) \to 0$ for all $x \neq 0$ but $\int_{-\infty}^{\infty} u(x,t)\,dx = 1$.

Consider now a smooth function $g(x)$, $g(-\infty) = 0$ and note that

$$g(x) = \int_{-\infty}^{\infty} \theta(x-y) g'(y)\,dy. \tag{3.1.13}$$

Really, the r.h.e. is $\int_{-\infty}^{x} g'(y)\,dy = g(x) - g(-\infty)$.

Also note that $U(x-y,t)$ solves the IVP with initial condition $U(x-y,+0) = \theta(x-y)$. Therefore $u(x,t) = \int_{-\infty}^{\infty} U(x-y,t) g'(y)\,dy$ solves the IVP

with initial condition $u(x,+0) = g(y)$. Integrating by parts with respect to y we arrive to $u(x,t) = \int_{-\infty}^{\infty} U_x(x-y,t)g(y)\,dy$ and finally to

$$u(x,t) = \frac{1}{2\sqrt{k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} g(y)\,dy. \tag{3.1.14}$$

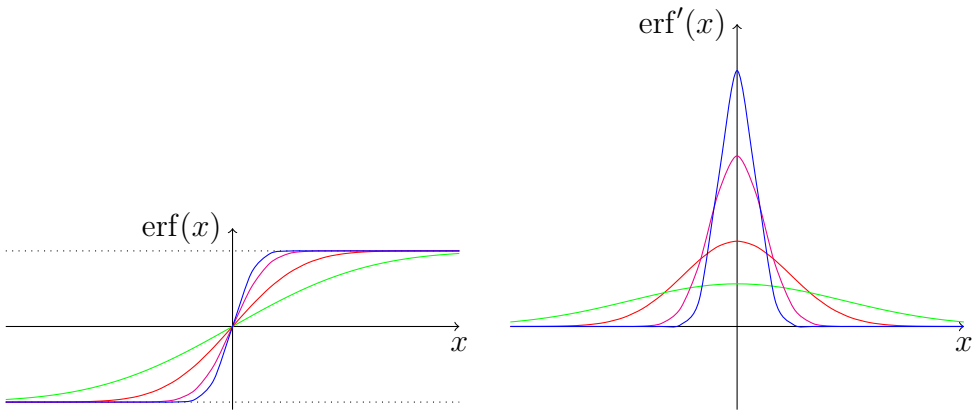
So we have proven:

Proposition 3.1.3. *Formula* (3.1.14) *gives us a solution of*

$$u_t = ku_{xx} \qquad -\infty < x < \infty, t > 0, \tag{3.1.15}$$
$$u|_{t=0} = g(x). \tag{3.1.16}$$

Remark 3.1.5. We will recreate the same formulae later using *Fourier transform*.

3.1.3 References



(a) erf(x) (b) erf$'(x)$

1. erf function
2. erf derivative

3.2 Heat equation (Miscellaneous)

3.2.1 1D Heat equation on half-line

In the previous section we considered heat equation

$$u_t = ku_{xx} \tag{3.2.1}$$

with $x \in \mathbb{R}$ and $t > 0$ and derived formula

$$u(x,t) = \int_{-\infty}^{\infty} G(x,y,t)g(y)\,dy. \tag{3.2.2}$$

with

$$G(x,y,t) = G_0(x-y,t) := \frac{1}{2\sqrt{k\pi t}} e^{-\frac{(x-y)^2}{4kt}} \tag{3.2.3}$$

for solution of IVP $u|_{t=0} = g(x)$.

Recall that $G(x,y,t)$ quickly decays as $|x-y| \to \infty$ and it tends to 0 as $t \to +0$ for $x \neq y$, but $\int G(x,y,t)\,dy = 1$.

Consider the same equation (3.2.1) on half-line with the homogeneous Dirichlet or Neumann boundary condition at $x = 0$: the method of continuation

$$u_D|_{x=0} = 0, \tag{D}$$

$$u_{N\,x}|_{x=0} = 0. \tag{N}$$

This solution also has a form (3.2.2) but with different function $G(x,y)$ (and obviously with the different domain of integration $[0,\infty)$):

$$G = G_D(x,y,t) = G_0(x-y,t) - G_0(x+y,t), \tag{3.2.4}$$

$$G = G_N x,y,t)| = G_0(x-y,t) + G_0(x+y,t) \tag{3.2.5}$$

for (D) and (N) respectively.

Both these functions satisfy equation (3.2.1) with respect to (x,t),

$$G_D|_{x=0} = 0, \tag{3.2.6}$$

$$G_{N\,x}|_{x=0} = 0. \tag{3.2.7}$$

tend to 0 as $t \to +0$, $x \neq y$

$$\int_0^\infty G_D(x,y,t)\,dy \to 1 \qquad \text{as } t \to +0, \tag{3.2.8}$$

$$\int_0^\infty G_D(x,y,t)\,dx = 1. \tag{3.2.9}$$

Further,

$$G(x,y,t) = G(y,x,t) \tag{3.2.10}$$

One can prove it by the method of continuation. Indeed, coefficients do not depend on x and equation contains only even order derivatives with respect to x. Recall from Subsection 2.6.3 that continuation is even under Neumann condition and odd under Dirichled condition.

3.2.2 Inhomogeneous boundary conditions

Consider now inhomogeneous boundary conditions

$$u_D|_{x=0} = p(t), \tag{3.2.11}$$

$$u_{N\,x}|_{x=0} = q(t). \tag{3.2.12}$$

Consider

$$0 = \int_\Pi G(x,y,t-\tau)\big(-u_\tau(y,\tau) + k u_{yy}(y,\tau)\big)\,d\tau' dy$$

with $\Pi = \{x > 0, 0 < \tau < t - \epsilon\}$. Integrating by parts with respect to τ in the first term and twice with respect to y in the second one we get

$$\begin{aligned} 0 = &\int_\Pi \big(-G_t(x,y,t-\tau) + kG_{yy}(x,y,t-\tau)\big)u(y,\tau)\,d\tau' dy \\ &- \int_0^\infty G(x,y,\epsilon)u(y,t-\epsilon)\,dy + \int_0^\infty G(x,y,t)u(y,0)\,dy + \\ &k\int_0^{t-\epsilon} \big(-G(x,y,t-\tau)u_y(y,\tau) + G_y(x,y,t-\tau)u(y,\tau)\big)_{y=0}\,d\tau. \end{aligned}$$

Note that, since G satisfies (3.2.1) with respect to (y,t) as well due to symmetry, the first line is 0.

In the second line the first term tends to $-u(x,t)$ because of properties of $G(x,y,t)$ (really, tends everywhere but for $x=y$ to 0 and its integral from 0 to ∞ tends to 1).

So we get

$$u(x,t)=\int_0^\infty G(x,y,t)\underbrace{u(y,0)}_{=g(y)}\,dy+$$
$$\int_0^t \big(-G(x,y,t-\tau)u_y(y,\tau)+G_y(x,y,t-\tau)u(y,\tau)\big)_{y=0}\,d\tau. \tag{3.2.13}$$

The first line gives in the r.h.e. us solution of the IBVP with 0 boundary condition. Let us consider the second line.

In the case of Dirichlet boundary condition $G(x,y,t)=0$ as $y=0$ and therefore we get here

$$k\int_0^t G_y(x,y,t-\tau)\underbrace{u(0,\tau)}_{=p(\tau)}\,d\tau;$$

In the case of Neumann boundary condition $G_y(x,y,t)=0$ as $y=0$ and therefore we get here

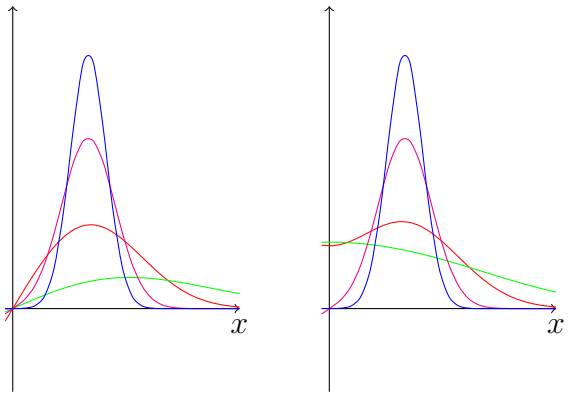
$$-k\int_0^t G(x,y,t-\tau)\underbrace{u(0,\tau)}_{=q(\tau)}\,d\tau.$$

So, (3.2.13) becomes

$$u_D(x,t)=\int_0^\infty G_D(x,y,t)g(y)\,dy+$$
$$k\int_0^t G_{D\,y}(x,0,t-\tau)p(\tau)\,d\tau; \tag{3.2.14}$$

and

$$u_N(x,t)=\int_0^\infty G_N(x,y,t)g(y)\,dy-$$
$$k\int_0^t G_{N\,y}(x,0,t-\tau)q(\tau)\,d\tau. \tag{3.2.15}$$



(a) (b)

Figure 3.1: Plots of $G_D(x, y, t)$ and $G_N(x, y, t)$ as $y = 0$ (for some values of t)

Remark 3.2.1. (a) If we consider a half-line $(-\infty, 0)$ rather than $(0, \infty)$ then the same terms appear on the right end ($x = 0$) albeit with the opposite sign;

(b) If we consider a finite interval (a, b) then there will be contributions from both ends;

(c) If we consider Robin boundary condition $(u_x - \alpha u)|_{x=0} = q(t)$ then formula (3.2.15) would work but G should satisfy the same Robin condition and we cannot consruct G by a method of continuation.

3.2.3 Inhomogeneous right-hand expression

Consider equation

$$u_t - ku_{xx} = f(x, t). \tag{3.2.16}$$

Either by Duhamel principle or just using the same calculations as above one can prove that its contribution would be

$$\int_0^t \int G(x, y, t - \tau) f(y, \tau)\, dydt \tag{3.2.17}$$

with the same G as was used for equation (3.2.1).

3.2.4 Multidimensional heat equation

Now we claim that for 2D and 3D heat equations

$$u_t = k\big(u_{xx} + u_{yy}\big), \tag{3.2.18}$$
$$u_t = k\big(u_{xx} + u_{yy} + u_{zz}\big), \tag{3.2.19}$$

similar formulae hold:

$$u = \iint G_2(x, y; x', y'; t) g(x', y')\, dx' dy', \tag{3.2.20}$$
$$u = \iiint G_3(x, y, z; x', y', z'; t) g(x', y', z')\, dx' dy' dz' \tag{3.2.21}$$

with

$$G_2(x, y; x', y'; t) = G_1(x, x', t) G_1(y, y', t), \tag{3.2.22}$$
$$G_3(x, y, z; x', y', z'; t) = G_1(x, x', t) G_1(y, y', t) G_1(z, z', t); \tag{3.2.23}$$

in particular for the whole $\mathbb{R}^n$

$$G_n(\mathbf{x}, \mathbf{x}'; t) = (2\sqrt{\pi k t})^{-n/2} e^{-\frac{|\mathbf{x}-\mathbf{x}'|}{4kt}}. \tag{3.2.24}$$

To justify our claim we note that

(a) G_n satisfies n-dimensional heat equation. Really, consider f.e. G_2:

$$\begin{aligned} G_{2\,t}(x, y; x', y'; t) =& G_{1\,t}(x, x', t) G_1(y, y', t) + G_1(x, x', t) G_{1.t}(y, y', t) = \\ & k G_{1\,xx}(x, x', t) G_1(y, y', t) + k G_1(x, x', t) G_{1\,yy}(y, y', t) = \\ & k\Delta\big(G_{1\,t}(x, x', t) G_1(y, y', t)\big) = k\Delta G_2(x, y; x', y'; t) \end{aligned}$$

(b) $G_n(\mathbf{x}, \mathbf{x}'; t)$ quickly decays as $|\mathbf{x}-\mathbf{x}'| \to \infty$ and it tends to 0 as $t \to +0$ for $\mathbf{x} \neq \mathbf{x}'$, but

(c) $\int G(\mathbf{x}, \mathbf{x}', t)\, dy \to 1$ as $t \to +0$;

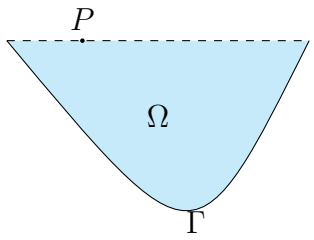
(d) $G(\mathbf{x}, \mathbf{x}', t) = G(\mathbf{x}', \mathbf{x}, t)$.

The last three properties are due to the similar properties of G_1.

Properties (a)-(d) imply integral representation (3.2.20) (or its n-dimensional variant).

3.2.5 Maximum principle

Consider heat eqution in the domain Ω like below



We claim that

Proposition 3.2.1 (maximum principle)**.** *Let u satisfy heat equation in Ω. Then*

$$\max_{\Omega} u = \max_{\Gamma} u. \tag{3.2.25}$$

Almost correct proof. Let (3.2.25) be wrong. Then there exist point $P = (\bar{x}, \bar{t}) \in \Omega \setminus \Gamma$ s.t. u reaches its maximum at P. Without any loss of the generality we can assume that P belongs to an upper lid of Ω. Then

$$u_t(P) \geq 0 \tag{3.2.26}$$

(really $u(\bar{x}, \bar{t}) \geq u(\bar{x}, t)$ for all $t : \bar{t} > t > \bar{t} - \epsilon$ and then $\big(u(\bar{x}, \bar{t}) - u(\bar{x}, t)\big)/(\bar{t} - t) \geq 0$ and as $t \nearrow \bar{t}$) we get (3.2.26).

Also $u_{xx}(P) \leq 0$ (really $u(x, \bar{t})$ reaches maximum as $x = \bar{x}$). This inequality combined with (3.2.26) almost contradict to heat equation (*almost* because there could be equalities). □

Correct proof. Note first that the above arguments prove (3.2.25) if u satisfies inequality $u_t - ku_{xx} < 0$ because then there will be contradiction.

Note that $v = u - \varepsilon t$ satisfies $v_t - kv_{xx} < 0$ for any $\varepsilon > 0$ and therefore

$$\max_{\Omega}(u - \varepsilon t) = \max_{\Gamma}(u - \varepsilon t).$$

Taking limit as $\varepsilon \to +0$ we get (3.2.25). □

Remark 3.2.2. (a) Sure, the same proof works for multidimensional heat equation.

(b) In fact, *either* in $\Omega \setminus \Gamma$ u is strictly less than $\max_\Gamma u$ *or* $u = \text{const}$. The proof is a bit more sophisticated.

Corollary 3.2.1 (minimum principle)**.**

$$\min_\Omega u = \min_\Gamma u. \tag{3.2.27}$$

Really, $-u$ also satisfies heat equation.

Corollary 3.2.2. *$u = 0$ everywhere on $\Gamma \implies u = 0$ everywhere on Ω.*

Really, then $\max_\Omega u = \min_\Omega u = 0$.

Corollary 3.2.3. *Let u, v both satisfy heat equation. Then $u = v$ everywhere on $\Gamma \implies u = v$ everywhere on Ω.*

Proof. Really, then $(u - v)$ satisfies heat equation. □

3.2.6 Problems

Crucial in many problems is formula (3.2.14) rewritten as

$$u(x,t) = \int_{-\infty}^{\infty} G(x,y,t)g(y)\,dy. \tag{3.2.28}$$

with

$$G(x,y,t) = \frac{1}{2\sqrt{k\pi t}} e^{-\frac{(x-y)^2}{4kt}} \tag{3.2.29}$$

This formula solves IVP for a heat equation

$$u_t = ku_{xx} \tag{3.2.30}$$

with the initial function $g(x)$.

In many problems below for a modified standard problem you need to derive a similar formula albeit with modified $G(x,y,t)$. Consider

$$\operatorname{erf}(z) = \sqrt{\frac{2}{\pi}} \int_0^z e^{-z^2/2}\,dz \tag{Erf}$$

as a standard function.

Problem 3.2.1. Using method of continuation obtain formula similar to (3.2.28)-(3.2.29) for solution of IBVP for a heat equation on $x > 0, t > 0$ with the initial function $g(x)$ and with

(a) Dirichlet boundary condition $u|_{x=0} = 0$;

(b) Neumann boundary condition $u_x|_{x=0} = 0$;

Problem 3.2.2. Using method of continuation obtain formula similar to (3.2.28)-(3.2.29) for solution of IBVP for a heat equation on $x > 0, t > 0$ with the initial function $g(x)$ and with

(a) Dirichlet boundary condition on both ends $u|_{x=0} = u|_{x=L} = 0$;

(b) Neumann boundary condition on both ends $u_x|_{x=0} = u_x|_{x=L} = 0$;

(c) Dirichlet boundary condition on one end and Neumann boundary condition on another $u|_{x=0} = u_x|_{x=L} = 0$.

Problem 3.2.3. Consider heat equation with a convection term

$$u_t + \underbrace{cu_x}_{\text{convection term}} = ku_{xx}. \tag{3.2.31}$$

(a) Prove that it is obtained from the ordinary heat equation with respect to U by a change of variables $U(x,t) = u(x+ct,t)$. Interpret (3.2.31) as equation describing heat propagation in the media moving to the right with the speed c.

(b) Using change of variables $u(x,t) = U(x-vt,t)$ reduce it to ordinary heat equation and using (3.2.28)-(3.2.29) for a latter write a formula for solution $u(x,t)$.

(c) Can we use the method of continuation *directly* to solve IBVP with Dirichlet or Neumann boundary condition at $x > 0$ for (3.2.31) on $\{x > 0, t > 0\}$? Justify your answer.

(d) Plugging $u(x,t) = v(x,t)e^{\alpha x+\beta t}$ with appropriate constants α, β reduce (3.2.31) to ordinary heat equation.

(e) Using (d) write formula for solution of such equation on the half-line or an interval in the case of Dirichlet boundary condition(s). Can we use this method in the case of Neumann boundary conditions? Justify your answer.

Problem 3.2.4. Using either formula (3.2.28)-(3.2.29) or its modification (if needed)

(a) Solve IVP for a heat equation (3.2.30) with $g(x) = e^{-\varepsilon|x|}$; what happens as $\varepsilon \to +0$?

(b) Solve IVP for a heat equation with convection (3.2.31) with $g(x) = e^{-\varepsilon|x|}$; what happens as $\varepsilon \to +0$?

(c) Solve IBVP with the Dirichlet boundary condition for a heat equation (3.2.31) with $g(x) = e^{-\varepsilon|x|}$; what happens as $\varepsilon \to +0$?

(d) Solve IBVP with the Neumann boundary condition for a heat equation (3.2.30) with $g(x) = e^{-\varepsilon|x|}$; what happens as $\varepsilon \to +0$?

Problem 3.2.5. Consider a solution of the diffusion equation $u_t = u_{xx}$ in $[0 \le x \le L, 0 \le t < \infty]$.

Let

$$M(T) = \max_{[0\le x\le L, 0\le t\le T]} u(x,t),$$
$$m(T) = \min_{[0\le x\le L, 0\le t\le T]} u(x,t).$$

(a) Does $M(T)$ increase or decrease as a function of T?

(b) Does $m(T)$ increase or decrease as a function of T?

Problem 3.2.6. The purpose of this exercise is to show that the maximum principle is not true for the equation $u_t = xu_{xx}$ which has a coefficient which changes sign.

(a) Verify that $u = -2xt - x^2$ is a solution.

(b) Find the location of its maximum in the closed rectangle $[-2 \le x \le 2, 0 \le t \le 1]$.

(c) Where precisely does our proof of the maximum principle break down for this equation?

Problem 3.2.7. (a) Consider the heat equation on $J = (-\infty, \infty)$ and prove that an *"energy"*

$$E(t) = \int_J u^2(x,t)\,dx \tag{3.2.32}$$

does not increase; further, show that it really decreases unless $u(x,t) =$ const;

(b) Consider the heat equation on $J = (0,l)$ with the Dirichlet or Neumann boundary conditions and prove that an $E(t)$ does not increase; further, show that it really decreases unless $u(x,t) =$ const;

(c) Consider the heat equation on $J = (0,l)$ with the Robin boundary conditions

$$u_x(0,t) - a_0 u(0,t) = 0, \tag{3.2.33}$$
$$u_x(L,t) + a_L u(L,t) = 0. \tag{3.2.34}$$

If $a_0 > 0$ and $a_l > 0$, show that the endpoints contribute to the decrease of $E(t) = \int_0^L u^2(x,t)\,dx$.

This is interpreted to mean that part of the *energy* is lost at the boundary, so we call the boundary conditions *radiating* or *dissipative*.

Hint. To prove decrease of $E(t)$ consider it derivative by t, replace u_t by ku_{xx} and integrate by parts.

Remark 3.2.3. In the case of heat (or diffusion) equation an *energy* given by (3.2.32) is rather mathematical artefact.

Problem 3.2.8. Find a self–similar solution u of

$$u_t = (uu_x)_x \qquad -\infty < x < \infty, t > 0 \tag{3.2.35}$$

with finite $\int_{-\infty}^{\infty} u\,dx$.

3.A Intro into project: Random Walks

3.A.1 Project: Walk Problem

Consider a 1D-grid with a step $h \ll 1$ and also consider grid in time with a step $\tau \ll 1$. So, $x_n = nh$ and $t_m = m\tau$.

Assume that probabilities to move to the left and right (to the next point) for one time tick are q_L and q_R respectively.

Then denoting by p_n^m the probability to be at time t_m at point x_n we get equation

$$p_n^m = p_{n-1}^{m-1} q_R + p_n^{m-1}(1 - q_L - q_R) + p_{n+1}^{m-1} q_L. \tag{3.A.1}$$

One can rewrite it as

$$p_n^m - p_n^{m-1} = p_{n-1}^{m-1} q_R - 2p_n^{m-1}(q_L + q_R) + p_{n+1}^{m-1} q_L = \\ K\big(p_{n-1}^{m-1} - p_n^{m-1} + p_{n-1}^{m-1}\big) - L\big(p_{n+1}^{m-1} - p_{n-1}^{m-1}\big) \tag{3.A.2}$$

where we used notations $K = \frac{1}{2}(q_L + q_R)$ and $L = \frac{1}{2}(q_R - q_L)$.

Task 3.A.1. Using Taylor formula and assuming that $p(x,t)$ is a smooth function prove that

$$\Lambda p := \frac{1}{h^2}\big(p_{n+1} - 2p_n + p_{n-1}\big) = \frac{\partial^2 p}{\partial x^2} + O(h^2), \tag{3.A.3}$$

$$Dp := \frac{1}{2h}\big(p_{n+1} - p_{n-1}\big) = \frac{\partial p}{\partial x} + O(h^2), \tag{3.A.4}$$

$$\frac{1}{\tau}\big(p^m - p_{m-1}\big) = \frac{\partial p}{\partial t} + O(\tau). \tag{3.A.5}$$

Then (3.A.2) becomes after we neglect small terms

$$\frac{\partial p}{\partial t} = \lambda \frac{\partial^2 p}{\partial x^2} - \mu \frac{\partial p}{\partial x} \tag{3.A.6}$$

where $K = \lambda\tau/h^2$, $L = \mu\tau/2h$.

Remark 3.A.1. This is a correct scaling or we will not get any PDE.

Remark 3.A.2. Here $p = p(x,t)$ is not a probability but *a probability density*: probability to be at moment t on interval $(x, x+dx)$ is $\mathsf{P}(x < \xi(t) < x + dx) = p(x,t)\,dx$. Since $\sum_{-\infty<n<infty} p_n^m = 1$ we have

$$\int_{-\infty}^{\infty} p(x,t)\,dx = 1. \tag{3.A.7}$$

Remark 3.A.3. The first term on the right of (3.A.6) is *a diffusion term*; in the case of symmetric walk $q_L = q_R$ only it is present:

$$\frac{\partial p}{\partial t} = \lambda \frac{\partial^2 p}{\partial x^2}. \tag{3.A.8}$$

The second term on the right of (3.A.6) is *a convection term*; moving it to the left and making change of coordinates $t_{new} = t$, $x_{new} = x - \mu t$ we get in this new coordinates equation (3.A.8). So this term is responsible for the shift with a constant speed μ (on the top of diffusion).

Remark 3.A.4. (3.A.2) is a *finite difference equation* which is a *finite difference approximation* for PDE (3.A.7). However this approximation is *stable* only if $\tau \le \frac{h^2}{2\lambda}$. This is a fact from *numerical analysis*.

Task 3.A.2 (Main task). Multidimensional case. Solution (in due time when we study). BVP. More generalization (later).

3.A.2 Absorption problem

Consider $1D$-walk (with the same rules) on a segment $[0, l]$ with both *absorbing* ends. Let p_n be a probability that our walk will end up at l if started from x_n. Then

$$p_n = p_{n-1} q_L + p_{n+1} q_R + p_n (1 - q_L - q_R). \tag{3.A.9}$$

Task 3.A.3. Prove limiting equation

$$0 = \lambda \frac{\partial^2 p}{\partial x^2} - \mu \frac{\partial p}{\partial x}. \tag{3.A.10}$$

Solve it under boundary conditions $p(0) = 0$, $p(l) = 1$. Explain these boundary conditions.

Remark 3.A.5. Here $p = p(x)$ is a probability and (3.A.7) does not hold.

Task 3.A.4 (Main task). Multidimensional case: in the domain with the boundary. Boundary conditions (there is a part Γ of the boundary and we are interested in the probability to end up here if started from given point). May be: Generalization: part of boundary is reflecting.

Chapter 4

Separation of variables and Fourier Series

In this Chapter we consider simplest separation of variables problems, arising simplest eigenvalue problems and corresponding Fourier series.

4.1 Separation of variables (the first blood)

Consider IBVP for 1D-wave equation on $(0,l)$:

$$u_{tt} - c^2 u_{xx} = 0, \qquad 0 < x < l \tag{4.1.1}$$

$$u|_{x=0} = u|_{x=l} = 0, \tag{4.1.2}$$

$$u|_{t=0} = g(x) \qquad u_t|_{t=0} = h(x). \tag{4.1.3}$$

4.1.1 Separation of variables

Let us skip temporarily initial conditions (4.1.3) and consider only (4.1.1)–(4.1.2) and look for a solution in a special form

$$u(x,t) = X(x)T(t) \tag{4.1.4}$$

with unknown functions $X(x)$ on $(0,l)$ and $T(t)$ on $(-\infty,\infty)$.

Remark 4.1.1. We are looking for *non-trivial solution* $u(x,t)$ which means that $u(x,t)$ is not identically 0. Therefore neither $X(x)$ nor $T(t)$ could be identically 0 either.

Plugging (4.1.4) into (4.1.1)–(4.1.2) we get

$$X(x)T''(t) = c^2X''(x)T(t),$$
$$X(0)T(t) = X(l)T(t) = 0,$$

which after division by $X(x)T(t)$ and $T(t)$ become

$$\frac{T''(t)}{T(t)} = c^2\frac{X''(x)}{X(x)}, \tag{4.1.5}$$
$$X(0) = X(l) = 0. \tag{4.1.6}$$

Recall, neither $X(x)$ nor $T(t)$ are identically 0.

In (4.1.5) the l.h.e. does not depend on x and the r.h.e. does not depend on t and since we have an identity we conclude that

Remark 4.1.2. Both expressions do not depend on x, t and therefore they are constant.

This is a crucial conclusion of the separation of variables method. We rewrite it as two equalities

$$\frac{T''(t)}{T(t)} = -c^2\lambda, \qquad \frac{X''(x)}{X(x)} = -\lambda$$

which in turn we rewrite as (4.1.7) and (4.1.8):

$$X'' + \lambda X = 0, \tag{4.1.7}$$
$$X(0) = X(l) = 0, \tag{4.1.6}$$
$$T'' + c^2\lambda T = 0. \tag{4.1.8}$$

4.1.2 Egenvalue problem

Consider BVP (for ODE) (4.1.7)–(4.1.6). We need to find its solution $X(x)$ which is not identically 0.

Definition 4.1.1. Such solutions are called *eigenfunctions* and corresponding numbers λ *eigenvalues* (compare with *eigenvectors* and *eigenvalues*.)

Proposition 4.1.1. (4.1.7)–(4.1.6) *has eigenvalues and eigenfunctions*

$$\lambda_n = \frac{\pi^2 n^2}{l^2} \qquad n = 1, 2, \dots, \tag{4.1.9}$$

$$X_n(x) = \sin(\frac{\pi n x}{l}). \tag{4.1.10}$$

Proof. (4.1.9) is a 2-nd order linear ODE with constant coefficients and to solve it one needs to consider *characteristic equation*

$$k^2 + \lambda = 0 \tag{4.1.11}$$

and therefore $k_{1,2} = \pm\sqrt{-\lambda}$ and $X = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$ (provided $\lambda \neq 0$). Plugging into $X(0) = 0$ and $X(l) = 0$ we get

$$\begin{aligned} A \qquad &+ B \qquad &= 0, \\ Ae^{\sqrt{-\lambda}l} &+ Be^{-\sqrt{-\lambda}l} &= 0 \end{aligned}$$

and this system has a non-trivial solution $(A, B) \neq 0$ if and only if its determinant is 0:

$$\begin{vmatrix} 1 & 1 \\ e^{\sqrt{-\lambda}l} & e^{-\sqrt{-\lambda}l} \end{vmatrix} = e^{-\sqrt{-\lambda}l} - e^{\sqrt{-\lambda}l} = 0 \iff e^{2\sqrt{-\lambda}l} \iff 2\sqrt{-\lambda}l = 2\pi i n$$

with $n = 1, 2, \dots$. Here we excluded $n = 0$ since $\lambda \neq 0$ and both n and $-n$ lead to the same λ and X. The last equation is equivalent to (4.1.9). Then $k_{1,2} = \frac{\pi n}{l} i$.

Meanwhlie $B = -A$ anyway and we get $X = 2Ai \sin(\frac{\pi n x}{l})$ i.e. (4.1.10) as factor does not matter.

So far we have not covered $\lambda = 0$. But then $k_{1,2} = 0$ and $X = A + Bx$ and plugging into (4.1.6) we get $A = A + Bl = 0 \implies A = B = 0$ and $\lambda = 0$ is not an eigenvalue. □

4.1.3 Simple solutions

After eigenvalue problem has been solved we plug $\lambda = \lambda_n$ into (4.1.8):

$$T'' + (\frac{c\pi n}{l})^2 T = 0 \tag{4.1.12}$$

which is also a 2-nd order linear ODE with constant coefficients.

Characteristic equation $k^2 + (\frac{c\pi n}{l})^2 = 0$ has solutions $k_{1,2} = \pm\frac{c\pi n}{l}i$ and therefore

$$T_n(t) = A_n \cos(\frac{c\pi n}{l}t) + BA_n \sin(\frac{c\pi n}{l}t) \tag{4.1.13}$$

and finally

$$u_n(x,t) = \underbrace{\big(A_n \cos(\frac{c\pi n}{l}t) + B_n \sin(\frac{c\pi n}{l}t)\big)}_{=T_n(t)} \cdot \underbrace{\sin(\frac{\pi n x}{l})}_{=X_n(x)} \qquad n = 1, 2, \dots \tag{4.1.14}$$

This *simple solution* (4.1.14) represents a *standing wave*
standing wave
which one can decompose into a sum of running waves
running waves decomposition
and the general discussion of standing waves could be found in wikipedia

4.1.4 General solutions

The sum of solutions of 4.1.1)-(4.1.2) is also a solution:

$$u(x,t) = \sum_{n=1}^{\infty} \big(A_n \cos(\frac{c\pi n t}{l}) + B_n \sin(\frac{c\pi n t}{l})\big) \cdot \sin(\frac{\pi n x}{l}). \tag{4.1.15}$$

We have an important question to answer:

Have we covered all solutions of (4.1.1)-(4.1.2)? – Yes, we did but we need to justify it.

Plugging (4.1.15) into (4.1.3) we get respectively

$$\sum_{n=1}^{\infty} A_n \sin(\frac{\pi n x}{l}) = g(x), \tag{4.1.16}$$

$$\sum_{n=1}^{\infty} \frac{c\pi n}{l} B_n \sin(\frac{\pi n x}{l}) = h(x). \tag{4.1.17}$$

How to find A_n and B_n? Do they exist? Are they unique?

What we got are *Fourier series* (actually sin-Fourier series). And we consider their theory in the several next sections.

4.2 Eigenvalue problem

4.2.1 Problems with explicit solutions

Example 4.2.1 (Dirichlet-Dirichlet; from Section 4.1). . Consider eigenvalue problem

$$X'' + \lambda X = 0 \qquad 0 < x < l, \tag{4.2.1}$$

$$X(0) = X(l) = 0 \tag{4.2.2}$$

as eigenvalues and corresponding eigenfunctions

$$\lambda_n = \big(\frac{\pi n}{l}\big)^2, \qquad n = 1, 2, \dots \tag{4.2.3}$$

$$X_n = \sin\big(\frac{\pi n}{l} x\big). \tag{4.2.4}$$

Example 4.2.2 (Neumann-Neumann). Eigenvalue problem

$$X'' + \lambda X = 0 \qquad 0 < x < l, \tag{4.2.5}$$

$$X'(0) = X'(l) = 0 \tag{4.2.6}$$

has eigenvalues and corresponding eigenfunctions

$$\lambda_n = \big(\frac{\pi n}{l}\big)^2, \qquad n = 0, 1, 2, \dots \tag{4.2.7}$$

$$X_n = \cos\big(\frac{\pi n}{l} x\big). \tag{4.2.8}$$

Example 4.2.3 (Dirichlet-Neumann). Consider eigenvalue problem

$$X'' + \lambda X = 0 \qquad 0 < x < l, \tag{4.2.9}$$

$$X(0) = X'(l) = 0 \tag{4.2.10}$$

has eigenvalues and corresponding eigenfunctions

$$\lambda_n = \big(\frac{\pi(2n+1)}{2l}\big)^2, \qquad n = 0, 1, 2, \dots \tag{4.2.11}$$

$$X_n = \sin\big(\frac{\pi(2n+1)}{2l} x\big) \tag{4.2.12}$$

while the same problem albeit with the ends reversed (i.e. $X'(0) = X(l) = 0$) has the same eigenvalues and eigenfunctions $\cos\big(\frac{\pi(2n+1)}{2l} x\big)$.

Example 4.2.4 (periodic). Consider eigenvalue problem

$$X'' + \lambda X = 0 \qquad 0 < x < l, \tag{4.2.13}$$

$$X(0) = X(l), \quad X'(0) = X'(l) \tag{4.2.14}$$

has eigenvalues and corresponding eigenfunctions

$$\lambda_0 = 0, \tag{4.2.15}$$

$$X_0 = 1, \tag{4.2.16}$$

$$\lambda_{2n-1} = \lambda_{2n} = \big(\frac{\pi n}{2l}\big)^2, \qquad n = 1, 2, \dots \tag{4.2.17}$$

$$X_{2n-1} = \cos\big(\frac{2\pi n}{l}x\big), \qquad X_{2n} = \sin\big(\frac{2\pi n}{l}x\big). \tag{4.2.18}$$

Alternatively, as all eigenvalues but 0 have multiplicity 2 one can select

$$\lambda_n = \big(\frac{2\pi n}{l}\big)^2, \qquad n = \dots, -2, -1, 0, 1, 2, \dots \tag{4.2.19}$$

$$X_n = \exp\big(\frac{2\pi n}{l} ix\big). \tag{4.2.20}$$

Example 4.2.5 (quasiperiodic). Consider eigenvalue problem

$$X'' + \lambda X = 0 \qquad 0 < x < l, \tag{4.2.21}$$

$$X(0) = e^{-ikl}X(l), \quad X'(0) = X'(l)e^{-ikl}X(l) \tag{4.2.22}$$

with $0 < k < \frac{2\pi}{l}$ has eigenvalues and corresponding eigenfunctions

$$\lambda_n = \big(\frac{2\pi n}{l} + k\big)^2, \qquad n = 0, 2, 4, \dots \tag{4.2.23}$$

$$X_n = \exp\big(\big[\frac{2\pi n}{l} + k\big] ix\big), \tag{4.2.24}$$

$$\lambda_n = \big(\frac{2\pi(n+1)}{l} - k\big)^2, \qquad n = 1, 3, 5, \dots \tag{4.2.25}$$

$$X_n = \exp\big(\big[\frac{2\pi(n+1)}{l} - k\big] ix\big). \tag{4.2.26}$$

This is the simplest example of problems appearing in the description of free electrons in the crystals; much more complicated and realistic example would be Schrdinger equation

$$X'' + \big(\lambda - V(x)\big)X = 0$$

or its $3D$-analog.

4.2.2 Problems with almost explicit solutions

Example 4.2.6 (Robin boundary conditions). Consider eignevalue problem

$$X'' + \lambda X = 0 \qquad 0 < x < l, \tag{4.2.27}$$

$$X'(0) = \alpha X(0), \quad X'(l) = -\beta X(l) \tag{4.2.28}$$

with $\alpha \geq 0$, $\beta \geq 0$ ($\alpha + \beta > 0$). Then

$$\lambda \int_0^l X^2\,dx = -\int_0^l X''X\,dx = \int_0^l X'^2\,dx - X'(l)X(l) + X'(0)X(0) = \int_0^l X'^2\,dx + \beta X(l)^2 + \alpha X(0)^2 \tag{4.2.29}$$

and $\lambda_n = \omega_n 2$ where $\omega_n > 0$ are roots of

$$\tan(\omega l) = \frac{(\alpha+\beta)\omega}{\omega^2 - \alpha\beta}; \tag{4.2.30}$$

$$X_n = \omega\cos(\omega_n x) + \alpha\sin(\omega_n x); \tag{4.2.31}$$

($n = 1, 2, \ldots$). Observe that

(a) As $\alpha, \beta \to +0$ $\quad \omega_n \to \frac{\pi(n-1)}{l}$.

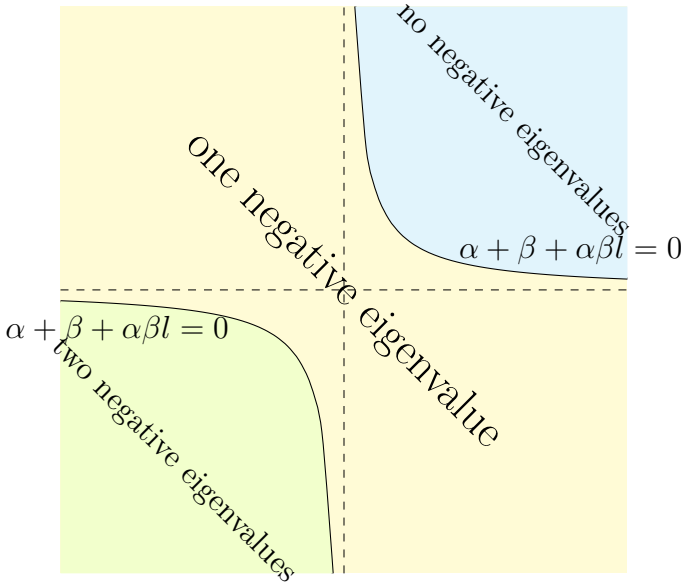
(b) As $\alpha, \beta \to +\infty$ $\quad \omega_n \to \frac{\pi n}{l}$.

(c) As $\alpha \to +0, \beta \to +\infty$ $\quad \omega_n \to \frac{\pi(n-\frac{1}{2})}{l}$.

Example 4.2.7 (Robin boundary conditions (continued)). However if α and/or β are negative, one or two negative eigenvalues $\lambda = -\gamma^2$ *can* also appear where

$$\tanh(\gamma l) = -\frac{(\alpha+\beta)\gamma}{\gamma^2 + \alpha\beta}, \tag{4.2.32}$$

$$X(x) = \gamma\cosh(\gamma x) + \alpha\sinh(\gamma x). \tag{4.2.33}$$



To investigate when it happens, consider the threshold case of eigenvalue $\lambda = 0$: then $X = cx + d$ and plugging into b.c. we have $c = \alpha d$ and $c = -\beta(d+lc)$; this system has non-trivial solution $(c, d) \neq 0$ iff $\alpha+\beta+\alpha\beta l = 0$. This hyperbola divides (α, β)-plane into three zones:

To calculate the number of negative eigenvalues one can either apply the general variational principle or analyze the case of $\alpha = \beta$; for both approaches see Appendix 4.A.

4.2.3 Problems to Sections 4.1, 4.2

"Solve equation graphically" means that you plot a corresponding function and points $(z_n, 0)$ where it intersects with OX will give us all the frequencies $\omega_n = \omega(z_n)$.

"Simple solution" $u(x, t) = X(x)T(t)$.

You may assume that all eigenvalues are real (which is the case).

Problem 4.2.1. Justify Example 4.2.6 and Example 4.2.7 Consider eignevalue

problem with Robin boundary conditions

$$X'' + \lambda X = 0 \qquad 0 < x < l, \tag{4.2.34}$$

$$X'(0) = \alpha X(0), \tag{4.2.35}$$

$$X'(l) = -\beta X(l), \tag{4.2.36}$$

with $\alpha, \beta \in \mathbb{R}$.

(a) **Prove** that positive eigenvalues are $\lambda_n = \omega_n^2$ and the corresponding eigenfunctions are X_n where $\omega_n > 0$ are roots of

$$\tan(\omega l) = \frac{(\alpha+\beta)\omega}{\omega^2 - \alpha\beta}; \tag{4.2.37}$$

$$X_n = \omega_n \cos(\omega_n x) + \alpha \sin(\omega_n x); \tag{4.2.38}$$

with $n = 1, 2, \ldots$. **Solve** this equation graphically.

(b) Prove that negative eigenvalues if there are any are $\lambda_n = -\gamma_n^2$ and the corresponding eigenfunctions are Y_n where $\gamma_n > 0$ are roots of

$$\tanh(\gamma l) = -\frac{(\alpha+\beta)\gamma}{\gamma^2 + \alpha\beta}, \tag{4.2.39}$$

$$Y_n(x) = \gamma_n \cosh(\gamma_n x) + \alpha \sinh(\gamma_n x). \tag{4.2.40}$$

Solve this equation graphically.

(c) To investigate how many negative eigenvalues are, consider the threshold case of eigenvalue $\lambda = 0$: then $X = cx + d$ and plugging into b.c. we have $c = \alpha d$ and $c = -\beta(d + lc)$; this system has non-trivial solution $(c, d) \neq 0$ iff $\alpha + \beta + \alpha\beta l = 0$. This hyperbola divides (α, β)-plane into three zones.

(d) *Prove* that eigenfunctions corresponding to different eigenvalues are orthogonal:

$$\int_0^l X_n(x) X_m(x)\, dx = 0 \qquad \text{as } \lambda_n \neq \lambda_m \tag{4.2.41}$$

where we consider now all eigenfunctions (no matter corresponding to positive or negative eigenvalues).

(e) **Bonus** Prove that eigenvalues are simple, i.e. all eigenfunctions corresponding to the same eigenvalue are proportional.

Problem 4.2.2. Analyse the same problem albeit with Dirichlet condition on the left end, : $X(0) = 0$.

Problem 4.2.3. Oscillations of the beam are described by equation

$$u_{tt} + Ku_{xxxx} = 0, \qquad 0 < x < l. \tag{4.2.42}$$

with $K > 0$.

If both ends clamped (that means having the fixed positions and directions) then the boundary conditions are

$$u(0,t) = u_x(0,t) = 0, \tag{4.2.43}$$
$$u(l,t) = u_x(l,t) = 0. \tag{4.2.44}$$

(a) *Find* equation describing frequencies and corresponding eigenfunctions (You may assume that all eigenvalues are real and positive).

(b) *Solve* this equation graphically.

(c) *Prove* that eigenfunctions corresponding to different eigenvalues are orthogonal.

(d) *Bonus* Prove that eigenvalues are simple, i.e. all eigenfunctions corresponding to the same eigenvalue are proportional.

HINT. Change coordinate system so that interval becomes $[-L, L]$ with $L = l/2$; consider separately even and odd eigenfunctions.

Problem 4.2.4. Consider oscillations of the beam with both ends free:

$$u_{xx}(0,t) = u_{xxx}(0,t) = 0, \tag{4.2.45}$$
$$u_{xx}(l,t) = u_{xxx}(l,t) = 0. \tag{4.2.46}$$

Follow previous problem but also consider eigenvalue 0.

Problem 4.2.5. Consider oscillations of the beam with the clamped left end and the free right end. Then boundary conditions are (4.2.42) and (4.2.46).

Note. In this case due to the lack of symmetry you cannot consider separately even and odd eigenfunctions.

Problem 4.2.6. Consider wave equation with the Neumann boundary condition on the left and "weird" b.c. on the right:

$$u_{tt} - c^2 u_{xx} = 0 \qquad 0 < x < l, \tag{4.2.47}$$

$$u_x(0,t) = 0, \tag{4.2.48}$$

$$(u_x + i\alpha u_t)(l,t) = 0 \tag{4.2.49}$$

with $\alpha \in \mathbb{R}$.

(a) **Separate** variables;

(b) **Find** "weird" eigenvalue problem for ODE;

(c) **Solve** this problem;

(d) **Find** simple solution $u(x,t) = X(x)T(t)$.

Hint. You may assume that all eigenvalues are real (which is the case).

Problem 4.2.7. Consider energy levels of the particle in the "rectangular well"

$$-u_{xx} + Vu = \lambda u \tag{4.2.50}$$

with $V(x) = \begin{cases} -H & |x| \le L, \\ 0 & |x| > 0 \end{cases}$

Hint. Solve equation for $|x| < L$ and for $|x| > L$ and solution must be continous (with its first derivative) as $|x| = L$: $u(L-0) = u(L+0)$, $u_x(L-0) = u_x(L+0)$ and the same at $-L$.

Hint. All eigenvalues belong to interval $(-H, 0)$.

Hint. Consider separately even and odd eigenfunctions.

4.3 Orthogonal systems

4.3.1 Examples

All systems we considered in the previous Section were orthogonal i.e.

$$(X_n, X_m) = 0 \qquad \forall m \neq n \tag{4.3.1}$$

with

$$(X,Y) := \int_0^l X(x)\bar{Y}(x)\,dx, \qquad \|X\|^2 := (X,X). \tag{4.3.2}$$

where $\bar{Y}$ means complex-conjugate to Y.

Exercise 4.3.1. Prove it by direct calculation.

Instead however we show that this nice property (and the fact that eigenvalues are real) is due to self-adjointness (the notion which we do not want to formulate at this time at least).

Consider X, Y satisfying Robin boundary conditions

$$X'(0) - \alpha X(0) = 0, \tag{4.3.3}$$
$$X'(l) + \beta X(l) = 0 \tag{4.3.4}$$

with $\alpha, \beta \in \mathbb{R}$ (so Y satisfies the same conditions). Note that

$$\begin{aligned}(X'', Y) = \int X''(x)\bar{Y}(x)\,dx = \\ - \int X'(x)\bar{Y}'(x)\,dx + X'(l)\bar{Y}(l) - X'(0)\bar{Y}(0) = \\ - (X', Y') - \beta X(l)\bar{Y}(l) - \alpha X(0)\bar{Y}(0).\end{aligned} \tag{4.3.5}$$

Therefore if we plug $Y = X \neq 0$ an eigenfunction, $X'' + \lambda X = 0$ we get $-\lambda \|X\|^2$ in the left-hand expression (with obviously real $\|X\|^2 \neq 0$) and also we get the real right expression (since $\alpha, \beta \in \mathbb{R}$); so λ must be real: *all eigenvalues are real*.

Further, for (X, Y'') we obtain the same equality albeit with α, β replaced by $\bar{\alpha}, \bar{\beta}$ and therefore due to assumption $\alpha, \beta \in \mathbb{R}$

$$(X'', Y) = (X, Y''). \tag{4.3.6}$$

But then if X, Y are eigenfunctions corresponding to *different* eigenvalues λ and μ we get from (4.3.6) that $-\lambda(X, Y) = -\mu(X, Y)$ and $(X, Y) = 0$ due to $\lambda \neq \mu$.

Remark 4.3.1. For periodic boundary conditions we cannot apply these arghuments to prove that $\cos(2\pi nx/l)$ and $\cos(2\pi nx/l)$ are orthogonal since they correspond to the same eigenvalue; we need to prove it directly.

4.3.2 Abstract orthogonal systems: definition

Consider *linear space* H, real or complex. From linear algebra course standard definition

1. $u+v=v+u \qquad \forall u,v\in \mathsf{H}$;
2. $(u+v)+w=u+(v+w) \qquad \forall u,v,w\in \mathsf{H}$;
3. $\exists 0\in \mathsf{H}:\ 0+u=u \qquad \forall u\in \mathsf{H}$;
4. $\forall u\in \mathsf{H}\exists(-u): u+(-u)=0$;
5. $\alpha(u+v)=\alpha u+\alpha v \qquad \forall u,v\in \mathsf{H} \quad \forall \alpha\in\mathbb{R}$;
6. $(\alpha+\beta)u=\alpha u+\beta u \qquad \forall u\in \mathsf{H} \quad \forall \alpha,\beta\in\mathbb{R}$;
7. $\alpha(\beta u)=(\alpha\beta)u \qquad \forall u\in \mathsf{H} \quad \forall \alpha,\beta\in\mathbb{R}$;
8. $1u=u \qquad \forall u\in \mathsf{H}$.

For complex linear space replace $\mathbb{R}$ by $\mathbb{C}$.
Assume that on H *inner product* is defined:

1. $(u+v,w)=(u,w)+(v,w) \qquad \forall u,v,w\in \mathsf{H}$;
2. $(\alpha u,v)=\alpha(u,v) \qquad \forall u,v\in \mathsf{H} \quad \forall \alpha\in\mathbb{R}$;
3. $(u,v)=(\bar{v,u}) \qquad \forall u,v\in \mathsf{H}$;
4. $\|u\|^2:=(u,u)\ge 0 \qquad \forall u\in \mathsf{H}$ (it implies that it is real–if we consider complex spaces) and $\|u\|=0 \iff u=0$.

Definition 4.3.1. (a) Finite dimensional real linear space with an inner product is called *Euclidean* space.

(b) Finite dimensional complex linear space with an inner product is called *Hermitian* space.

(c) Infinite dimensional linear space (real or complex) with an inner product is called *pre-Hilbert* space.

For Hilbert space we will need another property (completeness) which we add later.

Definition 4.3.2. (a) System $\{u_n\}$, $0\neq u_n\in \mathsf{H}$ (finite or infinite) is *orthogonal* if $(u_m,u_n)=0\ \forall m\neq n$;

(b) Orthogonal system is *orthonormal* if $\|u_n\|=1\ \forall n$, i.e. $(u_m,u_n)=\delta_{mn}$ – Kronecker symbol.

4.3.3 Orthogonal systems: approximation

Consider finite orthogonal system $\{u_n\}$. Let K be its *linear hull*: the set of linear combinations $\sum_n \alpha_n u_n$. Obviously K is a linear subspace of H. Let $v \in \mathsf{H}$ and we try to find the best approximation of v by elements of K, i.e. we are looking for $w \in \mathsf{K}$ s.t. $\|v-w\|$ minimal.

Theorem 4.3.1. *(a) There exists a unique minimizer;*

(b) This minimizer is an orthogonal projection of f to K, i.e. $w \in \mathsf{K}$ s.t. $(v-w)$ is orthogonal to all elements of K;

(c) Such orthogonal projection is unique and $w = \sum_n \alpha_n u_n$ with

$$\alpha_n = \frac{(v,u_n)}{\|u_n\|^2}. \tag{4.3.7}$$

(d) $\|v\|^2 = \|w\|^2 + \|v-w\|^2$.

(e) $v = w \iff \|v\|^2 = \|w\|^2$.

Proof. (c) Obviously $(v-w)$ is orthogonal to u_n iff (4.3.7) holds. If (4.3.7) holds for all n then $(v-w)$ is orthogonal to all u_n and therefore to all their linear combinations.

(d)-(e) In particular $(v-w)$ is orthogonal to w and then

$$\|v\|^2 = \|(v-w)+w\|^2 = \|v-w\|^2 + 2\,\mathrm{Re}\underbrace{(v-w,w)}_{=0} + \|w\|^2.$$

(a)-(b) Consider $w' \in \mathsf{K}$. Then $\|v-w'\|^2 = \|v-w\|^2 + \|w-w'\|^2$ because $(w-w') \in \mathsf{K}$ and therefore it is orthogonal to $(v-w)$. □

4.3.4 Orthogonal systems: approximation.

Now let $\{u_n\}_{n=1,2,\dots,}$ be infinite orthogonal system. Consider its finite subsystem with $n = 1, 2, \dots, N$, introduce K_N for it and consider orthogonal projection w_N of v on K_N. Then

$$w_N = \sum_{n=1}^{N} \alpha_N u_n$$

where α_n are defined by (4.3.7). Then according to (d) of Theorem 4.3.1

$$\|v\|^2 = \|v - w_N\|^2 + \|w_N\|^2 \geq \|w_N\|^2 = \sum_{n=1}^{N} |\alpha_n|^2 \|u_n\|^2.$$

Therefore series in the right-hand expression below converges

$$\|v\|^2 \geq \sum_{n=1}^{\infty} |\alpha_n|^2 \|u_n\|^2 \tag{4.3.8}$$

Really, recall that non-negative series can either converge or diverge to ∞.

Then w_N is a *Cauchy sequence*. Really, for $M > N$

$$\|w_N - w_M\|^2 = \sum_{n=N+1}^{M} |\alpha_n|^2 \|u_n\|^2 \leq \varepsilon_N$$

with $\varepsilon_N \to 0$ as $N \to \infty$ because series in (4.3.8) converges.

Now we want to conclude that w_N converges and to do this we must assume that every Cauchy sequence converges.

Definition 4.3.3. (a) H is *complete* if every Cauchy sequence converges in H.

(b) Complete pre-Hilbert space is called *Hilbert space*.

Remark 4.3.2. Every pre-Hilbert space could be completed i.e. extended to a complete space. From now on H is a Hilbert space.

Then we can introduce K– a closed linear hull of $\{u_n\}_{n=1,2,\dots}$ i.e. the space of

$$\sum_{n=1}^{\infty} \alpha_n u_n \tag{4.3.9}$$

with α_n satisfying

$$\sum_{n=1}^{\infty} |\alpha_n|^2 \|u_n\|^2 < \infty. \tag{4.3.10}$$

(Linear hull would be a space of finite linear combinations).

Let $v \in \mathsf{H}$. We want to find the best approximation of v by elements of K. But then we get immediately

Theorem 4.3.2. *If* H *is a Hilbert space then Theorem 4.3.1 holds for infinite systems as well.*

4.3.5 Orthogonal systems: completeness

Definition 4.3.4. Orthogonal system is *complete* if equivalent conditions below are satisfied:

(a) Its closed convex hull coincides with H.

(b) If $v \in \mathsf{H}$ is orthogonal to all u_n then $v = 0$.

Remark 4.3.3. Don't confuse completeness of spaces and completeness of orthogonal systems.

Our next goal is to establish completeness of some orthogonal systems and therefore to give a positive answer (in the corresponding frameworks) to the question in the end of the previous Section 4.2: can we decompose any function into eigenfunctions? Alternatively: Is the general solution a combination of simple solutions?

4.4 Ortogonal systems and Fourier series

4.4.1 Formulae

Consider system of functions

$$\Big\{\frac{1}{2}, \qquad \cos(\frac{\pi n x}{l}), \qquad \sin(\frac{\pi n x}{l}) \quad n = 1, \dots\Big\} \tag{4.4.1}$$

on interval $J := [x_0, x_1]$ with $(x_1 - x_0) = 2l$.

These are eigenfunctions of $X'' + \lambda X = 0$ with periodic boundary conditions $X(x_0) = X(x_1)$, $X'(x_0) = X'(x_1)$.

Proposition 4.4.1.

$$\int_J \cos(\frac{\pi m x}{l}) \cos(\frac{\pi n x}{l})\, dx = l\delta_{mn},$$
$$\int_J \sin(\frac{\pi m x}{l}) \sin(\frac{\pi n x}{l})\, dx = l\delta_{mn},$$
$$\int_J \cos(\frac{\pi m x}{l}) \sin(\frac{\pi n x}{l})\, dx = 0,$$

and

$$\int_J \cos(\frac{\pi m x}{l})\,dx = 0, \qquad \int_J \sin(\frac{\pi m x}{l})\,dx = 0, \qquad \int_J dx = 2l$$

for all $m, n = 1, 2, \dots$.

Proof. Easy; use formulae

$$\begin{aligned} 2\cos(\alpha)\cos(\beta) &= \cos(\alpha-\beta)+\cos(\alpha+\beta),\\ 2\sin(\alpha)\sin(\beta) &= \cos(\alpha-\beta)-\cos(\alpha+\beta),\\ 2\sin(\alpha)\cos(\beta) &= \sin(\alpha-\beta)+\sin(\alpha+\beta). \end{aligned}$$

□

Therefore according to the previous Section 4.3 we arrive to decomposition

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\bigl(a_n\cos(\frac{\pi n x}{l}) + b_n\sin(\frac{\pi n x}{l})\bigr) \tag{4.4.2}$$

with coefficients calculated according to (4.3.7)

$$a_n = \frac{1}{l}\int_J f(x)\cos(\frac{\pi n x}{l})\,dx \qquad n = 0, 1, 2, \dots, \tag{4.4.3}$$

$$b_n = \frac{1}{l}\int_J f(x)\sin(\frac{\pi n x}{l})\,dx \qquad n = 1, 2, \dots, \tag{4.4.4}$$

and satisfying *Parseval's equality*

$$\frac{l}{2}|a_0|^2 + \sum_{n=1}^{\infty} l\bigl(|a_n|^2 + |b_n|^2\bigr) = \int_J |f(x)|^2\,dx. \tag{4.4.5}$$

So far this is an *optional* result: provided we can decompose function $f(x)$.

4.4.2 Completeness of the system (4.4.1)

Now our goal is to prove that any function $f(x)$ on J could be decomposed into Fourier series (4.4.2). First we need

Lemma 4.4.1. *Let $f(x)$ be a piecewise-continuous function on J. Then*

$$\int_J f(x)\cos(\omega x)\,dx \to 0 \qquad \text{as } \omega \to \infty \tag{4.4.6}$$

and the same is true for $\cos(\omega x)$ *replaced by* $\sin(\omega x)$.

Proof. (a) Assume first that $f(x)$ is continuously differentiable on J. Then integrating by parts

$$\int_J f(x)\cos(\omega x)\,dx = \omega^{-1} f(x)\sin(\omega x)\big|_{x_0}^{x_1} - \omega^{-1}\int_J f(x)\sin(\omega x)\,dx = O(\omega^{-1}).$$

(b) Assume now only that $f(x)$ is continuous on J. Then it could be uniformly approximated by continuous functions (proof is not difficult but we skip it anyway):

$$\forall \varepsilon > 0 \exists f_\varepsilon \in C^1(J) : \forall x \in J |f(x) - f_\varepsilon(x)| \le \varepsilon.$$

Then obviously the difference between integrals (4.4.6) for f and for f_ε does not exceed $2l\varepsilon$; so choosing $\varepsilon = \varepsilon(\delta) = \delta/(4l)$ we make it $< \delta/2$. After ε is chosen and f_ε fixed we can choose ω_ε s.t. for $\omega > \omega_\varepsilon$ integral (4.4.6) for f_ε does not exceed $\delta/2$ in virtue of (a). Then integral (4.4.6) for f does not exceed δ.

(c) Integral (4.4.6) for interval J equals to the sum of integrals over intervals where f is continuous.

□

Now calculate coefficients according to (4.4.3)–(4.4.4) (albeit plug y instead of x) and plug into partial sum:

$$S_N(x) := \frac{1}{2}a_0 + \sum_{n=1}^{N}\Big(a_n\cos(\frac{\pi n x}{l}) + b_n\sin(\frac{\pi n x}{l})\Big) = \frac{1}{l}\int_J K_N(x,y) f(y)\,dy \tag{4.4.7}$$

with

$$K_N(x,y) = \frac{1}{2} + \sum_{n=1}^{N}\Big(\cos(\frac{\pi n y}{l})\cos(\frac{\pi n x}{l}) + \sin(\frac{\pi n y}{l})\sin(\frac{\pi n x}{l})\Big) = \frac{1}{2} + \sum_{n=1}^{N}\cos(\frac{\pi n(y-x)}{l}). \tag{4.4.8}$$

Note that

$$\sum_{n=1}^{N}\sin(\frac{1}{2}z)\cos(z) = \sum_{n=1}^{N}\big(\sin((n+\frac{1}{2})z) - \sin((n-\frac{1}{2})z) = \sin((N+\frac{1}{2})z) - \sin(\frac{1}{2}z)$$

and therefore

$$K_N(x,y) = \frac{\sin(k(N+\frac{1}{2})(x-y))}{\sin(k(x-y))}, \qquad k = \frac{\pi}{l}. \tag{4.4.9}$$

So

$$S_N(x) = \frac{1}{l}\int_J \frac{\sin(k(N+\frac{1}{2})(x-y))}{\sin(k(x-y))} f(y)\,dy \tag{4.4.10}$$

We cannot apply Lemma 4.4.1 to this integral immediately because of denominator.

Assume that x is internal point of J. Note that denominator vanishes on J only as $y = x$. Really, $\frac{\pi}{2l}(x-y) < \pi$. Also note that derivative of denominator does not vanish as $y = x$. Then $f(y)/\sin(k(x-y))$ is a piecewise continuous function of y provided all three conditions below are fulfilled:

(a) f is piecewise continuously differentiable function,

(b) f is continuous in x

(c) $f(x) = 0$ (we are talking about a single point).

In this case we can apply Lemma 4.4.1 and we conclude that $S_N(x) \to 0$ as $N \to \infty$. So,

If f satisfies (a)-(c) then $S_N(x) \to f(x)$ as $N \to \infty$.

Let us drop condition $f(x) = 0$. Then we can decompose

$$S_N(x) = \frac{1}{l}\int_J \frac{\sin(k(N+\frac{1}{2})(x-y))}{\sin(k(x-y))}\big(f(y)-f(x)\big)\,dy + \frac{1}{l}\int_J \frac{\sin(k(N+\frac{1}{2})(x-y))}{\sin(k(x-y)}f(x)\,dy \tag{4.4.11}$$

and the first integral tends to 0 due to Lemma 1. We claim that the second integral is identically equal $f(x)$. Indeed, we can move $f(x)$ out of integral and consider

$$\frac{1}{l}\int_J \frac{\sin(k(N+\frac{1}{2})(x-y))}{\sin(k(x-y))}\,dy = \frac{1}{l}\int_J \big(\frac{1}{2} + \sum_{n=1}^{N}\cos(\frac{\pi n(y-x)}{l})\big)\,dy \tag{4.4.12}$$

where integral of the first term equals l and integral of all other terms vanish.

4.4.3 Pointwise convergence

Therefore we arrive to

Theorem 4.4.1. *Let x be internal point of J (i.e. $x_0 < x < x_1$) and let*

(a) f be a piecewise continuously differentiable function,

(b) f be continuous in x. Then the Fourier series converges to $f(x)$ at x.

4.4.4 Pointwise convergence. II

Assume now that there is a jump at x. Then we need to be more subtle. First, we can replace $f(x)$ by its $2l$-periodic continuation from J to $\mathbb{R}$. Then

we can take any interval of the length $2l$ and result will be the same. So we take $[x-l, x+l]$. Now

$$\begin{aligned} S_N(x) = \frac{1}{l}\int_{J^-} \frac{\sin(k(N+\frac{1}{2})(x-y))}{\sin(k(x-y))}\big(f(y)-f(x-0)\big)\,dy+\\ \frac{1}{l}\int_{J^+} \frac{\sin(k(N+\frac{1}{2})(x-y))}{\sin(k(x-y))}\big(f(y)-f(x+0)\big)\,dy+\\ \frac{1}{l}\int_{J^-} \frac{\sin(k(N+\frac{1}{2})(x-y))}{\sin(k(x-y)}f(x-0)\,dy+\\ \frac{1}{l}\int_{J^+} \frac{\sin(k(N+\frac{1}{2})(x-y))}{\sin(k(x-y)}f(x+0)\,dy \end{aligned}$$

with $J^- = (x-l, l)$, $J^+ = (x, x+l)$. According to Lemma 4.4.1 again the first two integrals tend to 0 and we need to consider integrals

$$\frac{1}{l}\int_{J^-} \frac{\sin(k(N+\frac{1}{2})(x-y))}{\sin(k(x-y)}\,dy, \frac{1}{l}\int_{J^+} \frac{\sin(k(N+\frac{1}{2})(x-y))}{\sin(k(x-y)}\,dy.$$

Using back transformation like in (4.4.12) we conclude that both these integrals are equal to $\frac{1}{2}$. Therefore we proved

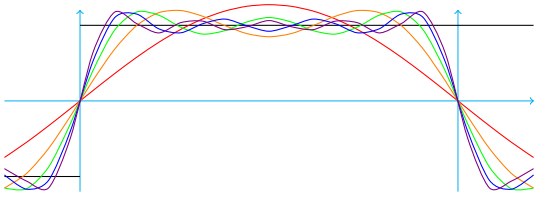
Theorem 4.4.2. *Let f be a piecewise continuously differentiable function. Then the Fourier series converges to*

(a) $f(x)$ *if x is internal point and f is continuous at x;*

(b) $\frac{1}{2}\big(f(x+0)+f(x-0)\big)$ *if x is internal point and f is discontinuous at x;*

(c) $\frac{1}{2}\big(f(x_0+0)+f(x_1-0)\big)$ *if $x = x_0$ or $x = x_1$. Recall that x_0 and x_1 are the ends.*

The last two statements are called *Stokes phenomenon*. Below are partial sums of sin-Fourier decomposition of $u(x) = 1$.



4.5 Other Fourier series

4.5.1 Fourier series for even and odd functions

In the previous Section 4.4 we proved the completeness of the system of functions

$$\Big\{\frac{1}{2}, \qquad \cos(\frac{\pi n x}{l}), \qquad \sin(\frac{\pi n x}{l}) \quad n=1,\dots\Big\} \tag{4.5.1}$$

on interval $J := [x_0, x_1]$ with $(x_1 - x_0) = 2l$. In other words we proved that any function $f(x)$ on this interval could be decomposed into Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\big(a_n \cos(\frac{\pi n x}{l}) + b_n \sin(\frac{\pi n x}{l})\big) \tag{4.5.2}$$

with coefficients calculated according to (4.3.7)

$$a_n = \frac{1}{l}\int_J f(x)\cos(\frac{\pi n x}{l})\,dx \qquad n=0,1,2,\dots, \tag{4.5.3}$$

$$b_n = \frac{1}{l}\int_J f(x)\sin(\frac{\pi n x}{l})\,dx \qquad n=1,2,\dots, \tag{4.5.4}$$

and satisfying Parseval's equality

$$\frac{l}{2}|a_0|^2 + \sum_{n=1}^{\infty} l\big(|a_n|^2 + |b_n|^2\big) = \int_J |f(x)|^2\,dx. \tag{4.5.5}$$

Now we consider some other orthogonal systems and prove their completeness. To do this we first prove

Lemma 4.5.1. *Let J be a symmetric interval: $J = [-l, l]$. Then*

(alph) $f(x)$ is even iff $b_n = 0$ $\quad \forall n = 1,2,\dots$.*

(alph) $f(x)$ is odd iff $a_n = 0$ $\quad \forall n = 0,1,2,\dots$.*

Proof. (a) Note that $\cos(\frac{\pi nx}{l})$ are even functions and $\sin(\frac{\pi nx}{l})$ are odd functions. Therefore if $b_n = 0 \qquad \forall n = 1, 2, \dots$ then only decomposition (4.5.2) contains only even functions and $f(x)$ is even. Conversely, if $f(x)$ is an even function then integrand in (4.5.4) is an odd function and its integral over symmetric interval is 0.

(b) Statement (b) is proven in the similar way.

□

4.5.2 cos-Fourier series

Let us consider function $f(x)$ on the interval $[0, l]$. Let us extend it as an even function on $[-l, l]$ so $f(x) := f(-x)$ for $x \in [-l, 0]$ and decompose it into full Fourier series (4.5.2); however sin-terms disappear and we arrive to decomposition

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{\pi nx}{l}). \tag{4.5.6}$$

This is decomposition with respect to orthogonal system

$$\Big\{\frac{1}{2}, \qquad \cos(\frac{\pi nx}{l}) \quad n = 1, \dots\Big\}. \tag{4.5.7}$$

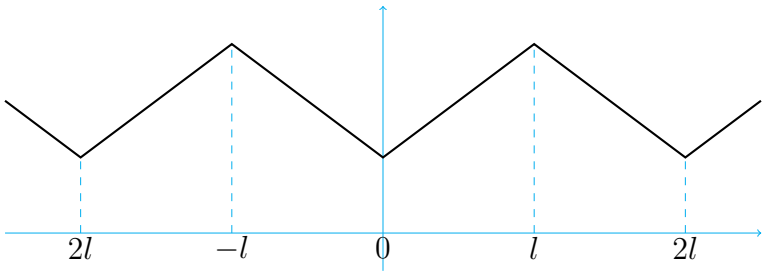
Its coefficients are calculated according to (4.5.3) but here integrands are even functions and we can take interval $[0, l]$ instead of $[-l, l]$ and double integrals:

$$a_n = \frac{2}{l}\int_0^l f(x)\cos(\frac{\pi nx}{l})\,dx \qquad n = 0, 1, 2, \dots \tag{4.5.8}$$

Also (4.5.5) becomes

$$\frac{l}{4}|a_0|^2 + \sum_{n=1}^{\infty}\frac{l}{2}|a_n|^2 = \int_0^l |f(x)|^2\,dx. \tag{4.5.9}$$

The sum of this Fourier series is $2l$-periodic. Note that even and then periodic continuation does not introduce new jumps.



4.5.3 sin-Fourier series

Let us consider function $f(x)$ on the interval $[0,l]$. Let us extend it as an odd function on $[-l,l]$ so $f(x) := -f(-x)$ for $x \in [-l,0]$ and decompose it into full Fourier series (4.5.2); however cos-terms disappear and we arrive to decomposition

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(\frac{\pi n x}{l}). \tag{4.5.10}$$

This is decomposition with respect to orthogonal system

$$\Big\{\sin(\frac{\pi n x}{l}) \quad n = 1, \dots\Big\}. \tag{4.5.11}$$

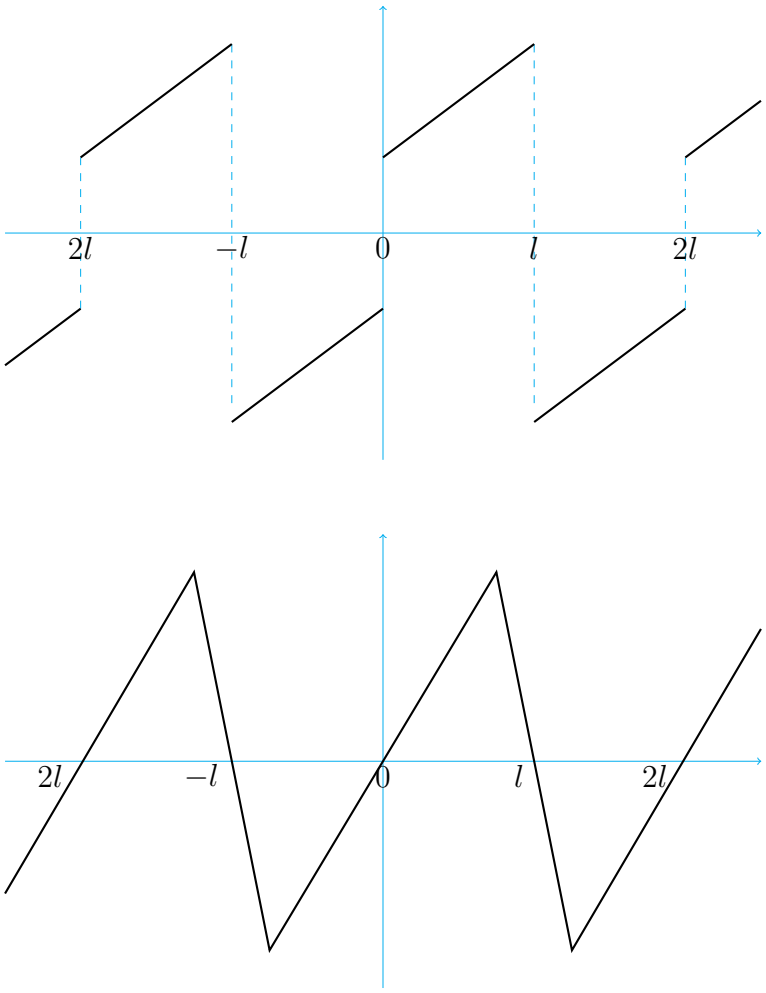
Its coefficients are calculated according to (4.5.4) but here integrands are even functions and we can take interval $[0,l]$ instead of $[-l,l]$ and double integrals:

$$b_n = \frac{2}{l}\int_0^l f(x)\sin(\frac{\pi n x}{l})\,dx \qquad n = 1,2,\dots \tag{4.5.12}$$

Also (4.5.5) becomes

$$\sum_{n=1}^{\infty} \frac{l}{2}|b_n|^2 = \int_0^l |f(x)|^2\,dx. \tag{4.5.13}$$

The sum of this Fourier series is $2l$-periodic. Note that odd and then periodic continuation does not introduce new jumps iff $f(0) = f(l) = 0$.



4.5.4 sin-Fourier series with half-integers

Let us consider function $f(x)$ on the interval $[0,l]$. Let us extend it as an even with respect to $x=l$ function on $[0,2l]$ so $f(x):=f(2l-x)$ for $x\in[l,2l]$; then we make an odd continuation to $[-2l,2l]$ and decompose it into full Fourier series (4.5.2) but with l replaced by $2l$; however cos-terms disappear and we arrive to decomposition

$$f(x)=\sum_{n=1}^{\infty} b'_n \sin(\frac{\pi n x}{2l}).$$

Then $f(2l-x)=\sum_{n=1}^{\infty} b'_n \sin(\frac{\pi n x}{2l})(-1)^{n+1}$ and since $f(x)=f(2l-x)$ due to original even continuation we conclude that $b'_n=0$ as $n=2m$ and we arrive to

$$f(x)=\sum_{n=0}^{\infty} b_n \sin(\frac{\pi(2n+1)x}{2l}) \tag{4.5.14}$$

with $b_n := b'_{2n+1}$ where we replaced m by n.

This is decomposition with respect to orthogonal system

$$\Big\{\sin(\frac{\pi(2n+1)x}{2l}) \quad n=1,\dots\Big\}. \tag{4.5.15}$$

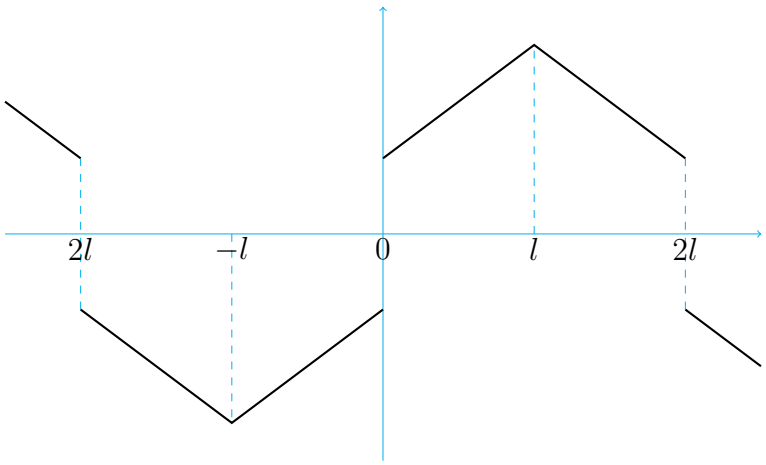
Its coefficients are calculated according to (4.5.12) (with l replaced by $2l$) but here we can take interval $[0,l]$ instead of $[0,2l]$ and double integrals:

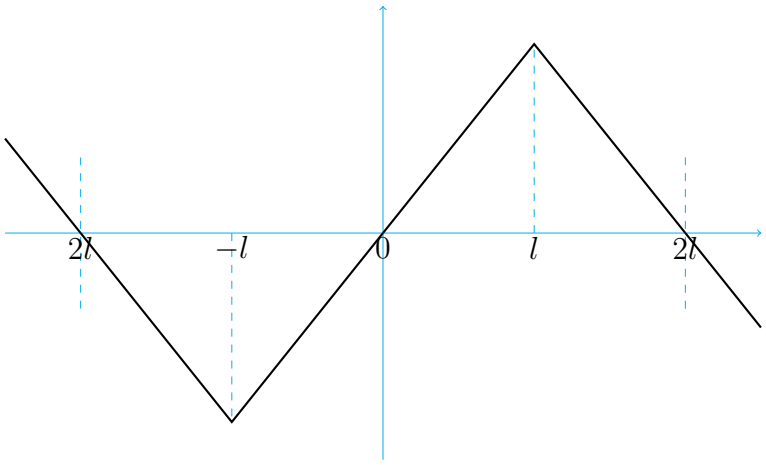
$$b_n=\frac{2}{l}\int_0^l f(x)\sin(\frac{\pi(2n+1)x}{2l})\,dx \qquad n=0,2,\dots \tag{4.5.16}$$

Also (4.5.13) becomes

$$\sum_{n=0}^{\infty}\frac{l}{2}|b_n|^2=\int_0^l |f(x)|^2\,dx. \tag{4.5.17}$$

The sum of this Fourier series is $4l$-periodic. Note that odd and then periodic continuation does not introduce new jumps iff $f(0)=0$.





4.5.5 Fourier series in complex form

Consider (4.5.2)–(4.5.5). Plugging

$$\cos(\frac{\pi nx}{l}) = \frac{1}{2}e^{\frac{i\pi nx}{l}} + \frac{1}{2}e^{-\frac{i\pi nx}{l}}$$
$$\sin(\frac{\pi nx}{l}) = \frac{1}{2i}e^{\frac{i\pi nx}{l}} - \frac{1}{2i}e^{-\frac{i\pi nx}{l}}$$

and separating terms with n and $-n$ and replacing in the latter $-n$ by $n = -1, -2, \dots$ we get

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{i\pi nx}{l}} \tag{4.5.18}$$

with $c_0 = \frac{1}{2}a_0$, $c_n = \frac{1}{2}(a_n - ib_n)$ as $n = 1, 2, \dots$ and $c_n = \frac{1}{2}(a_{-n} + ib_{-n})$ as $n = -1, -2, \dots$ which could be written as

$$c_n = \frac{1}{2l}\int_J f(x)e^{-\frac{i\pi nx}{l}}\,dx \qquad n = \dots, -2, -1, 0, 1, 2, \dots. \tag{4.5.19}$$

Parseval's equality becomes

$$2l\sum_{n=-\infty}^{\infty} |c_n|^2 = \int_J |f(x)|^2\,dx. \tag{4.5.20}$$

One can see easily that the system

$$\left\{X_n := e^{\frac{i\pi nx}{l}} \quad \dots, -2, -1, 0, 1, 2, \dots\right\} \tag{4.5.21}$$

on interval $J := [x_0, x_1]$ with $(x_1 - x_0) = 2l$ is orthogonal:

$$\int_J X_n(x)\bar{X}_m(x)\,dx = 2l\delta_{mn}. \tag{4.5.22}$$

Remark 4.5.1. All our formulae are due to Section 4.3 but we need the completeness of the systems and those are due to compleness of the system (4.5.1) established in 4.4.

Remark 4.5.2. Recall that with periodic boundary conditions all eigenvalues $(\frac{\pi n}{l})^2$ are of multiplicity 2 i.e. the corresponding *eigenspace* (consisting of all eigenfunctions with the given eigenvalue) has dimension 2 and $\{\cos(\frac{\pi nx}{l}), \sin(\frac{\pi nx}{l})\}$ and $\{e^{\frac{i\pi nx}{l}}, e^{-\frac{\pi nx}{l}}\}$ are just two different orthogonal basises in this eigenspace.

Remark 4.5.3. Fourier series in the complex form are from $-\infty$ to ∞ and this means that both sums $\sum_{n=0}^{\pm\infty} c_n e^{\frac{i\pi nx}{l}}$ must converge which is a stronger requirement than convergence of Fourier series in the trigonometric form. For piecewise differentiable function f Fourier series in the complex form converges at points where f is continuous but not at jump points where such series converges *only* in the sense of principal value:

$$\lim_{N\to+\infty} \sum_{n=-N}^{n=N} c_n e^{\frac{i\pi nx}{l}} = \frac{1}{2}\big(f(x+0) + f(x-0)\big). \tag{4.5.23}$$

4.5.6 Miscellaneous

We consider in appendices

(a) Multidimensional Fourier series

(b) Harmonic oscillator and Hermite functions

4.5.7 Problems to Sections 4.3, 4.4, 4.5

Some of the problems in this assignment could be solved based on the other problems and such solutions are much shorter than from the scratch; seeing and exploiting connections is a plus.

Here $\mathbb{N} = \{1, 2, 3, \dots\}$,

Problem 4.5.1. Decompose into full Fourier series on interval $[-l, l]$:

(a) e^{zx} where $z \in \mathbb{C}$; find "exceptional" values of z;

(b) $\cos(\omega x)$, $\sin(\omega x)$ where $0 < \omega \in \mathbb{R}$; fins "exceptional" values of ω;

(c) $\cosh(\eta x)$, $\sinh(\eta x)$ where $0 < \eta \in \mathbb{R}$;

Problem 4.5.2. Decompose into full Fourier series on interval $[-l, l]$ and sketch the graph of the sum of such Fourier series:

(a) x;

(b) $|x|$;

(c) x^2.

(d) For problem (b) with $l = 5$ plot 4 first partial sums like on the figure in the end of Section 4.4

Problem 4.5.3. Decompose into full Fourier series on interval $[-\pi, \pi]$ and sketch the graph of the sum of such Fourier series:

(a) $|\sin(x)|$;

(b) $|\cos(x)|$.

Problem 4.5.4. Decompose into sin Fourier series on interval $[0, \pi]$ and sketch the graph of the sum of such Fourier series:

(a) 1;

(b) x;

(c) $x(\pi - x)$;

(d) $\sin(mx)$ with $m \in \mathbb{N}$;

(e) $\cos(mx)$ with $m \in \mathbb{N}$;

(f) $\sin((m - \frac{1}{2})x)$ with $m \in \mathbb{N}$.

Problem 4.5.5. Decompose into cos Fourier series on interval $[0, \pi]$ and sketch the graph of the sum of such Fourier series:

(a) 1;

(b) x;

(c) $x(\pi - x)$;

(d) $\sin(mx)$ with $m \in \mathbb{N}$;

(e) $\cos(mx)$ with $m \in \mathbb{N}$;

(f) $\sin((m - \frac{1}{2})x)$ with $m \in \mathbb{N}$.

Problem 4.5.6. Decompose into Fourier series with respect to $\sin((n + \frac{1}{2})x)$ $(n = 0, 1, \ldots)$ on interval $[0, 2\pi]$ and sketch the graph of the sum of such Fourier series:

(a) 1;

(b) x;

(c) $x(\pi - x)$;

(d) $\sin(mx)$ with $m \in \mathbb{N}$;

(e) $\cos(mx)$ with $m \in \mathbb{N}$;

(f) $\sin((m - \frac{1}{2})x)$ with $m \in \mathbb{N}$.

4.A Calculation of negative eigenvalues in Robin problem

4.A.1 Variational approach

Analyzing Example hrefexample-4.2.6 and Example 4.2.7. We claim that

Theorem 4.A.1. *Eigenvalues are monotone functions of α, β.*

To prove it we need without proof to accept variational description of eigenvalues of self-adjoint operators bounded from below (very general theory) which in this case reads as:

Theorem 4.A.2. *Consider quadratic forms*

$$Q(u) = \int_0^l |u'|^2\,dx + \alpha|u(0)|^2 + \beta|u(l)|^2 \tag{4.A.1}$$

and

$$P(u) = \int_0^l |u|^2\,dx. \tag{4.A.2}$$

Then there are at least N eigenvalues which are less than λ if and only iff there is a subspace K of dimension N on which quadratic form

$$Q_\lambda(u) = Q(u) - \lambda P(u) \tag{4.A.3}$$

is negative definite *(i.e. $Q_\lambda(u) < 0$ for all $u \in \mathsf{K}$, $u \neq 0$).*

Note that $Q(u)$ is montone non-decreasing function of α, β. Therefore $N(\lambda)$ (the exact number of e.v. which are less than λ) is montone non-increasing function of α, β and therefore λ_N is montone non-decreasing function of α, β.

4.A.2 Case $\alpha = \beta$

The easiest way to deal with it would be to note that the hyperbola $\alpha + \beta + \alpha\beta l = 0$ has two branches and divides plane into 3 regions and due to continuity of eigenvalue in each of them the number of negative eigenvalues is the same.

Consider $\beta = \alpha$, it transects all three regions. Shift coordinate x to the center of interval, which becomes $[-L, L]$, $L = l/2$. Now problem becomes

$$X'' + \lambda X = 0, \tag{4.A.4}$$
$$X'(-L) = \alpha X(-L), \tag{4.A.5}$$
$$X'(L) = -\alpha X(L) \tag{4.A.6}$$

and therefore if X is an eigenfunction, then $Y(x) := X(-x)$ is an eigenfunction with the same eigenvalue.

Therefore we can consider separately eigenfunctions which are even functions, and which are odd functions–and those are described respectively by

$$X'' + \lambda X = 0, \tag{4.A.7}$$
$$X'(0) = 0, \tag{4.A.8}$$
$$X'(L) = -\alpha X(L) \tag{4.A.9}$$

and

$$X'' + \lambda X = 0, \tag{4.A.10}$$
$$X(0) = 0, \tag{4.A.11}$$
$$X'(L) = -\alpha X(L). \tag{4.A.12}$$

Since we are looking at $\lambda = -\gamma^2$ ($\gamma > 0$, we look at $X = \cosh(x\gamma)$ and $X = \sinh(X\gamma)$ respectively (see conditions (4.A.8), (4.A.11)) and then conditions (4.A.9), (4.A.12) tell us that

$$\alpha L = -(L\gamma)\tanh(L\gamma), \tag{4.A.13}$$
$$\alpha L = -(L\gamma)\coth(L\gamma) \tag{4.A.14}$$

respectively.

Both functions $z\tanh(z)$ and $z\coth(z)$ are monotone increasing for $z > 0$ with minima at $z = 0$ equal 0 and 1 respectively. Thus equation (4.A.13) has a single solution γ iff $\alpha < 0$ and (4.A.14) has a single solution γ iff $\alpha L < -1$.

Therefore as $\alpha l < 0$ there is one negative eigenvalue with an even eigenfunction and as $2\alpha l + (\alpha l)^2 < 0$ comes another negative eigenvalue with an odd eigenfunction.

Sure, one can apply a variational arguments above but analysis above has its own merit (mainly learning).

4.B Multidimensional Fourier series

4.B.1 2π-periodic case

Let function $u(\mathbf{x})$, $\mathbf{x} = (x_1, x_2, \dots, x_n)$ be 2π-periodic with respect to each variable $x_1, x_2, \dots, x_n$. Then

$$u(\mathbf{x}) = \sum_{\mathbf{m}\in\mathbb{Z}^n} c_{\mathbf{m}} e^{i\mathbf{m}\cdot\mathbf{x}} \tag{4.B.1}$$

with

$$c_{\mathbf{m}} = (2\pi)^{-n} \iiint_{\Omega} e^{-i\mathbf{m}\cdot\mathbf{x}} u(\mathbf{x})\, d^n x \tag{4.B.2}$$

and

$$\sum_{\mathbf{m}\in\mathbb{Z}^n} |c_{\mathbf{m}}|^2 = (2\pi)^{-n} \iiint_{\Omega} |u(\mathbf{x})|^2\, d^n x \tag{4.B.3}$$

where $\Omega = (0,1)^n$ is n-dimensional unit cube. Here and below we write n-dimensional integral as $\iiint$.

We need slightly generalize these formulae.

4.B.2 General case

Definition 4.B.1. Let Γ be *n-dimentional lattice*. It means that there are n linearly independent vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ and

$$\Gamma = \{(k_1\mathbf{e}_1 + k_2\mathbf{e}_2 + \dots + k_n\mathbf{e}_n : k_1, k_2, \dots, k_n \in \mathbb{Z}\} \tag{4.B.4}$$

Remark 4.B.1. Let Γ be The same lattice Γ is defined by vectors $\mathbf{e}'_1, \dots, \mathbf{e}'_n$ with $\mathbf{e}'_j = \sum_k \alpha_{jk}\mathbf{e}_k$ with integer coefficients if and only if the determinant of the matrix $(\alpha_{jk})_{j,k=1,\dots,n}$ of coefficients is equal ± 1.

Definition 4.B.2. Let Γ be Let Γ be *n-dimentional lattice*. We call $u(\mathbf{x})$ *periodic with respect to* Γ or simply Γ-*periodic* if

$$u(\mathbf{x}+\mathbf{y}) = u(\mathbf{x}) \qquad \forall \mathbf{y} \in \Gamma\ \forall \mathbf{x}. \tag{4.B.5}$$

In the previous Subsection $\Gamma = (2\pi\mathbb{Z})^n$. Let us change coordinate system so that Γ becomes $(2\pi\mathbb{Z})^n$, apply (4.B.1)–(4.B.3) and then change coordinate system back. We get

$$u(\mathbf{x}) = \sum_{\mathbf{m}\in\Gamma^*} c_{\mathbf{m}} e^{i\mathbf{m}\cdot\mathbf{x}} \tag{4.B.6}$$

with

$$c_{\mathbf{m}} = |\Omega|^{-1} \iiint_\Omega e^{-i\mathbf{m}\cdot\mathbf{x}} u(\mathbf{x})\, d^n x \tag{4.B.7}$$

and

$$\sum_{\mathbf{m}\in\Gamma^*} |c_{\mathbf{m}}|^2 = |\Omega|^{-1} \iiint_\Omega |u(\mathbf{x})|^2\, d^n x \tag{4.B.8}$$

where $|\Omega|$ is a volume of Ω and

Definition 4.B.3. (a) $\Omega = \{x_1\mathbf{e}_1 + \dots + x_n\mathbf{e}_n : 0 < x_1 < 1, \dots, 0 < x_n < 1\}$ is *an elementary cell*;

(b) $\Gamma^* = \{\mathbf{m} : \mathbf{m}\cdot\mathbf{y} \in 2\pi\mathbb{Z} \ \ \forall \mathbf{y} \in \Gamma\}$ is *a dual lattice*; it could be defined by vectors $\mathbf{e}_1^*, \dots, \mathbf{e}_n^*$ such that

$$\mathbf{e}_j^* \cdot \mathbf{e}_k = 2\pi\delta_{jk} \quad \forall j,k = 1,\dots,n \tag{4.B.9}$$

where δ_{jk} is a Kronecker symbol;

(c) $\Omega^* = \{k_1\mathbf{e}_1^* + \dots + k_n\mathbf{e}_n^* : 0 < k_1 < 1, \dots, 0 < k_n < 1\}$ is *a dual elementary cell*.

Remark 4.B.2. We prefer to use original coordinate system rather than one with coordinate vectors $(2\pi)^{-1}\mathbf{e}_1, \dots, (2\pi)^{-1}\mathbf{e}_n$ because the latter is not necessarily orthonormal and in it Laplacian will have a different form.

4.B.3 Special decomposition

These notions are important for studying the *band spectrum* of the Schrdinger operator $-\Delta + V(\mathbf{x})$ with periodic (with respect to some lattice Γ) potential in the whole space which has applications to the Physics of crystals. For this the following decomposition is used for functions $u(\mathbf{x})$ in the whole space $\mathbb{R}^n$

Theorem 4.B.1. *Let $u(\mathbf{x})$ be sufficiently fast decaying function on $\mathbb{R}^n$. Then*

$$u(\mathbf{x}) = \iiint_{\Omega^*} u(\mathbf{k};\mathbf{x})\, d^n\mathbf{k} \tag{4.B.10}$$

with

$$u(\mathbf{k};\mathbf{x}) = (2\pi)^{-n}|\Omega| \sum_{\mathbf{l}\in\Gamma} e^{-i\mathbf{k}\cdot\mathbf{l}} u(\mathbf{x}+\mathbf{l}). \tag{4.B.11}$$

Here $u(\mathbf{k};\mathbf{x})$ is quasiperiodic with quasimomentum $\mathbf{k}$

$$u(\mathbf{k};\mathbf{x}+\mathbf{y}) = e^{i\mathbf{k}\cdot\mathbf{y}} u(\mathbf{k};\mathbf{x}) \qquad \forall \mathbf{y} \in \Gamma \ \forall \mathbf{x}. \tag{4.B.12}$$

Proof. Observe that since u is sufficiently fast decaying series in (4.B.11) converges and one can see easily that it defines quasiperiodic with quasimomentum $\mathbf{k}$ function.

The proof of (4.B.10)is trivial as $\iiint_{\Omega^*} e^{-i\mathbf{k}\cdot\mathbf{l}}\, d^n\mathbf{k} = |\Omega^*|$ as $\mathbf{l} = 0$ and 0 as $0 \neq \mathbf{l} \in \Gamma$, and $|\Omega^*| = (2\pi)^n|\Omega|^{-1}$. □

4.C Harmonic Oscillator

Definition 4.C.1. *Quantum harmonic oscillator* is an operator on $\mathbb{R}$

$$L := -\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2 \tag{4.C.1}$$

It is defined in the space $L^2(\mathbb{R})$ of square integrable functions on $\mathbb{R}$.

Remark 4.C.1. Operator

$$L_{\alpha\beta} := -\frac{\alpha^2}{2}\partial_x^2 + \frac{\beta^2}{2}x^2 \tag{4.C.2}$$

can be reduced to (4.C.1) by change of variables $x := x\gamma$ with $\gamma = \sqrt{\beta/\alpha}$ and division by $\sqrt{\alpha\beta}$.

Observe that

$$L = \frac{1}{2}Z^*Z + \frac{1}{2} = \frac{1}{2}ZZ^* - \frac{1}{2} \tag{4.C.3}$$

with

$$Z := \partial_x + x, \qquad Z^* = -\partial_x + x \tag{4.C.4}$$

and Z^* is adjoint to Z: $(Zu, v) = (u, Z^*v)$.

Note that (4.C.3) implies that the lowest eigenvalue of L is $\frac{1}{2}$ with eigenfunction which is "annihilated" by Z, i.e. $u_0(x) := e^{-\frac{x^2}{2}}$.

To find other eigenvalues and eigenfunctions observe that $[Z^*, Z] = -2$ and therefore

$$LZ^* = Z^*(L+1), \qquad LZ = Z(L-1). \tag{4.C.5}$$

The first equality implies that if u is an eigenfunction with an eigenvalue λ, then Z^*u is an eigenfunction with an eigenvalue $\lambda + 1$; therefore we have a sequence of eigenvalues $\lambda_n = (n + \frac{1}{2})$, $n = 0, 1, 2, \dots$ and eigenfunctions u_n defined

$$u_n = Z^*u_{n-1}, \qquad n = 1, 2, \dots. \tag{4.C.6}$$

Theorem 4.C.1. *(a) There are no other than above eigenvalues and eigenfunctions;*

(b) $u_n(x) = H_n(x)e^{-\frac{x^2}{2}}$ where $H_n(x)$ is a polynomial of degree n (and it is even/odd for even/odd n);

(c) All $u_n(x)$ are orthogonal; $\|u_n\| = \sqrt{\pi n!}$.

(d) System $\{u_n\}$ is complete.

Proof. (a) The second of equalities (4.C.5) implies that if u is an eigenfunction with an eigenvalue λ, then Zu is an eigenfunction with an eigenvalue $\lambda - 1$; however since eigenvalues start from $\frac{1}{2}$ there are no eigenvalues in $(\frac{1}{2}, \frac{3}{2})$; so the next eigenvalue is $\frac{3}{2}$ and if u is a corresponding eigenfunction then $Zu = cu_0$. But then $Z^*Zu = cZ^*u_0$; but $Z^*Zu = (L - \frac{1}{2})u = u$ and $u = cZ^*u_0 = cu_1$. Continuing these arguments we conclude that there are no eigenvalues in $(\frac{3}{2}, \frac{5}{2})$; so the next eigenvalue is $\frac{5}{2}$ and $u = c_2$ and so on.

(b) By induction;

(c) Due to $L^* = L$ functions are orthogonal; on the other hand

$$\|u_n\|^2 = \|Z^*u\|^2 = (Z^*u_{n-1}, Z^*u_{n-1}) = (ZZ^*u, u) = ((L+\frac{1}{2})u_{n-1}, u_{n-1}) = (\lambda + \frac{1}{2})\|u_{n-1}\|^2 = n\|u\|^2$$

and by induction it is equal to $n!\|u_0\|^2 = n!\pi$.

Here we used the fact that $\|u_0\|^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \pi$. □

Definition 4.C.2. Functions u_n are *Hermite functions*, $H_n(x)$ are *Hermite polynomials*.

One can prove

$$H_n(x) = n! \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^m}{m!(n-2m)!}(2x)^{n-2m}. \tag{4.C.7}$$

Then

$$
\begin{aligned}
H_0(x) &= 1,\\
H_1(x) &= 2x,\\
H_2(x) &= 4x^2 - 2,\\
H_3(x) &= 8x^3 - 12x,\\
H_4(x) &= 16x^4 - 48x^2 + 12,\\
H_5(x) &= 32x^5 - 160x^3 + 120x,\\
H_6(x) &= 64x^6 - 480x^4 + 720x^2 - 120,\\
H_7(x) &= 128x^7 - 1344x^5 + 3360x^3 - 1680x,\\
H_8(x) &= 256x^8 - 3584x^6 + 13440x^4 - 13440x^2 + 1680,\\
H_9(x) &= 512x^9 - 9216x^7 + 48384x^5 - 80640x^3 + 30240x,\\
H_{10}(x) &= 1024x^{10} - 23040x^8 + 161280x^6 - 403200x^4 + 302400x^2 - 30240
\end{aligned}
$$

Remark 4.C.2. In the toy-model of QFT (Quantum Field Theory) u_n is considered as n-particle state, in particular u_0 is a vacuum state; operators $a = \frac{1}{\sqrt{2}}Z$ and $a^+ = \frac{1}{\sqrt{2}}Z^*$ are operators of annihilation and creation respectively, $N = a^+a = L - \frac{1}{2}$ is an operator of number of the particles (actually, it is true only for bosons).

4.C.0.1 References

http://en.wikipedia.org/wiki/Hermite_polynomials http://mathworld.wolfram.com/HermitePolynomial.html

See plots for Hermite polynomial and Hermite functions. Observe that $H_n(x)$ changes sign exactly n-times.

Chapter 5

Fourier transform

In this Chapter we consider Fourier transform which is the most useful of all integral transforms.

5.1 Fourier transform, Fourier integral

5.1.1 Heuristics

In Section 4.5 we wrote Fourier series in the complex form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{i\pi nx}{l}} \tag{5.1.1}$$

with

$$c_n = \frac{1}{2l} \int_{-l}^{l} f(x) e^{-\frac{i\pi nx}{l}}\, dx \qquad n = \ldots, -2, -1, 0, 1, 2, \ldots \tag{5.1.2}$$

and

$$2l \sum_{n=-\infty}^{\infty} |c_n|^2 = \int_{-l}^{l} |f(x)|^2\, dx. \tag{5.1.3}$$

From this form we *formally* without any justification deduct Fourier integral.

First we introduce

$$k_n := \frac{\pi n}{l} \qquad \text{and} \qquad \Delta k_n = k_n - k_{n-1} = \frac{\pi}{l} \tag{5.1.4}$$

and rewrite (5.1.1) as

$$f(x) = \sum_{n=-\infty}^{\infty} C(k_n) e^{ik_n x} \Delta k_n \tag{5.1.5}$$

with

$$C(k) = \frac{1}{2\pi} \int_{-l}^{l} f(x) e^{-ikx}\, dx \tag{5.1.6}$$

where we used $C(k_n) := c_n/(\Delta k_n)$; (5.1.3) should be rewritten as

$$\int_{-l}^{l} |f(x)|^2\, dx = 2\pi \sum_{n=-\infty}^{\infty} |C(k_n)|^2 \Delta k_n. \tag{5.1.7}$$

Now we *formally* set $l \to +\infty$; then integrals from $-l$ to l in the right-hand expression of (5.1.6) and the left-hand expression of (5.1.7) become integrals from $-\infty$ to $+\infty$.

Meanwhile, $\Delta k_n \to +0$ and *Riemannian sums* in the right-hand expressions of (5.1.5) and (5.1.7) become integrals:

$$f(x) = \int_{-\infty}^{\infty} C(k) e^{ikx}\, dk \tag{5.1.8}$$

with

$$C(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx; \tag{5.1.9}$$

(5.1.3) becomes

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = 2\pi \int_{-\infty}^{\infty} |C(k)|^2\, dk. \tag{5.1.10}$$

5.1.2 Definitions and Remarks

Definition 5.1.1. Formula (5.1.9) gives us a *Fourier transform* of $f(x)$, it usually is denoted by "hat":

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx; \tag{FT}$$

sometimes it is denoted by "tilde" ($\tilde{f}$), and seldom just by a corresponding capital letter $F(k)$.

Definition 5.1.2. Expression (5.1.8) is a *Fourier integral* aka *inverse Fourier transform*:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx}\, dk \tag{FI}$$

aka

$$\check{F}(x) = \int_{-\infty}^{\infty} F(k) e^{ikx}\, dk \tag{IFT}$$

Remark 5.1.1. Formula (5.1.10) is known as *Plancherel theorem*

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = 2\pi \int_{-\infty}^{\infty} |\hat{f}(k)|^2\, dk. \tag{PT}$$

Remark 5.1.2. (a) Sometimes expoments of $\pm ikx$ is replaced by $\pm 2\pi ikx$ and factor $1/(2\pi)$ dropped.

(b) Sometimes factor $\frac{1}{\sqrt{2\pi}}$ is placed in both Fourier transform and Fourier integral:

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx; \tag{FT*}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx}\, dk \tag{FI*}$$

Then FT and IFT differ only by i replaced by $-i$ and Plancherel theorem becomes

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |\hat{f}(k)|^2\, dk. \tag{PT*}$$

In this case Fourier transform and inverse Fourier transform differ only by $-i$ instead of i (very symmetric form) and both are *unitary operators*.

Remark 5.1.3. We can consider corresponding operator $LX = -X''$ in the space $L^2(\mathbb{R})$ of the square integrable functions on $\mathbb{R}$ but e^{ikx} are no more eigenfunctions since they do not belong to this space. In advanced Real Analysis such functions often are referred as *generalized eigenfunctions*.

Remark 5.1.4. (a) For justification see Appendix 5.1.4.

(b) Pointwise convergence is discussed in Appendix 5.1.5.

(c) Multidimensional Fourier transform and Fourier integral are discussed in Appendix 5.2.5.

5.1.3 cos- and sin-Fourier transform and integral

Applying the same arguments as in Section 4.5 we can rewrite formulae (5.1.8)–(5.1.10) as

$$f(x) = \int_0^\infty \big(A(k)\cos(kx) + B(k)\sin(kx)\big)\,dk \tag{5.1.11}$$

with

$$A(k) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(kx)\,dx, \tag{5.1.12}$$

$$B(k) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(kx)\,dx, \tag{5.1.13}$$

and

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \pi\int_0^\infty \big(|A(k)|^2 + |B(k)|^2\big)\,dk. \tag{5.1.14}$$

$A(k)$ and $B(k)$ are cos- and sin- Fourier transforms and

1. $f(x)$ is even function iff $B(k) = 0$;
2. $f(x)$ is odd function iff $A(k) = 0$.

Therefore

1. Each function on $[0,\infty)$ could be decomposed into cos-Fourier integral

$$f(x) = \int_0^\infty A(k)\cos(kx)\,dk \tag{5.1.15}$$

with

$$A(k) = \frac{2}{\pi}\int_0^\infty f(x)\cos(kx)\,dx. \tag{5.1.16}$$

2. Each function on $[0,\infty)$ could be decomposed into sin-Fourier integral

$$f(x) = \int_0^\infty B(k)\sin(kx)\,dk \tag{5.1.17}$$

with

$$B(k) = \frac{2}{\pi}\int_0^\infty f(x)\sin(kx)\,dx. \tag{5.1.18}$$

5.1.4 Justification

Let $u(x)$ be smooth fast decaying function; let us decompose it as in Section 4.B (but now we are in the simpler 1-dimensional framework and $\Gamma = 2\pi\mathbb{Z}$):

$$u(x) = \int_0^1 u(k;x)\,dk \tag{5.1.19}$$

with

$$u(k;x) = \sum_{m=-\infty}^{\infty} e^{-2\pi ikm} u(x+2\pi m). \tag{5.1.20}$$

Here $u(k;x)$ is *quasiperiodic with quasimomentum* k

$$u(k;x+2\pi n) = e^{iky}u(k;x) \qquad \forall n\in\mathbb{Z}\ \forall x\in\mathbb{R}. \tag{5.1.21}$$

Then $e^{-ikx}u(k;x)$ is periodic and one can decompose it into Fourier series

$$u(k;x) = \sum_{n=-\infty}^{\infty} e^{inx}e^{ikx}c_n(k) = \sum_{n=-\infty}^{\infty} e^{i(n+k)x}c_n(k) \tag{5.1.22}$$

(where we restored $u(k;x)$ multiplying by e^{ikx}) with

$$c_n(k) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i(n+k)x}u(k;x)\,dx \tag{5.1.23}$$

and

$$2\pi \sum_{n=-\infty}^{\infty} |c_n(k)|^2 = \int_0^{2\pi} |u(k;x)|^2\,dx. \tag{5.1.24}$$

Plugging (5.1.22) into (5.1.19) we get

$$u(x) = \int_0^1 \sum_{n=-\infty}^{\infty} c_n(k)e^{i(n+k)x}\,dk = \sum_{n=-\infty}^{\infty}\int_n^{n+1} C(\omega)e^{i\omega x} = \int_{-\infty}^{\infty} C(\omega)e^{i\omega x}\,d\omega$$

where $C(\omega) := c_n(k)$ with $n = \lfloor\omega\rfloor$ and $k = \omega - \lfloor\omega\rfloor$ which are respectively integer and fractional parts of ω. So, we got decomposition of $u(x)$ into Fourier integral.

Next, plugging (5.1.20) into (5.1.23) we get

$$C(\omega) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\omega x} \sum_{m=-\infty}^{\infty} e^{-2\pi i k m} u(x+2\pi m)\,dx = \frac{1}{2\pi}\int_{2\pi m}^{2\pi(m+1)} e^{-i\omega y} u(y)\,dy = \int_{-\infty}^{2\pi} e^{-i\omega y} u(y)\,dy$$

where we set $y = x + 2\pi m$. So, we got exactly formula for Fourier transform.

Finally, (5.1.24) implies

$$2\pi \sum_{n=-\infty}^{\infty} \int_0^1 |c_n(k)|^2\,dk = \int_0^{2\pi} \Big(\int_0^1 |u(k;x)|^2\,dk\Big)\,dx$$

where the left hand expression is exactly

$$2\pi \sum_{n=-\infty}^{\infty} \int_n^{n+1} |C(\omega)|^2\,d\omega = 2\pi \int_{-\infty}^{\infty} |C(\omega)|^2\,d\omega$$

and the right hand expression is

$$\int_0^{2\pi} \Big(\int_0^1 \sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} e^{2\pi i k(l-m)} u(x+2\pi m)\bar{u}(x+2\pi l)\,dk\Big)\,dx$$

and since $\int_0^1 e^{2\pi i k(l-m)}\,dk = \delta_{lm}$ (1 as $l=m$ and 0 otherwise) it is equal to

$$\int_0^{2\pi} \sum_{m=-\infty}^{\infty} |u(x+2\pi m)|^2\,dx = \int_{-\infty}^{\infty} |u(x)|^2\,dx.$$

So, we arrive to Plancherel theorem.

5.1.5 Discussion: pointwise convergence of Fourier integrals and series

Recall Theorem 4.4.2. Let f be a piecewise continuously differentiable function. Then the Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n \cos\big(\frac{\pi n x}{l}\big) + a_n \cos\big(\frac{\pi n x}{l}\big)\Big) \tag{5.1.25}$$

converges to

(b) $\frac{1}{2}\big(f(x+0)+f(x-0)\big)$ if x is internal point and f is discontinuous at x.

Exactly the same statement holds for Fourier Integral in the real form

$$\int_0^\infty \Big(A(k)\cos(kx)+B(k)\sin(kx)\Big)\,dk \tag{5.1.26}$$

where $A(k)$ and $B(k)$ are cos-and sin-Fourier transforms.

None of them however holds for Fourier series or Fourier Integral in the complex form:

$$\sum_{n=-\infty}^{\infty} c_n e^{i\frac{\pi n x}{l}}, \tag{5.1.27}$$

$$\int_{-\infty}^{\infty} C(k)e^{ikx}\,dk. \tag{5.1.28}$$

Why and what remedy do we have? If we consider definition of the partial sum of (5.1.25) and then rewrite in the complex form and similar deal with (5.1.28) we see that in fact we should look at

$$\lim_{N\to\infty}\sum_{n=-N}^{N} c_n e^{i\frac{\pi n x}{l}}, \tag{5.1.29}$$

$$\lim_{N\to\infty}\int_{-N}^{N} C(k)e^{ikx}\,dk. \tag{5.1.30}$$

Meanwhile convergence in (5.1.27) and (5.1.28) means more than this:

$$\lim_{M,N\to\infty}\sum_{n=-M}^{N} c_n e^{i\frac{\pi n x}{l}}, \tag{5.1.31}$$

$$\lim_{M,N\to\infty}\int_{-M}^{N} C(k)e^{ikx}\,dk \tag{5.1.32}$$

where M, N tend to ∞ independently. So the remedy is simple: understand convergence as in (5.1.29), (5.1.30) rather than as in (5.1.31), (5.1.32).

For integrals such limit is called *principal value* of integral and is denoted by

$$\operatorname{pv}\int_{-\infty}^{\infty} G(k)\,dk.$$

BTW similarly is defined

$$\operatorname{pv}\int_a^b G(k)\,dk := \lim_{\varepsilon\to+0}\Big(\int_a^{c-\varepsilon} G(k)\,dk + \int_{c+\varepsilon}^b G(k)\,dk\Big)$$

if there is a singularity at $c\in(a,b)$.Often instead of vp is used original (due to Cauchy) vp (valeur principale) and some other notations.

This is more general than the *improper integrals* studied in the end of Calculus I (which in turn generalize Riemann integrals). Those who took Complex Variables encountered such notion.

5.2 Properties of Fourier transform

5.2.1 Basic properties

In the previous Section 5.1 we introduced Fourier transform and Inverse Fourier transform

$$\hat f(k) = \frac{\kappa}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx \tag{FT}$$

$$\check F(x) = \frac{1}{\kappa}\int_{-\infty}^{\infty} F(k)e^{ikx}\,dk \tag{IFT}$$

with $\kappa=1$ (but here we will be a bit more flexible):

Theorem 5.2.1. *$F=\hat f \iff f=\check F$. (Already "proved")*

Theorem 5.2.2. *(a) Fourier transform: $f\mapsto\hat f$ is a linear operator $L^2(\mathbb{R},\mathbb{C})\to L^2(\mathbb{R},\mathbb{C})$;*

(b) Inverse Fourier transform: $F\mapsto\check F$ is an inverse operator (and also a linear operator) $L^2(\mathbb{R},\mathbb{C})\to L^2(\mathbb{R},\mathbb{C})$;

(c) If $\kappa=\frac{1}{\sqrt{2\pi}}$ these operators are unitary *i.e. preserve norm and an inner product:*

$$\|f\| = \Big(\int_{\mathbb{R}} |f(x)|^2\,dx\Big)^{\frac12}, \tag{5.2.1}$$

$$(f,g) = \int_{\mathbb{R}} f(x)\bar g(x)\,dx. \tag{5.2.2}$$

Proof. Easy. Preservation of inner product follows from preservation of norm. □

Remark 5.2.1. (a) Here $L^2(\mathbb{R},\mathbb{C})$ is a space of square integrable complexample-valued functions. Accurate definition requires a measure theory (studied in the course of Real Analysis). Alternatively one can introduce this space as a closure of the set of square integrable continuous functions but it also require a certain knowledge of Real Analysis.

(b) Properties (a) and (b) are obvious and (c) is due to Plancherel's theorem.

(c) In Quantum Mechanics Fourier transform is sometimes referred as "going to p-representation" (aka momentum representation) and Inverse Fourier transform is sometimes referred as "going to q-representation" (aka coordinate representation). In this case $\pm ikx$ is replaced by $\pm i\hbar^{-1}kx$ and 2π by $2\pi\hbar$.

Theorem 5.2.3. *(a)* $g(x)=f(x-a) \implies \hat{g}(k)=e^{-ika}\hat{f}(k)$;

(b) $g(x)=f(x)e^{ibx} \implies \hat{g}(k)=\hat{f}(k-b)$;

(c) $g(x)=f'(x) \implies \hat{g}(k)=ik\hat{f}(k)$;

(d) $g(x)=xf(x) \implies \hat{g}(k)=i\hat{f}'(k)$;

(e) $g(x)=f(\lambda x) \implies \hat{g}(k)=|\lambda|^{-1}\hat{f}(\lambda^{-1}k)$;

Proof. Here for brevity we do not write that all integrals are over $\mathbb{R}$ and set $\kappa=2\pi$.

(a) $\hat{g}=\int e^{-ikx}g(x)\,dx=\int e^{-ikx}f(x-a)\,dx=\int e^{-ik(x+a)}f(x)\,dx=e^{-ika}\hat{f}(k)$. We replaced x by $(x+a)$ in the integral.

(b) $\hat{g}=\int e^{-ikx}g(x)\,dx=\int e^{-ikx}e^{ibx}f(x)\,dx=\int e^{-i(k-b)x}f(x)\,dx=\hat{f}(k-b)$.

(c) $\hat{g}=\int e^{-ikx}g(x)\,dx=\int e^{-ikx}f'(x)\,dx \overset{\text{by parts}}{=} \int \bigl(e^{-ikx}\bigr)'f(x)\,dx=ik\hat{f}(k)$.

(d) $\hat{g}=\int e^{-ikx}g(x)\,dx=\int e^{-ikx}xf(x)\,dx=\int i\partial_k\bigl(e^{-ikx}\bigr)\,f(x)\,dx=i\hat{f}'(k)$.

(e) $\hat{g}=\int e^{-ikx}g(x)\,dx=\int e^{-ikx}f(\lambda x)\,dx=\int e^{-ik|\lambda|^{-1}x}f(x)\,\lambda^{-1}dx=\lambda^{-1}\hat{f}(\lambda^{-1}k)$. Here we replaced x by $\lambda^{-1}x$ in the integral and $|\lambda|^{-1}$ is an absolute value of Jacobian.

□

Corollary 5.2.1. *f is even (odd) iff $\hat{f}$ is even (odd).*

5.2.2 Convolution

Definition 5.2.1. *Convolution* of functions f and g is a function $f * g$:

$$(f * g)(x) := \int f(x-y)g(y)\,dy. \tag{5.2.3}$$

Theorem 5.2.4. *(a) $h = f * g \implies \hat{h}(k) = \frac{2\pi}{\kappa}\hat{f}(k)\hat{g}(k)$;*

(b) $h(x) = f(x)g(x) \implies \hat{h} = \kappa\hat{f} * \hat{g}$;

Proof. (a)

$$\hat{h}(x) = \frac{\kappa}{2\pi}\int e^{-ixk}h(x)\,dx = \frac{\kappa}{2\pi}\iint e^{-ixk}f(x-y)g(y)\,dxdy;$$

replacing in the integral $x := y + z$ we arrive to

$$\frac{\kappa}{2\pi}\iint e^{-i(y+z)k}f(z)g(y)\,dzdy = \frac{\kappa}{2\pi}\int e^{-izk}f(z)\,dz \times \int e^{-iyk}g(y)\,dz$$

which is equal to $\frac{2\pi}{\kappa}\hat{f}(k)\hat{g}(k)$.

(b) Similarly $\hat{f} * \hat{g}$ is a Fourier transform of $\frac{\kappa_1}{2\pi}fg$ where $\kappa_1 = \frac{2\pi}{\kappa}$.

□

5.2.3 Examples

Example 5.2.1. Let $f(x) = e^{-\alpha x}$ as $x > 0$ and $f(x) = 0$ as $x < 0$. Here $\operatorname{Re}\alpha > 0$.

$$\hat{f}(k) = \int_0^\infty e^{-(\alpha+ik)x}\,dx = -(\alpha+ik)^{-1}e^{-(\alpha+ik)x}\Big|_{x=0}^{x=\infty} = (\alpha+ik)^{-1}$$

provided $\kappa = 2\pi$.

In the general case $\hat{f}(k) = \frac{\kappa}{2\pi}(\alpha+ik)^{-1}$.

Example 5.2.2. Let $f(x) = e^{-\frac{\alpha}{2}x^2}$ with $\operatorname{Re}\alpha \geq 0$. Here even for $\operatorname{Re}\alpha = 0$ F.t. exists as integrals are converging albeit not absolutely.

Note that $f' = \alpha x f$. Applying Fourier transform and Theorem 5.2.3 (c),(d) to the left, right we get $ik\hat{f} = -i\alpha\hat{f}'$; solving it we arrive to $\hat{f} = Ce^{-\frac{1}{2\alpha}k^2}$.

To find C note that $C = \hat{f}(0) = \frac{\kappa}{2\pi}\int e^{-\frac{\alpha}{2}x^2}\,dx$ and for real $\alpha > 0$ we make a change of variables $x = \alpha^{-\frac{1}{2}}z$ and arrive to $C = \frac{\kappa}{\sqrt{2\pi\alpha}}$ because $\int e^{-z^2/2}\,dz = \sqrt{2\pi}$. Therefore

$$\hat{f}(k) = \frac{\kappa}{\sqrt{2\pi\alpha}}e^{-\frac{1}{2\alpha}k^2}.$$

Knowing complex variables one can justify it for complex α with $\operatorname{Re}\alpha \geq 0$; we take a correct branch of $\sqrt{\alpha}$ (condition $\operatorname{Re}\alpha \geq 0$ prevents going around origin). In particular, $(\pm i)^{\frac{1}{2}} = e^{\pm\frac{i\pi}{4}}$ and therefore for $\alpha = \pm i\beta$ with for $\beta > 0$ we get $f = e^{\mp\frac{i}{2\beta}x^2}$ and

$$\hat{f}(k) = \frac{\kappa}{2\sqrt{\pi\beta}}(1 \mp i)e^{\pm\frac{i}{2\beta}k^2x}.$$

5.2.4 Poisson summation formula

Theorem 5.2.5. *Let $f(x)$ be a continuous function on the line (∞, ∞) which vanishes for large $|x|$. Then for any $a > 0$*

$$\sum_{n=-\infty}^{\infty} f(an) = \sum_{n=-\infty}^{\infty} \frac{2\pi}{a}\hat{f}(\frac{2\pi}{a}n). \tag{5.2.4}$$

Proof. Observe that function

$$g(x) = \sum_{n=-\infty}^{\infty} f(x + an)$$

is periodic with period a. Note that the Fourier coefficients of $g(x)$ on the interval $(-\frac{a}{2}, \frac{a}{2})$ are $b_m = \frac{2\pi}{a}\hat{f}(\frac{2\pi}{a})$, where $\hat{f}(k)$ is the Fourier transform of $f(x)$.

Finally, in the Fourier series of $g(x)$ on $(-\frac{a}{2}, \frac{a}{2})$ plug $x = 0$ to obtain $g(0) = \sum_m b_m$ which coincides with (5.2.4). □

5.2.5 Multidimensional Fourier transform, Fourier integral

Definition 5.2.2. Multidimensional Fourier transform is defined as

$$\hat{f}(\mathbf{k}) = \Big(\frac{\kappa}{2\pi}\Big)^n \iiint_{\mathbb{R}^n} f(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}}\, d^n x \tag{FT}$$

$$\check{F}(x) = \left(\frac{1}{\kappa}\right)^n \iiint_{\mathbf{R}^n} F(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}\, d^n \mathbf{k} \tag{IFT}$$

with $\kappa = 1$ (but here we will be a bit more flexible).

All the main properties of 1-dimensional Fourier transform are preserved (with obvious modifications) but some less obvious modifications are mentioned:

Remark 5.2.2. Theorem 5.2.3(e) is replaced by

$$g(\mathbf{x}) = f(Q\mathbf{x}) \implies \hat{g}(\mathbf{k}) = |\det Q|^{-1} \hat{f}(Q^{*-1}\mathbf{k}) \tag{5.2.5}$$

where Q is a non-degenerate linear transformation.

Remark 5.2.3. Example 5.2.2 is replaced by the following: Let $f(x) = e^{-\frac{1}{2}A\mathbf{x}\cdot\mathbf{x}}$ where A is a symmetric (but not necessarily real matrix) $A^T = A$ with positive definite real part:

$$\operatorname{Re}(A\mathbf{x}\cdot\mathbf{x}) \geq \epsilon |\mathbf{x}|^2 \qquad \forall \mathbf{x}$$

with $\varepsilon > 0$. One can prove that inverse matrix A^{-1} has the same property and

$$\hat{f}(\mathbf{k}) = \left(\frac{\kappa}{\sqrt{2\pi}}\right)^n |\det A|^{-\frac{1}{2}} e^{-\frac{1}{2}A^{-1}\mathbf{k}\cdot\mathbf{k}}.$$

Remark 5.2.4. Poisson summation formula (Theorem 5.2.5) is replaced by

$$\sum_{\mathbf{m}\in\Gamma} f(\mathbf{m}) = \sum_{\mathbf{k}\in\Gamma^*} (2\pi)^n |\Omega|^{-1} \hat{f}(\mathbf{k}). \tag{5.2.6}$$

(in notations of Section 4.B).

5.2.6 Problems to Sections 5.1, 5.2

Some of the problems could be solved based on the other problems and properties of Fourier transform (see Section 5.2) and such solutions are much shorter than from the scratch; seeing and exploiting connections is a plus.

Problem 5.2.1. Let F be an operator of Fourier transform: $f(x) \to \hat{f}(k)$. Prove that

(a) $F^*F = FF^* = I$;

(b) $(F^2 f)(x) = f(-x)$ and therefore $F^2 f = f$ for even function f and $F^2 = -f$ for odd function f;

(c) $F^4 = I$;

(d) If f is a real-valued function then $\hat{f}$ is real-valued if and only if f is even and $i\hat{f}$ is real-valued if and only if f is odd.

Problem 5.2.2. Let $\alpha > 0$. Find Fourier transforms of

(a) $e^{-\alpha|x|}$;

(b) $e^{-\alpha|x|}\cos(\beta x)$, $e^{-\alpha|x|}\sin(\beta x)$ with $\beta > 0$;

(c) $xe^{-\alpha|x|}$ with $\beta > 0$;

(d) $xe^{-\alpha|x|}\cos(\beta x)$, $xe^{-\alpha|x|}\sin(\beta x)$ with $\beta > 0$.

Problem 5.2.3. Let $\alpha > 0$. Find Fourier transforms of

(a) $(x^2 + \alpha^2)^{-1}$;

(b) $x(x^2 + \alpha^2)^{-1}$;

(c) $(x^2 + \alpha^2)^{-1}\cos(\beta x)$, $(x^2 + \alpha^2)^{-1}\sin(\beta x)$;

(d) $x(x^2 + \alpha^2)^{-1}\cos(\beta x)$, $x(x^2 + \alpha^2)^{-1}\sin(\beta x)$.

Problem 5.2.4. Let $\alpha > 0$. Based on Fourier transform of $e^{-\alpha x^2/2}$ find Fourier transforms of

(a) $e^{-\alpha x^2/2}\cos(\beta x)$, $e^{-\alpha x^2/2}\sin(\beta x)$;

(b) $xe^{-\alpha x^2/2}\cos(\beta x)$, $xe^{-\alpha x^2/2}\sin(\beta x)$.

Problem 5.2.5. Find Fourier transforms of

(a) $f(x) = \begin{cases} 1 & |x| \le a, \\ 0 & |x| > a; \end{cases}$

(b) $f(x) = \begin{cases} x & |x| \le a, \\ 0 & |x| > a; \end{cases}$

(c) Using (a) calculate $\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\, dx$.

Problem 5.2.6. (a) Prove the same properties as in 5.2.1 for multidimensional Fourier tramsform (see Subsection 5.2.5.

(b) Prove that f if multidimensional function f has a rotational symmetry (that means $f(Q\mathbf{x}) = f(\mathbf{x})$ for all orthogonal transform Q) then $\hat{f}$ also has a rotational symmetry (and conversely).

Note. Equivalently f has a rotational symmetry if $f(\mathbf{x})$ depend only on $|\mathbf{x}|$.

Problem 5.2.7. Find multidimenxional Fourier transforms of

(a) $f(x) = \begin{cases} 1 & |\mathbf{x}| \le a, \\ 0 & |\mathbf{x}| > a; \end{cases}$

(b) $f(x) = \begin{cases} a - |\mathbf{x}| & |\mathbf{x}| \le a, \\ 0 & |\mathbf{x}| > a, \end{cases};$

(c) $f(x) = \begin{cases} a^2 - |\mathbf{x}|^2 & |\mathbf{x}| \le a, \\ 0 & |\mathbf{x}| > a, \end{cases};$

(d) $f(x) = e^{-\alpha|\mathbf{x}|}$.

Hint. Using Problem 5.2.6(b) observe that we need to find only $\hat{f}(0, \dots, 0, k)$ and use appropriate coordinate system (polar as $n = 2$, or spherical as $n = 3$ and so one).

Note. This problem could be solved as $n = 2$, $n = 3$ or $n \ge 2$ (any).

5.3 Applications of Fourier transform to PDEs

In the previous Section 5.1 and Section 5.2 we introduced Fourier transform and Inverse Fourier transform and established some of its properties; we also calculated some Fourier transforms. Now we going to apply to PDEs.

5.3.1 Heat equation

Consider problem

$$u_t = ku_{xx}, \qquad t>0,\ -\infty<x<\infty, \tag{5.3.1}$$

$$u|_{t=0} = g(x). \tag{5.3.2}$$

Making partial Fourier transform with respect to $x \mapsto \xi$ (so $u(x,t) \mapsto \hat{u}(\xi,t)$) we arrive to

$$\hat{u}_t = -k\xi^2\hat{u}, \tag{5.3.3}$$

$$\hat{u}|_{t=0} = \hat{g}(\xi). \tag{5.3.4}$$

Indeed, $\partial_x \mapsto i\xi$ and therefore $\partial_x^2 \mapsto -\xi^2$.

Note that (5.3.3) is an ODE and solving it we arrive to $\hat{u} = A(\xi)e^{-k\xi^2 t}$; plugging into (5.3.4) we find that $A(\xi) = \hat{g}(\xi)$ and therefore

$$\hat{u}(\xi,t) = \hat{g}(\xi)e^{-k\xi^2 t}. \tag{5.3.5}$$

The right-hand expression is a product of two Fourier transforms, one is $\hat{g}(\xi)$ and another is Fourier transform of IFT of $e^{-k\xi^2 t}$ (reverse engineering?).

If we had $e^{-\xi^2/2}$ we would have IFT equal to $\sqrt{2\pi}e^{-x^2/2}$; but we can get from $e^{-\xi^2/2}$ to $e^{-k\xi^2 t}$ by scaling $\xi \mapsto (2kt)\{\frac{1}{2}\}\xi$ and therefore $x \mapsto (2kt)^{-\frac{1}{2}}x$ (and we need to multiply the result by by $(2kt)^{-\frac{1}{2}}$); therefore $e^{-k\xi^2 t}$ is a Fourier transform of $\frac{\sqrt{2\pi}}{\sqrt{2kt}}e^{-x^2/4kt}$.

Again: $\hat{u}(\xi,t)$ is a product of FT of g and of $\frac{\sqrt{2\pi}}{\sqrt{2kt}}e^{-x^2/4kt}$ and therefore u is the convolution of these functions (multiplied by $1/(2\pi)$):

$$u(x,t) = g * \frac{1}{\sqrt{4\pi kt}}e^{-\frac{x^2}{4kt}} = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} g(x')e^{-\frac{(x-x')^2}{4kt}}\,dx'. \tag{5.3.6}$$

We recovered formula which we had already.

Remark 5.3.1. Formula (5.3.5) shows that the problem is really ill-posed for $t<0$.

5.3.2 Schrödinger equation

Consider problem

$$u_t = iku_{xx}, \qquad t > 0,\ -\infty < x < \infty, \tag{5.3.7}$$

$$u|_{t=0} = g(x). \tag{5.3.8}$$

Making partial Fourier transform with respect to $x \mapsto \xi$ (so $u(x,t) \mapsto \hat{u}(\xi,t)$) we arrive to

$$\hat{u}_t = -ik\xi^2\hat{u}, \tag{5.3.9}$$

$$\hat{u}|_{t=0} = \hat{g}(\xi). \tag{5.3.10}$$

Note that (5.3.9) is an ODE and solving it we arrive to $\hat{u} = A(\xi)e^{-ik\xi^2 t}$; plugging into (5.3.10) we find that $A(\xi) = \hat{g}(\xi)$ and therefore

$$\hat{u}(\xi,t) = \hat{g}(\xi)e^{-ik\xi^2 t}. \tag{5.3.11}$$

The right-hand expression is a product of two Fourier transforms, one is $\hat{g}(\xi)$ and another is Fourier transform of IFT of $e^{-ik\xi^2 t}$ (reverse engineering?).

As it was explained in Section 5.2 that we need just to plug $-ik$ instead of k (as $t > 0$) into the formulae we got before; so instead of $\frac{1}{\sqrt{4\pi kt}}e^{-\frac{x^2}{4kt}}$ we get $\frac{1}{\sqrt{-4\pi kit}}e^{-\frac{x^2}{-4kit}} = \frac{1}{\sqrt{4\pi kt}}e^{\frac{\pi i}{4}-\frac{ix^2}{4kt}}$ because we need to take a correct branch of $\sqrt{-i} = e^{-\frac{i\pi}{4}}$. As $t < 0$ we need to replace t by $-t$ and i by $-i$ resulting in $\frac{1}{\sqrt{4\pi k|t|}}e^{-\frac{\pi i}{4}-\frac{ix^2}{4kt}}$.

Therefore

$$u(x,t) = \frac{1}{\sqrt{4\pi k|t|}}\int_{-\infty}^{\infty} g(x')e^{\pm\frac{i\pi}{4}-\frac{i(x-x')^2}{4kt}}\,dx' \tag{5.3.12}$$

as $\pm t > 0$.

Remark 5.3.2. remark-5.3.2 Formula (5.3.11) shows that the problem is well-posed for both $t > 0$ and $t < 0$.

5.3.3 Laplace equation in half-plane

Consider problem

$$\Delta u := u_{xx} + u_{yy} = 0, \qquad y > 0,\ -\infty < x < \infty, \tag{5.3.13}$$

$$u|_{y=0} = g(x). \tag{5.3.14}$$

This problem definitely is not uniquely solvable (f.e. $u = y$ satisfies homogeneous boundary condition) and to make it uniquely solvable we need to add condition $|u| \le M$.

Making partial Fourier transform with respect to $x \mapsto \xi$ (so $u(x,t) \mapsto \hat{u}(\xi,t)$) we arrive to

$$\hat{u}_{yy} - \xi^2 \hat{u} = 0, \tag{5.3.15}$$

$$\hat{u}|_{y=0} = \hat{g}(\xi). \tag{5.3.16}$$

Note that (5.3.15) is an ODE and solving it we arrive to

$$\hat{u}(\xi, y) = A(\xi)e^{-|\xi|y} + B(\xi)e^{|\xi|y}. \tag{5.3.17}$$

Indeed, characteristic equation $\alpha^2 - \xi^2$ has two roots $\alpha_{1,2} = \pm|\xi|$; we take $\pm|\xi|$ instead of just $\pm\xi$ because we need to control signs.

We discard the second term in the right-hand expression of (5.3.17) because it is unbounded. However if we had Cauchy problem (i.e. $u|_{y=0} = g(x)$, $u_y|_{y=0} = h(x)$) we would not be able to do this and this problem will be ill-posed.

So, $\hat{u} = A(\xi)e^{-|\xi|y}$ and (5.3.16) yields $A(\xi) = \hat{g}(\xi)$:

$$\hat{u}(\xi, y) = \hat{g}(\xi)e^{-|\xi|y}. \tag{5.3.18}$$

Now we need to find the IFT of $e^{-|\xi|y}$. This calculations are easy (do them!) and IFT is $2y(x^2+y^2)^{-1}$. Then

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} g(x')\frac{y}{(x-x')^2+y^2}\,dx'. \tag{5.3.19}$$

Remark 5.3.3. Setting $y = 0$ we see that $u|_{y=0} = 0$. Contradiction?–No, we cannot just set $y = 0$. We need to find a limit as $y \to +0$, and note that $\frac{y}{(x-x')^2+y^2} \to 0$ except as $x' = x$ and $\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-x')^2+y^2}\,dx' = 1$ so the limit will be $g(x)$ as it should be.

5.3.4 Laplace equation in half-plane. II

Replace Dirichlet boundary condition by Robin boundary condition

$$\Delta u := u_{xx} + u_{yy} = 0, \qquad y > 0,\ -\infty < x < \infty, \tag{5.3.13}$$

$$(u_y - \alpha u)|_{y=0} = h(x). \tag{5.3.20}$$

Then (5.3.16) should be replaced by

$$(\hat{u}_y - \alpha\hat{u})|_{y=0} = \hat{h}(\xi). \tag{5.3.21}$$

and then

$$A(\xi) = -(|\xi| + \alpha)^{-1}\hat{h}(\xi) \tag{5.3.22}$$

and

$$\hat{u}(\xi, y) = -\hat{h}(\xi)(|\xi| + \alpha)^{-1}e^{-|\xi|y}. \tag{5.3.23}$$

The right-hand expression is a nice function provided $\alpha > 0$ (and this is correct from the physical point of view) and therefore everything is fine (but we just cannot calculate explicitely IFT of $(|\xi| + \alpha)^{-1}e^{-|\xi|y}$).

Consider Neumann boundary condition i.e. set $\alpha = 0$. Then we have a trouble: $-\hat{h}(\xi)(|\xi| + \alpha)^{-1}e^{-|\xi|y}$ could be singular at $\xi = 0$ and to avoid it we assume that $\hat{h}(0) = 0$. This means exactly that

$$\int_{-\infty}^{\infty} h(x)\,dx = 0 \tag{5.3.24}$$

and this condition we really need and it is justified from the physical point of view: f.e. if we are looking for stationary heat distribution and we have heat flow defined, we need to assume that the total flow is 0 (otherwise the will be no stationary distribution!).

So we need to calculate IFT of $-|xi|^{-1}e^{-|\xi|y}$. Note that derivative of this with respect to y is $e^{-|\xi|y}$ which has an IFT $\frac{1}{\pi}\frac{y}{x^2+y^2}$; integrating with respect to y we get $\frac{1}{2\pi}\log(x^2+y^2) + c$ and therefore

$$u(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(x')\log\big((x-x')^2 + y^2\big)\,dx' + C. \tag{5.3.25}$$

Remark 5.3.4. (a) Here C is an arbitrary constant. Again, the same physical interpretation: knowing heat flow we define solution up to a constant as the total heat energy is arbitrary.

(b) Formula (5.3.25) gives us a solution which can grow as $|x| \to \infty$ even if h is fast decaying there (or even if $h(x) = 0$ as $|x| \ge c$). However as $|x| \gg 1$ and h is fast decaying $\big((x-x')^2 + y^2\big) \approx \big(x^2 + y^2\big)$ (with a small error) and growing part of u is $\frac{1}{2\pi}\log\big(x^2+y^2\big)\int_{-\infty}^{\infty} h(x')\,dx'$ which is 0 precisely because of condition (5.3.24).

5.3.5 Laplace equation in strip

Consider problem

$$\Delta u := u_{xx} + u_{yy} = 0, \qquad 0 < y < b,\ -\infty < x < \infty, \tag{5.3.26}$$

$$u|_{y=0} = g(x), \tag{5.3.27}$$

$$u|_{y=b} = h(x). \tag{5.3.28}$$

Then we get (5.3.17) again

$$\hat{u}(\xi, y) = A(\xi)e^{-|\xi|y} + B(\xi)e^{|\xi|y} \tag{5.3.17}$$

but with two boundary condition we cannot diacard anything; we get instead

$$A(\xi) + B(\xi) = \hat{g}(\xi), \tag{5.3.29}$$

$$A(\xi)e^{-|\xi|b} + B(\xi)e^{|\xi|b} = \hat{h}(\xi) \tag{5.3.30}$$

which implies

$$A(\xi) = \frac{e^{|\xi|b}}{2\sinh(|\xi|b)}\hat{g}(\xi) - \frac{1}{2\sinh(|\xi|b)}\hat{h}(\xi),$$

$$B(\xi) = -\frac{e^{-|\xi|b}}{2\sinh(|\xi|b)}\hat{g}(\xi) + \frac{1}{2\sinh(|\xi|b)}\hat{h}(\xi)$$

and therefore

$$\hat{u}(\xi, y) = \frac{\sinh(|\xi|(b-y))}{\sinh(|\xi|b)}\hat{g}(\xi) + \frac{\sinh(|\xi|y)}{\sinh(|\xi|b)}\hat{h}(\xi). \tag{5.3.31}$$

One can see easily that $\frac{\sinh(|\xi|(b-y))}{\sinh(|\xi|b)}$ and $\frac{\sinh(|\xi|y)}{\sinh(|\xi|b)}$ are bounded as $0 \le y \le b$ and fast decaying as $|\xi| \to \infty$ as $y \ge \epsilon$ ($y \le b - \epsilon$ respectively) with arbitrarily small $\epsilon > 0$.

Exercise 5.3.1. Investigate other boundary conditions (Robin, Neumann, mixed–Dirichlet at $y = 0$ and Neumann at $y = b$ and so on.).

5.3.6 1D wave equation

Consider problem

$$u_{tt} = c^2 u_{xx}, \qquad -\infty < x < \infty, \tag{5.3.32}$$

$$u|_{t=0} = g(x), \tag{5.3.33}$$

$$u_t|_{t=0} = h(x). \tag{5.3.34}$$

Making partial Fourier transform with respect to $x \mapsto \xi$ we arrive to

$$\hat{u}_{tt} = -c^2\xi^2 \hat{u}_{xx}, \tag{5.3.35}$$

$$\hat{u}|_{t=0} = \hat{g}(\xi), \tag{5.3.36}$$

$$\hat{u}_t|_{t=0} = \hat{h}(\xi). \tag{5.3.37}$$

Then characteristic equation for ODE (5.3.35) is $\alpha^2 = -c^2\xi^2$ and $\alpha_{1,2} = \pm ic\xi$,

$$\hat{u}(\xi, t) = A(\xi)\cos(c\xi t) + B(\xi)\sin(c\xi t)$$

with initial conditions implying $A(\xi) = \hat{g}(\xi)$, $B(\xi) = 1/(ci\xi)\cdot\hat{h}(\xi)$ and

$$\hat{u}(\xi, t) = \hat{g}(\xi)\cos(c\xi t) + \hat{h}(\xi)\cdot\frac{1}{c\xi}\sin(c\xi t). \tag{5.3.38}$$

Rewriting $\cos(c\xi t) = \frac{1}{2}\bigl(e^{ic\xi t} + e^{-ic\xi t}\bigr)$ and recalling that multiplication of FT by $e^{i\xi b}$ is equivalen to to shifting original to the left by b we conclude that $\hat{g}(\xi)\cos(c\xi t)$ is a Fourier transform of $\frac{1}{2}\bigl(g(x+ct) + g(x-ct)\bigr)$.

If we denote H as a primitive of h then $\hat{h}(\xi)\cdot\frac{1}{c\xi}\sin(c\xi t) = \hat{H}(\xi)\cdot\frac{1}{c}i\sin(c\xi t)$ which in virtue of the same arguments is FT of $\frac{1}{2c}\bigl(H(x+ct) - H(x-ct)\bigr) = \frac{1}{2c}\int_{x-ct}^{x+ct} h(x')\,dx'$.

Therefore

$$u(x,t) = \frac{1}{2}\bigl(g(x+ct) + g(x-ct)\bigr) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(x')\,dx' \tag{5.3.39}$$

and we arrive again to d'Alembert formula.

5.3.7 Multidimensional equations

Multidimensional equations are treated in the same way:

5.3.7.1 Heat and Schrdinger equations

We make partial FT (with respect to spatial variables) and we get

$$\hat{u}(\xi,t) = \hat{g}(\xi)e^{-k|\xi|^2 t},$$
$$\hat{u}(\xi,t) = \hat{g}(\xi)e^{-ik|\xi|^2 t}$$

respectively where $\xi = (\xi_1,\dots,\xi_n)$, $|\xi| = (\xi_1^2+\dots\xi_n^2)^{\frac{1}{2}}$; in this case $e^{-k|\xi|^2 t} = \prod_{j=1}^n e^{-k|\xi_j|^2 t}$ is a product of functions depending on different variables, so IFT will be again such product and we have IFT equal

$$\prod_{j=1}^n \frac{1}{\sqrt{4\pi kt}} e^{-|x_j|^2/4kt} = (4\pi kt)^{-\frac{n}{2}} e^{-|x|^2/4kt}$$

for heat equation and similarly for Schrdinger equation and we get a solution as a multidimensional convolution. Here $x = (x_1,\dots,x_n)$ and $|x| = (x_1^2 + \dots + x_n^2)^{\frac{1}{2}}$.

5.3.7.2 Wave equation

We do the same but now

$$\hat{u}(\xi,t) = \hat{g}(\xi)\cos(c|\xi|t) + \hat{h}(\xi)\cdot\frac{1}{c|\xi|}\sin(c|\xi|t).$$

Finding IFT is not easy.

5.3.7.3 Laplace equation

We consider it in $\mathbb{R}^n \times I \ni (x;y)$ with either $I = \{y : y > 0\}$ or $I = \{y : 0 < y < b\}$ and again make partial FT with respect to x but not y.

5.3.8 Problems to Section 5.3

Problem 5.3.1. (a) Consider Dirichlet problem

$$u_{xx} + u_{yy} = 0, \qquad -\infty < x < \infty, y > 0, \tag{5.3.40}$$
$$u|_{y=0} = f(x). \tag{5.3.41}$$

Make Fourier transform by x, solve problem for ODE for $\hat{u}(k,y)$ which you get as a result and write $u(x,y)$ as a Fourier integral.

(b) Consider Neumann problem

$$u_{xx} + u_{yy} = 0, \qquad -\infty < x < \infty, y > 0, \tag{5.3.42}$$
$$u_y|_{y=0} = f(x). \tag{5.3.43}$$

Make Fourier transform by x, solve problem for ODE for $\hat{u}(k, y)$ which you get as a result and write $u(x, y)$ as a Fourier integral. What condition must satisfy f?

Problem 5.3.2. (a) Consider Dirichlet problem

$$u_{xx} + u_{yy} = 0, \qquad -\infty < x < \infty, 0 < y < 1, \tag{5.3.44}$$
$$u|_{y=0} = f(x), \quad u|_{y=1} = g(x). \tag{5.3.45}$$

Make Fourier transform by x, solve problem for ODE for $\hat{u}(k, y)$ which you get as a result and write $u(x, y)$ as a Fourier integral.

(b) Consider Dirichlet-Neumann problem

$$u_{xx} + u_{yy} = 0, \qquad -\infty < x < \infty, 0 < y < 1, \tag{5.3.46}$$
$$u|_{y=0} = f(x), \quad u_y|_{y=1} = g(x). \tag{5.3.47}$$

Make Fourier transform by x, solve problem for ODE for $\hat{u}(k, y)$ which you get as a result and write $u(x, y)$ as a Fourier integral.

(c) Consider Neumann problem

$$u_{xx} + u_{yy} = 0, \qquad -\infty < x < \infty, 0 < y < 1, \tag{5.3.48}$$
$$u_y|_{y=0} = f(x), \quad u_y|_{y=1} = g(x). \tag{5.3.49}$$

Make Fourier transform by x, solve problem for ODE for $\hat{u}(k, y)$ which you get as a result and write $u(x, y)$ as a Fourier integral. What condition must satisfy f, g?

Problem 5.3.3.

$$u_{xx} + u_{yy} = 0, \qquad -\infty < x < \infty, y > 0, \tag{5.3.50}$$
$$(u_y + \alpha u)|_{y=0} = f(x). \tag{5.3.51}$$

Make Fourier transform by x, solve problem for ODE for $\hat{u}(k, y)$ which you get as a result and write $u(x, y)$ as a Fourier integral. What condition (if any) must satisfy f?

Hint. Consider separately $\alpha > 0$ and $\alpha < 0$.

Problem 5.3.4. (a) Consider problem

$$\Delta^2 = 0, \qquad -\infty < x < \infty, y > 0, \tag{5.3.52}$$

$$u|_{y=0} = f(x), \qquad u_y|_{y=0} = g(x). \tag{5.3.53}$$

Make Fourier transform by x, solve problem for ODE for $\hat{u}(k, y)$ which you get as a result and write $u(x, y)$ as a Fourier integral.

(b) Consider problem

$$\Delta^2 = 0, \qquad -\infty < x < \infty, y > 0, \tag{5.3.54}$$

$$u_{yy}|_{y=0} = f(x), \qquad \Delta u_y|_{y=0} = g(x). \tag{5.3.55}$$

Make Fourier transform by x, solve problem for ODE for $\hat{u}(k, y)$ which you get as a result and write $u(x, y)$ as a Fourier integral. What condition must satisfy f, g?

Chapter 6

Separation of variables

In this Chapter we continue study separation of variables which we started in Chapter 4 but interrupted to explore Fourier series and Fourier transform.

6.1 Separation of variables for heat equation

6.1.1 Dirichlet boundary conditions

Consider problem

$$u_t = ku_{xx}, \qquad t>0,\ 0<x<l, \tag{6.1.1}$$

$$u|_{x=0} = u|_{x=l} = 0. \tag{6.1.2}$$

$$u|_{t=0} = g(x). \tag{6.1.3}$$

Let us consider a simple solution $u(x,t) = X(x)T(t)$; then separating variables we arrive to $\frac{T'}{T} = k\frac{X''}{X}$ which implies $X'' + \lambda X = 0$,

$$T' = -k\lambda T \tag{6.1.4}$$

(explain, how). We also get boundary conditions $X(0) = X(l) = 0$ (explain, how).

So, we have eigenvalues $\lambda_n = (\frac{\pi n}{l})^2$ and eigenfunctions $X_n = \sin(\frac{\pi n x}{l})$ $(n = 1, 2, \dots)$ and equation (6.1.4) for T, which results in

$$T_n = A_n e^{-k\lambda_n t} \tag{6.1.5}$$

and therefore a simple solution is

$$u_n = A_n e^{-k\lambda_n t} \sin(\frac{\pi n x}{l}) \tag{6.1.6}$$

and we look for a general solution in the form

$$u = \sum_{n=1}^{\infty} A_n e^{-k\lambda_n t} \sin(\frac{\pi n x}{l}). \tag{6.1.7}$$

Again, taking in account initial condition (6.1.3) we see that

$$u = \sum_{n=1}^{\infty} A_n \sin(\frac{\pi n x}{l}). \tag{6.1.8}$$

and therefore

$$A_n = \frac{2}{l} \int_0^l g(x) \sin(\frac{\pi n x}{l})\, dx. \tag{6.1.9}$$

6.1.2 Corollaries

(a) Formula (6.1.6) shows that the problem is really ill-posed for $t < 0$.

(b) Formula (6.1.9) shows that as $t \to +\infty$

$$u = O(e^{-k\lambda_1 t}); \tag{6.1.10}$$

(c) Moreover we have as $t \to +\infty$

$$u = A_1 e^{-k\lambda_1 t} X_1(x) e^{-k\lambda_1 t} + O(e^{-k\lambda_2 t}). \tag{6.1.11}$$

Consider now inhomogeneous problem with the right-hand expression and boundary conditions independent on t:

$$u_t = k u_{xx} + f(x), \qquad t > 0,\ 0 < x < l, \tag{6.1.12}$$

$$u|_{x=0} = \phi, \qquad u|_{x=l} = \psi, \tag{6.1.13}$$

$$u|_{t=0} = g(x). \tag{6.1.14}$$

Let us discard initial condition and find a *stationary solution* $u = v(x)$:

$$v'' = -\frac{1}{k} f(x), \qquad 0 < x < l, \tag{6.1.15}$$

$$v(0) = \phi, \qquad v(l) = \psi. \tag{6.1.16}$$

Then (6.1.15) implies

$$v(x) = -\frac{1}{k}\int_0^x \int_0^{x'} f(x'')\, dx'' dx' + A + Bx = \int_0^x (x - x') f(x')\, dx' + A + Bx$$

where we used formula of n-th integral (you must know it from the 1st year calculus)

$$I_n(x) = \frac{1}{(n-1)!}\int_a^x (x - x')^{n-1} f(x')\, dx' \qquad n = 1, 2, \dots \tag{6.1.17}$$

for $I_n := \int_a^x I_{n-1}(x')\, dx'$, $I_0(x) := f(x)$.

Then satisfying b.c. $A = \phi$ and $B = \frac{1}{l}(\psi - \phi + \frac{1}{k}\int_0^l (l - x') f(x')\, dx')$ and

$$v(x) = \int_0^x G(x, x') f(x')\, dx' + \phi(1 - \frac{x}{l}) + \psi\frac{x}{l} \tag{6.1.18}$$

with

$$G(x, x') = \frac{1}{k}\begin{cases} x'(1 - \dfrac{x}{l}) & 0 < x' < x, \\ x(1 - \dfrac{x'}{l}) & x < x' < l. \end{cases} \tag{6.1.19}$$

Returning to the original problem we note that $u - v$ satisfies (6.1.1)–(6.1.3) with $g(x)$ replaced by $g(x) - v(x)$ and therefore $u - v = O(e^{-k\lambda_1 t})$. So

$$u = v + O(e^{-k\lambda_1 t}). \tag{6.1.20}$$

In other words, solution stabilizes to the stationary solution. For more detailed analysis of BVP for ODEs see 6.A.

6.1.3 Other boundary conditions

Similar approach works in the cases of boundary conditions we considered before:

(a) Dirichlet on one and and Neumann on the other $u|_{x=0} = u_x|_{x=l} = 0$;

(b) Neumann on both ends $u_x|_{x=0} = u_x|_{x=l} = 0$;

(c) Periodic $u|_{x=l} = u|_{x=0}$, $u_x|_{x=l} = u_x|_{x=0}$;

(d) Dirichlet on one and and Robin on the other $u|_{x=0} = (u_x+\beta u)|_{x=l} = 0$;

(e) Robin on both ends $(u_x - \alpha u)|_{x=0} = (u_x + \beta u)|_{x=l} = 0$

but in (d), (e) we cannot find eigenvalues explicitly.

6.1.4 Corollaries

All corollaries remain valid as long as $\lambda_1 > 0$ which happens in cases (a), (d) with $\beta \geq 0$, (e) with $\alpha \geq 0, \beta \geq 0$ except $\alpha = \beta = 0$.

Let us consider what happens when $\lambda_1 = 0$ (cases (b) and (c)).

First, solution of the problem with r.h.e. and b.c. equal to 0 does not decay as $t \to +\infty$, instead

$$u = A_1 + O(e^{-k\lambda_2 t}) \tag{6.1.21}$$

because in (b) and (c) $X_1(x) = 1$.

Second, solution of stationary problem exists *only conditionally*: iff

$$\frac{1}{k}\int_0^l f(x)\,dx - \phi + \psi = 0 \tag{6.1.22}$$

in the case of Neumann b.c. on both ends $u_x|_{x=0} = \phi$, $u_x|_{x=l} = \psi$ and

$$\frac{1}{k}\int_0^l f(x)\,dx = 0 \tag{6.1.23}$$

in the case of periodic b.c.

To cover the case when (6.1.22) or (6.1.23) fails (i.e. total heat flow is not 0 so there is no balance) it is sufficient to consider the case $f = p$, $\phi = \psi = 0$; then $u = pt$ with

$$p = \frac{1}{l}\int_0^l f(x)\,dx \tag{6.1.24}$$

and in the general case

$$u = pt + A_1 + O(e^{-k\lambda_2 t}). \tag{6.1.25}$$

6.2 Separation of variables: Misc equations

In the previous Section 6.1 we considered heat equation.

6.2.1 Schrödinger equation

Consider problem

$$u_t = iku_{xx}, \qquad t>0,\ 0<x<l, \tag{6.2.1}$$

$$(\alpha_0 u_x - \alpha u)|_{x=0} = (\beta_0 u_x + \beta u)|_{x=l} = 0. \tag{6.2.2}$$

$$u|_{t=0} = g(x) \tag{6.2.3}$$

where *either* $\alpha_0 = 0$, $\alpha = 1$ and we have a Dirichlet boundary condition *or* $\alpha_0 = 1$ and we have either Neumann or Robin boundary condition and the same at $x = l$.

Let us consider a simple solution $u(x,t) = X(x)T(t)$; then separating variables we arrive to $\frac{T'}{T} = ik\frac{X''}{X}$ which implies $X'' + \lambda X = 0$,

$$T' = -ik\lambda T \tag{6.2.4}$$

(explain, how). We also get boundary conditions $(\alpha_0 X' - \alpha X) = (\beta_0 X' + \beta X)(l) = 0$ (explain, how).

So, we have eigenvalues λ_n and eigenfunctions X_n ($n = 1, 2, \ldots$) and equation (6.2.4) for T, which results in

$$T_n = A_n e^{-ik\lambda_n t} \tag{6.2.5}$$

and therefore a simple solution is

$$u_n = A_n e^{-ik\lambda_n t} X_n(x) \tag{6.2.6}$$

and we look for a general solution in the form

$$u = \sum_{n=1}^{\infty} A_n e^{-ik\lambda_n t} X_n(x). \tag{6.2.7}$$

Again, taking in account initial condition (6.2.3) we see that

$$u = \sum_{n=1}^{\infty} A_n X_n(x). \tag{6.2.8}$$

and therefore

$$A_n = \frac{1}{\|X_n\|^2}\int_0^l g(x)X_n(x)\,dx. \tag{6.2.9}$$

Remark 6.2.1. (a) Formula (6.2.6) shows that the problem is well-posed for $t<0$ and $t>0$.

(b) However there is no stabilization.

(c) What we got is a finite interval version of analysis of Subsection 5.3.2.

6.2.2 1D wave equation

Consider problem

$$u_{tt} = c^2 u_{xx}, \qquad 0<x<l, \tag{6.2.10}$$

$$(\alpha_0 u_x - \alpha u)|_{x=0} = (\beta_0 u_x + \beta u)|_{x=l} = 0. \tag{6.2.11}$$

$$u|_{t=0} = g(x), \qquad u_t|_{t=0} = h(x). \tag{6.2.12}$$

Actually we started from this equation in Section 4.1 but now we consider more general boundary conditions. Now we have

$$T'' = -k\lambda T \tag{6.2.13}$$

and we have

$$T_n = \begin{cases} A_n\cos(\omega_n t) + B_n\sin(\omega_n t) & \omega_n = c\lambda_n^{\frac12} \text{as } \lambda_n>0,\\ A_n + B_n t & \text{as } \lambda_n = 0,\\ A_n\cosh(\eta_n t) + B_n\sinh(\eta_n t) & \eta_n = c(-\lambda_n)^{\frac12} \text{as } \lambda_n>0, \end{cases} \tag{6.2.14}$$

and respectively we get

$$u = \sum_n T_n(t)X_n(x) \tag{6.2.15}$$

and we find from initial conditions

$$A_n = \frac{1}{\|X_n\|^2}\int_0^l g(x)X_n(x)\,dx, \tag{6.2.16}$$

$$B_n = \frac{1}{\|X_n\|^2}\int_0^l h(x)X_n(x)\,dx \times \begin{cases} \dfrac{1}{\omega_n} & \text{as } \lambda_n>0,\\ 1 & \text{as } \lambda_n = 0,\\ \dfrac{1}{\eta_n} & \text{as } \lambda_n<0. \end{cases} \tag{6.2.17}$$

6.2.3 Laplace equation in half-strip

Consider problem

$$\Delta u := u_{xx} + u_{yy} = 0, \qquad y > 0,\ 0 < x < l, \tag{6.2.18}$$

$$(\alpha_0 u_x - \alpha u)|_{x=0} = (\beta_0 u_x + \beta u)|_{x=l} = 0, \tag{6.2.19}$$

$$u|_{y=0} = g(x). \tag{6.2.20}$$

To make it uniquely solvable we need to add condition $|u| \le M$.

Again separating variables $u(x,y) = X(x)Y(y)$ we get

$$Y'' = \lambda Y \tag{6.2.21}$$

and therefore *assuming that* $\lambda > 0$ we get

$$Y = Ae^{-\sqrt{\lambda}y} + Be^{\sqrt{\lambda}y} \tag{6.2.22}$$

We discard the second term in the right-hand expression of (6.2.22) because it is unbounded. However if we had Cauchy problem (i.e. $u|_{y=0} = g(x)$, $u_y|_{y=0} = h(x)$) we would not be able to do this and this problem will be ill-posed.

So, $u = Ae^{-\sqrt{\lambda}y}$ and (6.2.20) and

$$u_n = A_n e^{-\sqrt{\lambda_n}y} X_n(x) \tag{6.2.23}$$

and *assuming that all* $\lambda_n > 0$ we get

$$u = \sum_n A_n e^{-\sqrt{\lambda_n}y} X_n(x) \tag{6.2.24}$$

and (6.2.19) yields

$$A_n = \frac{1}{\|X_n\|^2} \int_0^l g(x) X_n(x)\, dx. \tag{6.2.25}$$

Remark 6.2.2. (a) If there is eigenvalue $\lambda = 0$ we have $Y = A + By$ and as we are looking for a bounded solution we discard the second term again; so *our analysis remain valid as all* $\lambda_n \ge 0$.

(b) If we have a problem

$$\Delta u := u_{xx} + u_{yy} = 0, \qquad y > 0,\ 0 < x < l, \tag{6.2.26}$$

$$(\alpha_0 u_x - \alpha u)|_{x=0} = \phi(y), \tag{6.2.27}$$

$$(\beta_0 u_x + \beta u)|_{x=l} = \psi(y), \tag{6.2.28}$$

$$u|_{y=0} = g(x) \tag{6.2.29}$$

with $g(x) = 0$ we could reduce it by the method of continuation to the problem in the whole strip and solve it by Fourier transform Subsecion (see 5.3.5).

(c) In the general case we can find $u = u_{(1)}+u_{(2)}$ where $u_{(1)}$ solves problem with $g = 0$ and $u_{(2)}$ solves problem with $\phi = \psi = 0$ (explain how it follows from the linearity).

(d) One can replace Dirichlet boundary condition $u|_{y=0}$ by Robin boundary condition $(u_y - \gamma u)|_{y=0} = g(x)$ ($\gamma \geq 0$) but there is an exceptional case: there is an eigenvalue $\lambda_0 = 0$ and as $y = 0$ we have Neumann boundary condition.

(e) In this exceptional case (usually as we have Neumann b.c. everywhere– as $x = 0$, $x = l$, $y = 0$) a required solution simply does not exists unless $\int_0^l g(x)X_0(x)\,dx = 0$.

6.2.4 Laplace equation in rectangle

Consider problem

$$\Delta u := u_{xx} + u_{yy} = 0, \qquad 0 < y < b,\ 0 < x < a, \tag{6.2.30}$$

$$(\alpha_0 u_x - \alpha u)|_{x=0} = (\beta_0 u_x + \beta u)|_{x=a} = 0, \tag{6.2.31}$$

$$u|_{y=0} = g(x), \tag{6.2.32}$$

$$u|_{y=b} = h(x). \tag{6.2.33}$$

Then we get (6.2.21) and (6.2.22) again but with two b.c. we cannot diacard anything; we get instead

$$A_n + B_n = g_n, \tag{6.2.34}$$

$$A_n e^{-\sqrt{\lambda_n}b} + B_n e^{\sqrt{\lambda_n}b} = h_n \tag{6.2.35}$$

where g_n and h_n are Fourier coefficients of g and h respectively, which implies

$$A_n = \frac{e^{\sqrt{\lambda_n}b}}{2\sinh(\sqrt{\lambda_n}b)}g_n - \frac{1}{2\sinh(\sqrt{\lambda_n}b)}h_n),$$
$$B_n = -\frac{e^{-\sqrt{\lambda_n}b}}{2\sinh(\sqrt{\lambda_n}b)}g_n + \frac{1}{2\sinh(\sqrt{\lambda_n}b)}h_n$$

and therefore

$$Y_n(y) = \frac{\sinh(\sqrt{\lambda_n}(b-y))}{\sinh(\sqrt{\lambda_n}b)}g_n + \frac{\sinh(\sqrt{\lambda_n}y)}{\sinh(\sqrt{\lambda_n}b)}h_n. \tag{6.2.36}$$

One can see easily that $\frac{\sinh(\sqrt{\lambda_n}(b-y))}{\sinh(\sqrt{\lambda_n}b)}$ and $\frac{\sinh(\sqrt{\lambda_n}y)}{\sinh(\sqrt{\lambda_n}b)}$ are bounded as $0 \le y \le b$.

Exercise 6.2.1. exercise-6.2.1 Investigate other boundary conditions (Robin, Neumann, mixed).

Remark 6.2.3. (a) There is an exeptional case: there is an eigenvalue $\lambda_0 = 0$ and as $y = 0$ and $y = b$ we have Neumann boundary conditions. Then solution does not exist unless $\int_0^a h(x)X_0(x)\,dx - \int_0^a g(x)X_0(x)\,dx = 0$.

(b) We can consider general b.c. with $(\alpha_0 u_x - \alpha u)|_{x=0} = \phi(y)$, $(\beta_0 u_x + \beta u)|_{x=a} = \psi(y)$. Then we can find $u = u_{(1)} + u_{(2)}$ where $u_{(1)}$ solves problem with $g = h = 0$ and $u_{(2)}$ solves problem with $\phi = \psi = 0$ (explain how it follows from the linearity). The second problem is also "our" problem with x and y permutted.

(c) Assume that we have Neumann b.c. everywhere–as $x = 0$, $x = a$, $y = 0$, $y = b$. Then solution does not exist unless

$$\int_0^a h(x)\,dx - \int_0^a g(x)\,dx + \int_0^b \psi(y)\,dy - \int_0^b \phi(y)\,dy = 0 \tag{6.2.37}$$

which means that the total heat flow is 0. How from two assumptions we can get one? Well, we just need to consider $\phi = g = 0$, $\psi = \frac{y}{b}$, $h = -\frac{x}{a}$ (explain why) but there is a solution $u = \frac{y}{b} - \frac{x}{a}$ for that.

6.3 Laplace operator in different coordinates

6.3.1 Laplace operator in polar coordinates

In the next several lectures we are going to consider Laplace equation in the disk and similar domains and separate variables there but for this purpose we need to express Laplace operator in polar coordinates. Recall that (from 1st year Calculis) polar coordinates are (r,θ) connected with Cartesian coordinates by $x = r\cos(\theta)$, $y = \sin(\theta)$ and inversely

$$\begin{cases} r = \sqrt{x^2+y^2}, \\ \theta = \arctan\big(\frac{y}{x}\big); \end{cases}$$

surely the second formula is not exactly correct as changing $(x,y) \to (-x,-y)$ does not change it ratio but replaces θ by $\theta+\pi$ (or $\theta-\pi$) as θ is defined modulo $2\pi n$ with $n \in \mathbb{Z}$. It does not really maater as we are interested only in derivatives:

$$r_x = \cos(\theta),\ r_y = \sin(\theta),\ \theta_x = -r^{-1}\sin(\theta),\ \theta_y = r^{-1}\cos(\theta). \tag{6.3.1}$$

Exercise 6.3.1. Prove (6.3.1).

Then by chain rule

$$\begin{cases} \partial_x = \cos(\theta)\partial_r - r^{-1}\sin(\theta)\partial_\theta, \\ \partial_x = \sin(\theta)\partial_r + r^{-1}\cos(\theta)\partial_\theta \end{cases} \tag{6.3.2}$$

and therefore

$$\Delta = \partial_x^2 + \partial_y^2 = \big(\cos(\theta)\partial_r - r^{-1}\sin(\theta)\partial_\theta\big)^2 + \big(\sin(\theta)\partial_r + r^{-1}\cos(\theta)\partial_\theta\big)^2$$

and after tedious calculations one can get

$$\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2. \tag{6.3.3}$$

Exercise 6.3.2. Do it.

Instead we want to use a different method requiring much less error prone calculations but more delicate arguments (useful in more complicated cases).

Note first the identity

$$\iint \Delta u \cdot v \, dxdy = -\iint \nabla u \cdot \nabla v \, dxdy \tag{6.3.4}$$

provided $v = 0$ near Γ (boundary of $\mathcal{D}$) and integrals are taken over $\mathcal{D}$.

Now let us express the left- and right-hand expression in polar coordinates. Recall that polar coordinates are *orthogonal* (i.e. level lines of r (circles) and level lines of θ (rays from the origin) are orthogonal in the points where they intersect) and the distance between two close points ds can be calculated as

$$ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2 \tag{6.3.5}$$

and therefore area element is $dA = dxdy = rdrd\theta$.

But what about $\nabla u \cdot \nabla v$? We claim that

$$\nabla u \cdot \nabla v = u_r v_r + \frac{1}{r^2} u_\theta v_\theta. \tag{6.3.6}$$

Indeed, ∇u is a vector of the different nature than $d\mathbf{s} = (dx, dy)$. They are connected by $du = \nabla u \cdot d\mathbf{s}$ and when we change coordinates $d\mathbf{s} = Q d\mathbf{s}$ with some matrix Q and since

$$du = \nabla u \cdot d\mathbf{s} = \nabla u' \cdot d\mathbf{s}' = \nabla u' \cdot Q d\mathbf{s} = Q^T \nabla u' \cdot d\mathbf{s}$$

we conclude that $d\mathbf{s}' = Q^{T\,-1} \nabla u$ where ^{T} means transposed matrix. Such dual vectors mathematicians call *covectors*.

Remark 6.3.1. (a) While mathematicians talk about *vectors* and *covectors* physicists often call them *covariant* and *contravariant* vectors.

(b) Also there are notions of pseudo-vectors (and pseudo-covectors) and pseudo-scalars which change signs when right-oriented coordinate system is changed to the left-oriented one. F.e. if we restrict ourselves to Cartesian coordinates, vector-product of two vectors is a pseudo-vector, and oriented volume is a pseudo-scalar. Curl of a vector field is a pseudo-vector field. Intensity of magnetic field is a pseudo-vector.

(c) However for more general coordinate systems there are also *densities* which in addition to usual transformations "reflect" the change of volume.

For us here important is only the difference between vectors and covectors.

Therefore (6.3.4) becomes

$$\iint \Delta u \cdot v \, r dr d\theta = -\iint \big(u_r v_r + \frac{1}{r^2} u_\theta v_\theta\big) r \, dr d\theta = \\ -\iint \big(r u_r v_r + \frac{1}{r} u_\theta v_\theta\big) \, dr d\theta = \iint \big(\big(r u_r\big)_r + \big(\frac{1}{r} u_\theta\big)_\theta\big) v \, dr d\theta$$

where we integrated by parts. This identity

$$\iint r\Delta u \cdot v \, dr d\theta = \iint \big(\big(r u_r\big)_r + \big(\frac{1}{r} u_\theta\big)_\theta\big) v \, dr d\theta.$$

holds for any v vanishing near Γ and therefore we can nix integration and v:

$$r\Delta u = \big(r u_r\big)_r + \big(\frac{1}{r} u_\theta\big)_\theta.$$

Exercise 6.3.3. Think about this. Finally we get

$$\Delta u = r^{-1}\big(r u_r\big)_r + \big(\frac{1}{r} u_\theta\big)_\theta$$

which is exactly (6.3.3).

It may look too complicated for polar coordinates but in more general cases this approach is highly beneficial.

6.3.2 Laplace operator in spherical coordinates

Spherical coordinates are ρ (radius), ϕ (latitude) and θ (longitude):

$$\begin{cases} x = \rho \sin(\phi) \cos(\theta), \\ y = \rho \sin(\phi) \sin(\theta) \\ z = \rho \cos(\phi). \end{cases}$$

Conversely

$$\begin{cases} \rho = \sqrt{x^2+y^2+z^2}, \\ \phi = \arctan\big(\dfrac{\sqrt{x^2+y^2}}{z}\big), \\ \theta = \arctan\big(\dfrac{y}{x}\big); \end{cases}$$

and using chain rule and "simple" calculations becomes rather challenging.

Instead we recall that these coordinates are also orthogonal: if we fix ϕ and θ we get rays from origin, which are orthogonal to the speres which we get if we fix r. On the spheres if we fix θ we get meridians and if we fix ϕ we get parallels and those are also orthogonal. Then

$$ds^2 = dx^2+dy^2+dy^2+dz^2 = d\rho^2+\rho^2 d\phi^2+\rho^2\sin^2(\phi)d\theta^2 \qquad (6.3.5)'$$

where $d\rho$, $\rho d\phi$ and $\rho\sin(\phi)d\theta$ are distances along rays, meridians and parallels and therefore volume element is $dV = dxdydz = \rho^2\sin(\theta)d\rho d\phi d\theta$.

Therefore

$$\nabla u\cdot\nabla v = u_\rho v_\rho + \frac{1}{\rho^2}u_\phi v_\phi + \frac{1}{\rho^2\sin(\phi)}u_\theta v_\theta. \qquad (6.3.6)'$$

Plugging this into

$$\iiint \Delta u\cdot v\,dxdydz = -\iiint \nabla u\cdot\nabla v\,dxdydz \qquad (6.3.4)'$$

we get

$$\begin{aligned} &\iiint \Delta u\cdot v\rho^2\sin(\phi)\,d\rho d\phi d\theta = \\ &\quad -\iiint \big(u_\rho v_\rho + \frac{1}{\rho^2}u_\phi v_\phi + \frac{1}{\rho^2\sin(\phi)}u_\theta v_\theta\big)\rho^2\sin(\phi)\,d\rho d\phi d\theta = \\ &\qquad \iiint \Big(\big(\rho^2\sin(\phi)u_\rho\big)_\rho + \big(\sin(\phi)u_\phi\big)_\phi + \big(\frac{1}{\sin(\phi)}u_\theta\big)_\theta\Big)v\,d\rho d\phi d\theta. \end{aligned}$$

Then we can nix integration and factor v:

$$\Delta u\cdot\rho^2\sin(\phi) = \big(\rho^2\sin(\phi)u_\rho\big)_\rho + \big(\sin(\phi)u_\phi\big)_\phi + \big(\frac{1}{\sin(\phi)}u_\theta\big)_\theta$$

and then

$$\Delta u = \frac{1}{\rho^2 \sin(\phi)}\Big(\big(\rho^2 \sin(\phi) u_\rho\big)_\rho + \big(\sin(\phi) u_\phi\big)_\phi + \big(\frac{1}{\sin(\phi)} u_\theta\big)_\theta\Big)$$

and finally

$$\Delta = \partial_\rho^2 + \frac{2}{\rho}\partial_\rho + \frac{1}{\rho^2}\big(\partial_\phi^2 + \cot(\phi)\partial_\phi\big) + \frac{1}{\rho^2 \sin^2(\phi)}\partial_\theta^2. \tag{6.3.7}$$

(compare with (6.3.3))

Definition 6.3.1.

$$\Lambda := \partial_\phi^2 + \cot(\phi)\partial_\phi + \frac{1}{\sin^2(\phi)}\partial_\theta^2 \tag{6.3.8}$$

is a spherical Laplacian (aka Laplace-Beltrami operator on the sphere).

6.3.3 Special knowledge: Generalization

If the length element is

$$ds^2 = \sum_{j,k} g_{jk} dq^j dq^k \tag{6.3.5}"$$

where $q = q^1, \dots, q^n)$ are new coordinates and we prefer to write dq^j rather than dq_j (to half-follow Einstein' notations), g_{jk} is symmetric matrix, then

$$\nabla u \cdot \nabla v = \sum_{j,k} g^{jk} u_{q^j} u_{q^k} \tag{6.3.6}"$$

where (g^{jk}) is an inverse matrix to (g_{jk}): $\sum_k g^{jk} g_{kl} = \sum_k g_{lk} g^{kj} = \delta_l^j$.

Then volume element is $|\det(g_{jk})|^{\frac{1}{2}}\, dq_1 \cdots dq_n$ and

$$\Delta u = |\det(g_{jk})|^{-\frac{1}{2}} \sum_{j,k} \frac{\partial}{\partial q^j}\big(|\det(g_{jk})|^{\frac{1}{2}} g^{jk} \frac{\partial u}{\partial q^k}\big) \tag{6.3.9}$$

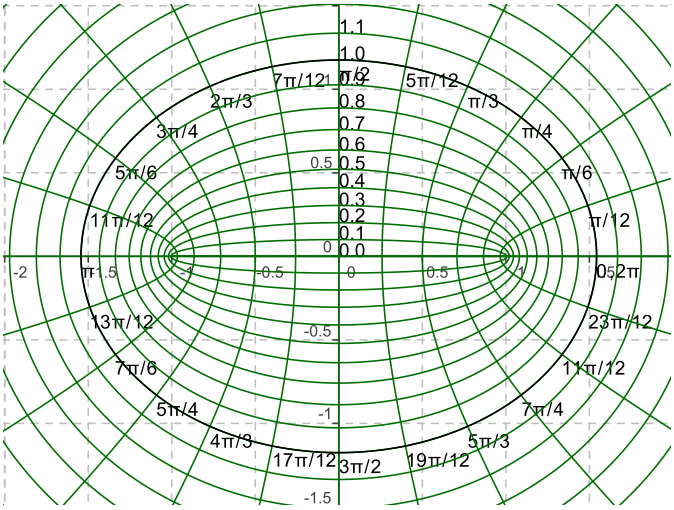
Remark 6.3.2. Formula (6.3.9) defines Laplace operator on *Riemannian manifolds* (like surfaces in 3D) where Cartesian coordinates do not exist at all. Such manifolds are studied in the *Riemannian geometry* and are used f.e. in *General relativity* (actually GR uses *pseudo-Riemannian manifolds*).

6.3.4 Secret knowledge: elliptic and parabolic coordinates

Elliptic coordinates on $\mathbb{R}^2$ are (σ, τ):

$$\begin{cases} x = c\cosh(\sigma)\cos(\tau), \\ y = c\sinh(\sigma)\sin(\tau). \end{cases} \tag{6.3.10}$$

Level lines $\sigma = \text{const}$ are ellipses with foci at $(-c, 0)$ and $(c, 0)$ and level lines $\tau = \text{const}$ are hyperbolae with the same focai; so we have confocal ellipses and hyperbolae.



These coordinates are not only orthogonal but they are *conformal* (ds^2 is proportional to $d\sigma^2 + d\tau^2$)

$$ds^2 = \big(\sinh^2(\sigma) + \sin^2(\tau)\big)(d\sigma^2 + d\tau^2) \tag{6.3.11}$$

and therefore

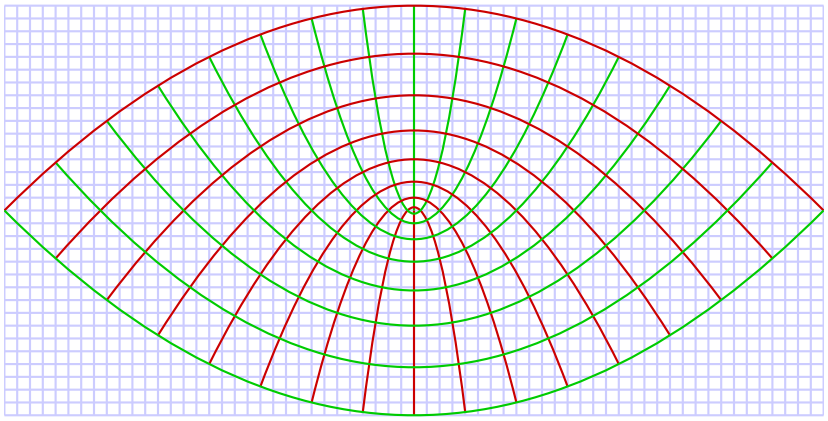
$$\Delta = \frac{1}{c^2\big(\sinh^2(\sigma) + \sin^2(\tau)\big)}(\partial_\sigma^2 + \partial_\tau^2). \tag{6.3.12}$$

Elliptic cylindrical coordinates in $\mathbb{R}^3$ are obtained by adding z to elliptic coordinates.

Parabolic coordinates on $\mathbb{R}^2$ are (σ, τ):

$$\begin{cases} x = \sigma\tau, \\ y = \dfrac{1}{2}(\sigma^2 - \tau^2). \end{cases} \tag{6.3.13}$$

Level lines $\sigma = \text{const}$ and $\tau = \text{const}$ are confocal parabolae.



These coordinates are also *conformal*

$$ds^2 = (\sigma^2 + \tau^2)(d\sigma^2 + d\tau^2) \tag{6.3.14}$$

and therefore

$$\Delta = \frac{1}{\sigma^2 + \tau^2}(\partial_\sigma^2 + \partial_\tau^2). \tag{6.3.15}$$

Three-dimensional parabolic coordinates are obtained by rotating the two-dimensional system about the symmetry axis of the parabolae.

Parabolic cylindrical coordinates in $\mathbb{R}^3$ are obtained by adding z to parabolic coordinates.

6.4 Laplace operator in the disk: separation of variables

6.4.1 Separation of variables

So, consider problem

$$\begin{aligned} &\Delta u = 0 && \text{as } \ x^2 + y^2 < a^2, \\ &u = g && \text{at } \ x^2 + y^2 = a^2. \end{aligned}$$

In the polar coordinates it becomes

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \qquad \text{as } \ r < a, \tag{6.4.1}$$

$$u = g(\theta) \qquad \text{at } \ r = a. \tag{6.4.2}$$

First let us forget for a while about (6.4.2) and consider a simple solution $u = R(r)\Theta(\theta)$ to equation (6.4.1). Then

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0.$$

To separate r and θ we must divide by $R\Theta$ and multiply by r^2:

$$\underbrace{\frac{r^2R'' + rR'}{R}} + \underbrace{\frac{\Theta''}{\Theta}} = 0.$$

Then repeating our usual separation of variables magic spell both expressions are constant, say λ and $-\lambda$:

$$r^2R'' + rR' - \lambda R = 0, \tag{6.4.3}$$

$$\Theta'' + \lambda\Theta = 0. \tag{6.4.4}$$

Now we need boundary conditions to Θ and those are periodic:

$$\Theta(2\pi) = \Theta(0), \qquad \Theta'(2\pi) = \Theta'(0). \tag{6.4.5}$$

We already know solution to (6.4.4)–(6.4.5):

$$\lambda_0 = 0, \qquad \lambda_n = n^2, \qquad n = 1, 2, \dots. \tag{6.4.6}$$

$$\Theta_0 = \frac{1}{2} \qquad \Theta_{n,1} = \cos(n\theta), \qquad \Theta_{n,2} = \sin(n\theta). \tag{6.4.7}$$

Equation (6.4.3) is an Euler equation, we are looking for solutions $R = r^m$. Then

$$m(m-1) + m - \lambda = 0 \implies m^2 = \lambda \tag{6.4.8}$$

Plugging $\lambda_n = n^2$ into (6.4.3) we get $m = \pm n$ and therefore $R = Ar^n + Br^{-n}$ as $n \neq 0$ and $R = A + B\log r$ as $n = 0$. Therefore

$$u = \frac{1}{2}\big(A_0 + B_0 \log r\big) + \sum_{n=1}^{\infty}\Big(\big(A_n r^n + B_n r^{-n}\big)\cos(n\theta) + \big(C_n r^n + D_n r^{-n}\big)\sin(n\theta)\Big). \tag{6.4.9}$$

Here we assembled simple solutions together.

As we are looking for solutions in the disk $\{r < a\}$ we should discard terms singular as $r = 0$; namely we should set $B_0 = 0$, $B_n = D_n = 0$ for $n = 1, 2, \dots$ and therefore

$$u = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n\Big(A_n \cos(n\theta) + C_n \sin(n\theta)\Big). \tag{6.4.10}$$

If we consider equation outside of the disk (so as $r > a$) we need to impose condition $\max|u| < \infty$ and discard terms singular as $r = \infty$; namely we should sety $B_0 = 0$, $A_n = C_n = 0$ for $n = 1, 2, \dots$ and therefore

$$u = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^{-n}\Big(B_n \cos(n\theta) + D_n \sin(n\theta)\Big). \tag{6.4.11}$$

Finally, if we consider equation outside in the annulus (aka ring) $\{a < r < b\}$ we need b.c. on both circles $\{r = a\}$ and $\{r = b\}$ and we discard no terms.

6.4.2 Poisson formula

But we are dealing with the disk. Plugging (6.4.10) into (6.4.2) we get

$$g(\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n\Big(A_n \cos(n\theta) + C_n \sin(n\theta)\Big) \tag{6.4.12}$$

and therefore

$$A_n = \frac{1}{\pi}a^{-n}\int_0^{2\pi} g(\theta')\cos(n\theta')\,d\theta',$$
$$C_n = \frac{1}{\pi}a^{-n}\int_0^{2\pi} g(\theta')\sin(n\theta')\,d\theta'.$$

Plugging into (6.4.10) we get

$$u(r,\theta) = \int_0^{2\pi} G(r,\theta,\theta')g(\theta')\,d\theta' \tag{6.4.13}$$

with

$$G(r,\theta,\theta') := \frac{1}{2\pi}\Big(1+2\sum_{n=1}^{\infty} r^n a^{-n}\big(\cos(n\theta)\cos(n\theta')+\sin(n\theta)\sin(n\theta)\big)\Big) =$$
$$\frac{1}{2\pi}\Big(1+2\sum_{n=1}^{\infty} r^n a^{-n}\cos(n(\theta-\theta'))\Big) =$$
$$\frac{1}{2\pi}\Big(1+2\operatorname{Re}\sum_{n=1}^{\infty}\big(ra^{-1}e^{i(\theta-\theta')}\big)^n\Big) =$$
$$\frac{1}{2\pi}\Big(1+2\operatorname{Re}\frac{ra^{-1}e^{i(\theta-\theta')}}{1-ra^{-1}e^{i(\theta-\theta')}}\Big)$$

where we summed geometric progression with the factor $z = ra^{-1}e^{i(\theta-\theta')}$ (then $|z| = ra^{-1} < 1$).

Multiplying numerator and denominator by $a^2 - rae^{-i(\theta-\theta')}$ we get $rae^{-i(\theta-\theta')} - r^2$ and $a^2 - ra[e^{-i(\theta-\theta')} + e^{i(\theta-\theta')}] + r^2a^{-2} = a^2 - 2ra\cos(\theta-\theta') + r^2$ and therefore we get

$$G(r,\theta,\theta') = \frac{1}{2\pi}\Big(1+2\frac{ra\cos(\theta-\theta')}{a^2-2ra\cos(\theta-\theta')+r^2}\Big)$$

and finally

$$G(r,\theta,\theta') = \frac{1}{2\pi}\frac{a^2-r^2}{a^2-2ra\cos(\theta-\theta')+r^2}. \tag{6.4.14}$$

Recall that $r < a$.

Formula (6.4.13)–(6.4.14) is *Poisson formula*.

Exercise 6.4.1. Prove that in the center of the disk (as $r = 0$)

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} g(\theta')\,d\theta' \tag{6.4.15}$$

and the right-hand expression is a mean value of u over circumference $\{r = a\}$.

Exercise 6.4.2. (a) Using (6.4.11) instead of (6.4.10) prove that for the problem

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \qquad \text{as } \ a < r, \tag{6.4.16}$$

$$u = g(\theta) \qquad \text{at } \ r = a, \tag{6.4.17}$$

$$\max|u| < \infty \tag{6.4.18}$$

solution is given (for $r > a$) by (6.4.13) but with

$$G(r,\theta,\theta') = \frac{1}{2\pi}\frac{r^2 - a^2}{a^2 - 2ra\cos(\theta-\theta') + r^2}. \tag{6.4.19}$$

(b) As a corollary, prove that

$$u(\infty) := \lim_{r\to\infty} u = \frac{1}{2\pi}\int_0^{2\pi} g(\theta')\,d\theta). \tag{6.4.20}$$

6.5 Laplace operator in the disk. II

6.5.1 Neumann problem

Consider now Neumann problem

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \qquad \text{as } \ r < a, \tag{6.5.1}$$

$$u_r = h(\theta) \qquad \text{at } \ r = a. \tag{6.5.2}$$

Plugging in (6.5.2) expression (6.4.10)

$$u = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n\Big(A_n\cos(n\theta) + C_n\sin(n\theta)\Big) \tag{6.4.10}$$

we get

$$\sum_{n=1}^{\infty} na^{n-1}\Big(A_n\cos(n\theta) + C_n r\sin(n\theta)\Big) = h(\theta) \tag{6.5.3}$$

and feel trouble!

(a) We cannot satisfy (6.5.3) unless $h(\theta)$ has "free" coefficient equal to 0 i.e.

$$\int_0^{2\pi} h(\theta)\,d\theta = 0. \tag{6.5.4}$$

(b) Even if (6.5.4) holds we cannot find A_0 so in the end solution is defined up to a constant.

So to cure (a) assume that (6.5.4) is fulfilled and to fix (b)impose condition

$$\iint_{\mathcal{D}} u(x,y)\,dxdy = 0. \tag{6.5.5}$$

(Indeed, it is $\iint u(r,\theta)\,rdrd\theta = \pi a^2 A_0$). Then $A_0 = 0$ and

$$A_n = \frac{1}{\pi n} a^{1-n} \int_0^{2\pi} h(\theta')\cos(n\theta')\,d\theta',$$
$$C_n = \frac{1}{\pi n} a^{1-n} \int_0^{2\pi} h(\theta')\sin(n\theta')\,d\theta.$$

Plugging into (6.4.10) we have

$$u(r,\theta) = \int_0^{2\pi} G(r,\theta,\theta')g(\theta')\,d\theta' \tag{6.5.6}$$

with

$$G(r,\theta,\theta') := \frac{1}{\pi}\Big(\sum_{n=1}^{\infty} \frac{1}{n} r^n a^{1-n}\big(\cos(n\theta)\cos(n\theta') + \sin(n\theta)\sin(n\theta)\big)\Big) =$$
$$\frac{1}{\pi}\Big(\sum_{n=1}^{\infty} \frac{1}{n} r^n a^{1-n}\cos(n(\theta-\theta'))\Big) =$$
$$\frac{a}{\pi}\Big(\operatorname{Re}\sum_{n=1}^{\infty} \frac{1}{n}\big(ra^{-1}e^{i(\theta-\theta')}\big)^n\Big) =$$
$$-\frac{a}{\pi}\operatorname{Re}\log\big(1 - ra^{-1}e^{i(\theta-\theta')}\big)$$

where we used that $\sum_{n=1}^{\infty} \frac{1}{n} z^n = -\log(1-z)$ (indeed, if we denote it by $f(z)$ then $f'(z) = \sum_{n=1}^{\infty} z^{n-1} = (1-z)^{-1}$ and $f(0) = 0$) and plugged $z = ra^{-1}e^{i(\theta-\theta')}$ with $|z| < 1$. The last expression equals

$$-\frac{a}{2\pi}\log\bigl[a^{-2}\bigl(1-ra^{-1}e^{i(\theta-\theta')}\bigr)\bigl(1-ra^{-1}e^{-i(\theta-\theta')}\bigr)\bigr] = -\frac{a}{2\pi}\log\bigl[a^{-2}\bigl(a^2-2ar\cos(\theta-\theta')+r^2\bigr)\bigr].$$

with

$$G(r,\theta,\theta') = -\frac{a}{2\pi}\log\bigl[a^{-2}\bigl(a^2-2ar\cos(\theta-\theta')+r^2\bigr)\bigr]. \tag{6.5.7}$$

Recall that $r<a$.

Considering outside of the disk we should use (6.4.11)

$$u = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^{-n}\Bigl(B_n\cos(n\theta)+D_n\sin(n\theta)\Bigr). \tag{6.4.11}$$

Again we need to impose condition (6.5.4); condition (6.5.5) is now replaced by

$$\lim_{r\to\infty} u = 0. \tag{6.5.8}$$

Then $A_0=0$ and

$$B_n = -\frac{1}{\pi}\frac{1}{n}a^{n+1}\int_0^{2\pi} h(\theta')\cos(n\theta')\,d\theta',$$
$$D_n = -\frac{1}{\pi}\frac{1}{n}a^{n+1}\int_0^{2\pi} h(\theta')\sin(n\theta')\,d\theta'.$$

Plugging into (6.4.11) we get (6.5.6) with

$$G(r,\theta,\theta') = \frac{a}{2\pi}\log\bigl[r^{-2}\bigl(a^2-2ar\cos(\theta-\theta')+r^2\bigr)\bigr]. \tag{6.5.9}$$

6.5.2 Laplace in the sector

Consider now our equation in the sector $\{r<a,\ 0<\theta<\alpha\}$ and impose 0 Dirichlet boundary conditions at radial parts of the boundary and non-zero on the circular part. Consider now Neumann problem

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0 \qquad \text{as } \ r<a,\ 0<\theta<\alpha \tag{6.5.10}$$

$$u(r,0)=u(r,\alpha)=0 \qquad 0<r<a, \tag{6.5.11}$$

$$u=h(\theta) \qquad \text{at } \ r=a,\ 0<\theta<\alpha. \tag{6.5.12}$$

Remark 6.5.1. We can consider different b.c. on these three parts of the boundary, trouble is when Neumann is everywhere.

Then separating variables as in Section 6.4 we get

$$\Theta'' + \lambda\Theta = 0,$$
$$\Theta(0) = \Theta(\alpha) = 0$$

and therefore

$$\lambda_n = \big(\frac{\pi n}{\alpha}\big)^2, \qquad \Theta_n = \sin\big(\frac{\pi n\theta}{\alpha}\big) \qquad n = 1, 2, \dots$$

and plugging into (6.4.3)

$$r^2R'' + rR' - \lambda R = 0 \tag{6.4.3}$$

we get

$$R_n = A_n r^{\frac{\pi n}{\alpha}} + B_n r^{-\frac{\pi n}{\alpha}} \tag{6.5.13}$$

and therefore

$$u = \sum_{n=1}^{\infty} \big(A_n r^{\frac{\pi n}{\alpha}} + B_n r^{-\frac{\pi n}{\alpha}}\big) \sin\big(\frac{\pi n\theta}{\alpha}\big) \tag{6.5.14}$$

where for sector $\{r < a,\ 0 < \theta < \alpha\}$ we should set $_n = 0$ (for domain $\{r > a,\ 0 < \theta < \alpha\}$ we should set $A_n = 0$ and for domain $\{a < r < b,\ 0 < \theta < \alpha\}$ we don't nix anything). The rest is easy except we don't get nice formula like Poisson formula.

6.A Linear second order ODEs

6.A.1 Introduction

This is not a required reading but at some moment you would like to see how problems we discuss here for PDEs are solved for ODEs (consider it as a toy-model)

We consider ODE

$$Ly := y'' + a_1(x)y + a_2(x)y' = f(x). \tag{6.A.1}$$

Let $\{y_1(x), y_2(x)\}$ be a fundamental system of solutions of the corresponding homogeneous equation

$$Ly := y'' + a_1(x)y + a_2(x)y' = 0). \tag{6.A.2}$$

Recall that then Wronskian

$$W(y_1, y_2; x) := \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \tag{6.A.3}$$

does not vanish.

6.A.2 Cauchy problem (aka IVP)

Consider equation (6.A.1) with the initial conditions

$$y(x_0) = b_1, \qquad y'(x_0) = b_2. \tag{6.A.4}$$

Without any loss of the generality one can assume that

$$\begin{aligned} &y_1(x_0) = 1, \quad y_1'(x_0) = 0, \\ &y_2(x_0) = 0, \quad y_1'(x_0) = 1. \end{aligned} \tag{6.A.5}$$

Indeed, replacing $\{y_1(x), y_2(x)\}$ by $\{z_1(x), z_2(x)\}$ with $z_j = \alpha_{j1}y_1 + \alpha_{j2}y_2$ we reach (6.A.5) by solving the systems

$$\begin{aligned} &\alpha_{11}y_1(x_0) + \alpha_{12}y_2(x_0) = 1, \qquad && \alpha_{21}y_1(x_0) + \alpha_{22}y_2(x_0) = 0 \\ &\alpha_{11}y_1'(x_0) + \alpha_{12}y_2'(x_0) = 0, \qquad && \alpha_{21}y_1'(x_0) + \alpha_{22}y_2'(x_0) = 1 \end{aligned}$$

which have unique solutions because $W(y_1, y_2; x_0) \neq 0$.

Then the general solution to (6.A.2) is $y = C_1y_1 + C_2y_2$ with constants C_1, C_2. To find the general solution to (6.A.1) we apply method of variations of parameters; then

$$\begin{aligned} &C_1'y_1 + C_2'y_2 = 0, \\ &C_1'y_1' + C_2'y_2' = f(x) \end{aligned} \tag{6.A.6}$$

and then

$$C_1' = -\frac{1}{W}y_2f, \qquad C_2' = \frac{1}{W}y_12f \tag{6.A.7}$$

and

$$\begin{aligned} C_1(x) &= -\int_{x_0}^{x} \frac{1}{W(x')} y_2(x') f(x')\, dx' + c_1, \\ C_2(x) &= \int_{x_0}^{x} \frac{1}{W(x')} y_1(x') f(x')\, dx' + c_2 \end{aligned} \tag{6.A.8}$$

and

$$y(x) = \int_{x_0}^{x} G(x;x') f(x')\, dx' + b_1 y_1(x) + b_2 y_2(x) \tag{6.A.9}$$

with

$$G(x;x') = \frac{1}{W(x')}\big(y_2(x)y_1(x') - y_1(x)y_2(x')\big) \tag{6.A.10}$$

and $c_1 = b_1$, $c_2 = b_2$ found from initial data.

Definition 1. $G(x,x')$ is a *Green function* (called in the case of IVP also *Cauchy function*).

This formula (6.A.9) could be rewritten as

$$y(x) = \int_{x_0}^{x} G(x;x') f(x')\, dx' + G'_x(x;x_0)b_1 + G(x;x_0)b_2. \tag{6.A.11}$$

6.A.2.1 BVP

Consider equation (6.A.1) with the boundary conditions

$$y(x_1) = b_1, \qquad y(x_2) = b_2 \tag{6.A.12}$$

where $x_1 < x_2$ are the ends of the segment $[x_1, x_2]$.

Consider first homogeneous equation (6.A.2); then $y = c_1y_1 + c_2y_2$ and (6.A.12) becomes

$$\begin{aligned} c_1y_1(x_1) + c_2y_2(x_1) &= b_1, \\ c_1y_1(x_2) + c_2y_2(x_2) &= b_2 \end{aligned}$$

and this system is solvable for any b_1, b_2 and this solution is unique if and only if determinant is not 0:

$$\begin{vmatrix} y_1(x_1) & y_2(x_1) \\ y_1(x_2) & y_2(x_2) \end{vmatrix} \neq 0. \tag{6.A.13}$$

Assume that this condition is fulfilled. Then without any loss of the generality one can assume that

$$y_1(x_1) = 1, \quad y_1(x_2) = 0, \quad y_2(x_1) = 0, \quad y_2(x_2) = 1; \tag{6.A.14}$$

otherwise as before we can replace them by their linear combinations. Consider inhomogeneous equation. Solving it by method of variations of parameters we have again (6.A.7) but its solution we write in a form slightly different from (6.A.8)

$$\begin{aligned} C_1(x) &= -\int_{x_1}^{x} \frac{1}{W(x')} y_2(x') f(x')\, dx' + c_1, \\ C_2(x) &= -\int_{x}^{x_2} \frac{1}{W(x')} y_1(x') f(x')\, dx' + c_2. \end{aligned} \tag{6.A.15}$$

Then

$$y(x) = \int_{x_1}^{x_2} G(x;x') f(x')\, dx' + c_1 y_1(x) + c_2 y_2(x) \tag{6.A.16}$$

where

$$G(x;x') = -\frac{1}{W(x')} \begin{cases} y_2(x') y_1(x) & x_1 < x' < x, \\ y_1(x') y_2(x) & x < x' < x_2. \end{cases} \tag{6.A.17}$$

From boundary conditions one can check easily that $c_1 = b_1$, $c_2 = b_2$. One can also $y_1(x) = -G'_{x'}(x;x')|_{x'=x_1}$, $y_2(x) = -G'_{x'}(x;x')|_{x'=x_2}$ and therefore

$$\begin{aligned} y(x) = \int_{x_1}^{x_2} & G(x;x') f(x')\, dx' \\ & - G'_{x'}(x;x')|_{x'=x_1} b_1 + G'_{x'}(x;x')|_{x'=x_2} b_2. \end{aligned} \tag{6.A.18}$$

Definition 2. $G(x,x')$ is a *Green function*.

6.A.3 BVP. II

Assume now that (6.A.13) is violated. Then we cannot expect that the problem is uniquely solvable but let us salvage what we can. Without any loss of the generality we can assume now that

$$y_2(x_1) = y_2(x_2) = 0; \tag{6.A.19}$$

Using for a solution the same formulae (6.A.8), (6.A.10) but with x_0 replaced by x_1, plugging into boundary conditions and using (6.A.19) we have

$$c_1 y_1(x_1) = b_1, \qquad \bigl(c_1 - \int_{x_1}^{x_2} \frac{1}{W(x')} y_2(x')\, dx'\bigr) y_1(x_2) = b_2$$

which could be satisfied if and only iff

$$\int_{x_1}^{x_2} \frac{1}{W(x')} y_2(x')\,dx' - \frac{b_1}{y_1(x_1)} + \frac{b_2}{y_1(x_2)} = 0 \tag{6.A.20}$$

but solution is not unique: it is defined modulo $c_2 y_2(x)$.

Remark 1 More general boundary conditions

$$\alpha_1 y'(x_1) + \beta_1 y(x_1) = b_1, \qquad \alpha_2 y'(x_2) + \beta_2 y(x_2) = b_2 \tag{6.A.21}$$

could be analyzed in a similar way.

6.6 Problems to Chapter 6

Problem 6.6.1. (a) Find the solutions that depend only on r of the equation

$$\Delta u := u_{xx} + u_{yy} + u_{zz} = k^2 u,$$

where k is a positive constant. (*Hint:* Substitute $u = v/r$)

(b) Find the solutions that depend only on r of the equation

$$\Delta u := u_{xx} + u_{yy} + u_{zz} = -k^2 u,$$

where k is a positive constant. (*Hint:* Substitute $u = v/r$)

Problem 6.6.2. (a) Try to find the solutions that depend only on r of the equation

$$\Delta u := u_{xx} + u_{yy} = k^2 u,$$

where k is a positive constant. What ODE should satisfy $u(r)$?

(b) Try to find the solutions that depend only on r of the equation

$$\Delta u := u_{xx} + u_{yy} = -k^2 u,$$

where k is a positive constant. What ODE should satisftfy $u(r)$?

Problem 6.6.3. (a) Solve

$$\Delta := u_{xx} + u_{yy} = 0 \qquad \text{in } r < a$$
$$u|_{r=a} = f(\theta).$$

where we use polar coordinates (r, θ) and $f(\theta) = \begin{cases} 1 & 0 < \theta < \pi \\ -1 & \pi < \theta < 2\pi. \end{cases}$

(b) Solve

$$\begin{aligned} &\Delta := u_{xx} + u_{yy} = 0 && \text{in } r > a \\ &u|_{r=a} = f(\theta), \\ &\max|u| < \infty. \end{aligned}$$

where we use polar coordinates (r, θ) and $f(\theta) = \begin{cases} 1 & 0 < \theta < \pi \\ -1 & \pi < \theta < 2\pi. \end{cases}$

Problem 6.6.4. (a) Solve

$$\begin{aligned} &\Delta := u_{xx} + u_{yy} = 0 && \text{in } r < a \\ &u_r|_{r=a} = f(\theta) \end{aligned}$$

where we use polar coordinates (r, θ) and $f(\theta) = \begin{cases} 1 & 0 < \theta < \pi \\ -1 & \pi < \theta < 2\pi. \end{cases}$

(b) Solve

$$\begin{aligned} &\Delta := u_{xx} + u_{yy} = 0 && \text{in } r > a \\ &u_r|_{r=a} = f(\theta), \\ &\max|u| < \infty. \end{aligned}$$

where we use polar coordinates (r, θ) and $f(\theta) = \begin{cases} 1 & 0 < \theta < \pi \\ -1 & \pi < \theta < 2\pi. \end{cases}$

Problem 6.6.5. Describe all real-valued solutions of biharmonic equation

$$u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0 \tag{6.6.1}$$

which one can obtain by a method of separation $u(x, y) = X(x)Y(y)$.

Chapter 7

Laplace equation

7.1 General properties of Laplace equation

7.1.1 Existence and unicity

Consider problem

$$\Delta u - cu = f \qquad \text{in } \mathcal{D}, \tag{7.1.1}$$

$$u = 0 \qquad \text{on } \Gamma_- \tag{7.1.2}$$

$$\partial_\nu u - \alpha u = 0 \qquad \text{on } \Gamma_+ \tag{7.1.3}$$

where $\mathcal{D}$ is a connected bounded domain, Γ its boundary (smooth), consisting of two non-intersecting parts Γ_- and Γ_+, and ν a unit interior normal to Γ, $\partial_\nu u := \nabla u \cdot \nu$ is a normal derivative of u, c and α real valued functions.

Then

$$\begin{aligned} -\int_{\mathcal{D}} fu\,dxdy &= -\int_{\mathcal{D}} (u\Delta u - cu^2)\,dxdy \\ &= \int_{\mathcal{D}} (|\nabla u|^2 + cu^2)\,dxdy + \int_{\Gamma} u\partial_\nu u\,ds \\ &= \int_{\mathcal{D}} (|\nabla u|^2 + cu^2)\,dxdy + \int_{\Gamma_+} \alpha u^2\,ds \end{aligned}$$

as we can integrate over Γ_+. Therefore assuming that

$$c \geq 0, \qquad \alpha \geq 0 \tag{7.1.4}$$

we conclude that $f = 0 \implies \nabla u = 0$ and then $u = \text{const}$ and unless

$$c \equiv 0, \quad \alpha \equiv 0, \qquad \Gamma_- = \emptyset \tag{7.1.5}$$

we conclude that $u = 0$.

So, if (7.1.4) is fulfilled but (7.1.5) fails problem (7.1.1)–(7.1.3) has no more than one solution (explain why). One can prove that the solution exists (sorry, we do not have analytic tools for this).

Theorem 7.1.1. *If* (7.1.4) *is fulfilled but* (7.1.5) *fails problem* (7.1.1)–(7.1.3) *is uniquely solvable.*

Assume now that (7.1.5) is fulfilled. Then $u = C$ is a solution with $f = 0$. So, problem has no more than one solution modulo constant. Also

$$\int_{\mathcal{D}} f \, dxdy = \int_{\mathcal{D}} \Delta u \, dxdy = -\int_{\Gamma} \partial_\nu u \, ds$$

and therefore solution of

$$\Delta u = f \qquad \text{in } \mathcal{D}, \tag{7.1.6}$$

$$\partial_\nu u = h \qquad \text{on } \Gamma \tag{7.1.7}$$

does not exist unless

$$\int_{\mathcal{D}} f \, dxdy + \int_{\Gamma} h \, ds = 0. \tag{7.1.8}$$

One can prove that under assumption (7.1.8) the solution exists (sorry, we do not have analytic tools for this).

Theorem 7.1.2. *If* (7.1.5) *is fulfilled problem* (7.1.6)–(7.1.7) *has a solution iff* (7.1.8) *is fulfilled and this solution is unique modulo constant.*

7.2 Potential theory and around

7.2.1 Gauss formula and its applications

Consider Gauss formula

$$\int_{\Omega} \nabla \cdot \mathbf{U} \, dV = -\int_{\Sigma} \mathbf{U} \cdot \nu \, dS \tag{7.2.1}$$

where Ω is a bounded domain with the boundary Σ, dV is a volume lement, dS is an area element, $d\nu$ is a unit interior normal to Σ , $\mathbf{U}$ is a vector field and $\nabla \cdot \mathbf{U}$ its divergence.

Remark 7.2.1. Usually ν would denote an exterior normal and then there would be no sign $-$ on the right of (7.2.1).

Let us plug $\mathbf{U} = \nabla u$ into (7.2.1). We get

$$\int_\Omega \Delta u \, dV = -\int_\Sigma \frac{\partial u}{\partial \nu} \, dS \tag{7.2.2}$$

where $\Delta u = \nabla \cdot \nabla u$ is Laplacian of u, $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$ is a derivative of u in direction ν.

Let us plug $\mathbf{U} = \nabla u$ into (7.2.1). We get

$$\int_\Omega \big(w\Delta u + \nabla u \cdot \nabla w\big) \, dV = -\int_\Sigma w \nabla u \cdot \nu \, dS. \tag{7.2.3}$$

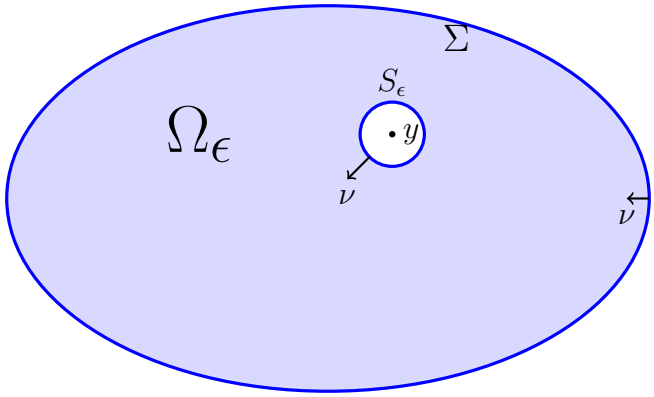
Antisymmetrizing (7.2.3) by u, w (permutting u, w and subtracting from original formula) we get

$$\int_\Omega \big(w\Delta u - u\Delta w\big) \, dV = \int_\Sigma \big(u\frac{\partial w}{\partial \nu} - w\frac{\partial u}{\partial \nu}\big) \, dS. \tag{7.2.4}$$

Consider now point y not on Σ and a function $w = |x-y|^{2-n}$ where n is a dimension (as $n = 2$ we take $w = -\log|x-y|$). As $y \in \Omega$ e cannot plug it in our formulae as w is singular at $x = y$. So we consider B_ϵ ball of small radius ϵ with a center y, $\Omega_\epsilon = \Omega \setminus B_\epsilon$ domain Ω with removed B_ϵ and $\Sigma_\epsilon = \Sigma \cup S_\epsilon$ its boundary, where S_ϵ is the sphere of radius ϵ with a center y. We get

$$\begin{aligned}\int_{\Omega_\epsilon} \big(w\Delta u - u\Delta w\big) \, dV = \int_\Sigma \big(u\frac{\partial w}{\partial \nu} - w\frac{\partial u}{\partial \nu}\big) \, dS \\ + \int_{S_\epsilon} \big(u\frac{\partial w}{\partial \nu} - w\frac{\partial u}{\partial \nu}\big) \, dS.\end{aligned} \tag{7.2.5}$$

Let us consider the last term in (7.2.5) as $\epsilon \to 0$. Note that on S_ϵ $w = \epsilon^{2-n}$ and therefore $|\int_{S_\epsilon} w\frac{\partial u}{\partial \nu} \, dS|$ does not exceed $C\epsilon^{n-1} \times \epsilon^{2-n} = C\epsilon$ so it tends to 0. We used that the area of S_ϵ is $\sigma_n \epsilon^{n-1}$ (where $\sigma_2 = 2\pi$, $\sigma_3 = 4\pi$).



For $\int_{S_\epsilon} u\frac{\partial w}{\partial \nu}\,dS$ we need more sophisticated arguments because $\frac{\partial w}{\partial \nu} = (n-2)\epsilon^{1-n}$ on S_ϵ (really, here ν is a radial direction away from y; as $n=2$ we get $-\epsilon^{-1}$). Consider

$$\int_{S_\epsilon} u\frac{\partial w}{\partial \nu}\,dS = \int_{S_\epsilon} \big(u-u(y)\big)\frac{\partial w}{\partial \nu}\,dS + u(y)\int_{S_\epsilon}\frac{\partial w}{\partial \nu}\,dS \tag{7.2.6}$$

(old trick to add and subtract constant). An absolute value of the first term does not exceed $C\max_{x\in S_\epsilon}|u(x)-u(y)| \to 0$ as $\epsilon \to 0$. The first term just equals $(n-2)\sigma_n u(y)$.

Exercise 7.2.1. Prove that $\Delta w = 0$ as $x \neq y$.

Therefore (7.2.5) becomes

$$\int_\Omega w\Delta u\,dV = \int_\Sigma \big(u\frac{\partial w}{\partial \nu} - w\frac{\partial u}{\partial \nu}\big)\,dS + (n-2)\sigma_n u(y)$$

where the term on the left is a usual improper integral and we have proven

Theorem 7.2.1. *As Ω is bounded domain with a boundary Σ, and $y \in \Omega$ (and in particular, $y \notin \Sigma$)*

$$u(y) = \int_\Omega G(x,y)\Delta u(x)\,dV + \int_\Sigma \big(-u(x)\frac{\partial G}{\partial \nu_x}(x,y) + G(x,y)\frac{\partial u}{\partial \nu}(x)\big)\,dS \tag{7.2.7}$$

with

$$G(x,y)=\begin{cases} -\dfrac{1}{(n-2)\sigma_n}|x-y|^{2-n} & n\neq 2,\\ -\dfrac{1}{4\pi}|x-y|^{-1} & n=3,\\ \dfrac{1}{2\pi}\log|x-y| & n=2,\\ \dfrac{1}{2}|x-y| & n=1. \end{cases} \tag{7.2.8}$$

7.2.2 Potential

Definition 7.2.1. $G(x,y)$ is a *potential*. In particular, as $n=3$ we get a *Coulomb potential* (aka *Newton potential*), and as $n=2$ we get a *logarithmic potential*.

Therefore if we know that $\Delta u = f$ in Ω, $u|_\Sigma = g$ and $\frac{\partial u}{\partial \nu}=h$ (which is *overdetermined problem* as only one boundary condition is in fact needed) we get

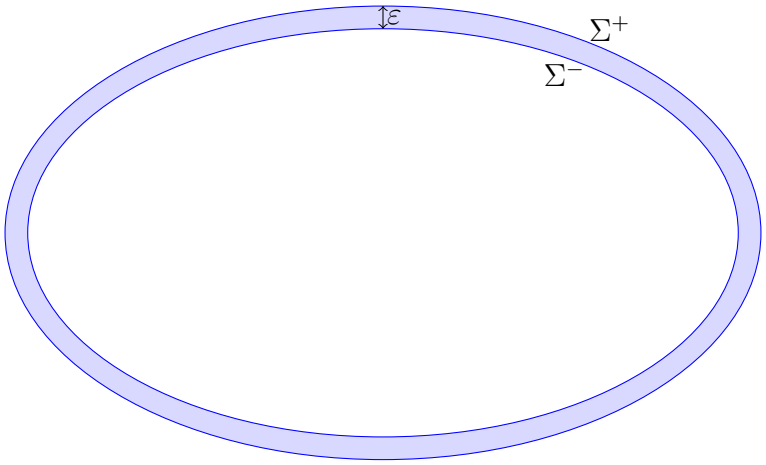
$$\begin{aligned} u(y)=&\int_\Omega G(x,y)f(x)\,dV \\ &+\int_\Sigma G(x,y)h(x)\,dS \\ &-\int_\Sigma g(x)\frac{\partial G}{\partial \nu_x}\,dS. \end{aligned} \tag{7.2.9}$$

Definition 7.2.2. (a) The first term in the right-hand expression is a potential created by a charge with density f,

(b) The second term in the right-hand expression is a potential created by a charge with surface density h (aka *single layer potential*),

(c) The third term in the right-hand expression is a double layer potential.

Remark 7.2.2. In fact we can consider two surfaces $\Sigma^-- = \Sigma$ and Σ^+ on the distance exactly ε from Σ on its "outer" side and density $-\varepsilon^{-1}g$ on Σ^- and εg on Σ^+ (we can extend g and as long as this extension is smooth it does not matter how), consider corresponding single layer potential, and then tend $\varepsilon\to+0$.



Remark 7.2.3. It is known that volume on n-dimensional ball is

$$\omega_n = \begin{cases} \dfrac{\pi^{n/2}}{(n/2)!} & \text{for even } \ n, \\ \dfrac{2^{(n+1)/2}\pi^{(n-1)/2}}{n!!} & \text{for odd } \ n, \end{cases}$$

where $m!! = m(m-2)(m-4)\cdots$ (the last factor 2 or 1 for even and odd m) and $\sigma_n = n\omega_n$.

7.2.3 Laplace equation in $\mathbb{R}^n$

This formula (7.2.9) as we expand Ω to the whole space becomes $\mathbb{R}^n$

Theorem 7.2.2. *Let $n \geq 3$. If f decays fast enough at infinity then*

$$u(y) = \int_{\mathbb{R}^n} G(x,y)\Delta u(x)\, dV \tag{7.2.10}$$

solves equation

$$\Delta u = f \tag{7.2.11}$$

with conditions at infinity

$$u = O(r^{2-n}), \tag{7.2.12}$$

$$\partial_r u = O(r^{1-n}) \qquad \text{as} \quad r = |x| \to \infty \tag{7.2.13}$$

7.2.4 Mean-value theorem

Consider Ω a ball of radius r with a center in y. Let u be a harmonic function (i.e. $\Delta u = 0$) in Ω. Then

(a) The first term in the right-hand expression of (7.2.7) is 0,

(b) Breaking the second term of (7.2.7) into

$$-\int_\Sigma u(x)\frac{\partial G}{\partial \nu_x}(x,y)\,dS + \int_\Sigma G(x,y)\frac{\partial u}{\partial \nu}(x)\,dS$$

in the second part we can drag factor $G(x,y)$ out of integral where remains $\int_\Sigma \frac{\partial u}{\partial \nu}\,dS = -\int_V \Delta u\,dV = 0$

(c) In the first part we can drag factor $\frac{\partial G}{\partial \nu_x}$ out of integral where remains $\int_\Sigma u\,dS$ and we get

$$u(y) = \frac{1}{\sigma_n r^{n-1}}\int_\Sigma u(x)\,dS. \tag{7.2.14}$$

So we proved statement (a) of

Theorem 7.2.3. *(a) If u is harmonic in the ball $B(y,r)$ of radius r then in its center y the value of u is a mean value of u over the sphere $S(y,r)$ bounding this ball.*

(b) If u is harmonic in the ball $B(y,r)$ of radius r then in its center the value of u is a mean value of u over this ball.

Proof. To prove (b) one should note from (a) that $\int_{S(y,\rho)} u(x)\,dS = \sigma_n \rho^{n-1} u(y)$ for any $\rho \in (0,r)$ and then

$$\int_{B(y,r)} u(x)\,dx = \int_0^r \Big(\int_{S(y,\rho)} u(x)\,dS\Big)\,dr = u(y)\times\sigma_n\int_0^r \rho^{n-1}\,d\rho$$

and that $\sigma_n \int_0^r \rho^{n-1}\,d\rho$ is a volume of $B(y,r)$. □

7.2.5 Maximum and minimum principle

Theorem 7.2.4. *If u is a harmonic function in the bounded domain Ω with a boundary Σ then*

(a) $\max_\Omega u = \max_\Sigma u$ *and* $\min_\Omega u = \min_\Sigma u$*;*

(b) If Ω is connected and $u(y) = \max_\Omega u$ (or $u(y) = \min_\Omega u$) for some inner point y then $u = \text{const}$.

Proof. Assume that $u(y) = \max_\Omega u$ for some inner point y. Consider the largest ball $B(y, r)$ containing in Ω. By mean-value theorem $u(y)$ is a mean-value of u over this ball but it is also a maximal value of u in this ball and therefore $u(x) = u(y)$ in this ball. Then any point on $S(y, r)$ could be used as a "new y" and we can continue. Eventually each point of x which could be connected to y by a continuous curve inside of Ω will be covered and then in this point $u(x) = u(y)$. So, in the connected component of Ω containing y $u = u(y)$ and it will be true on its border. Then $\max_\Sigma u \geq u(y) = \max_\Omega u$; but $\max_\Omega u \geq \max_\Sigma u$ and therefore (a) has been proven.

It also proves (b) as now we assume that Ω is connected. □

7.2.6 Unicity for Dirichlet Problem

Theorem 7.2.5. *(a) Let Ω be a bounded domain. Then solution of Dirichlet problem in Ω for Laplace equation is unique.*

(b) Let Ω be an unbounded domain. Then solution of Dirichlet problem in Ω for Laplace equation is unique under condition at infinity: $|u| \to 0$ as $|x| \to \infty$.

Proof. Consider u solving $\Delta u = 0$, $u|_\Sigma = 0$) and satisfying condition at infinity in (b)).

(a) Due to maximum and minimum principle

$$0 = \min_\Sigma u \leq u(y) \leq \max_\Sigma u = 0 \qquad \text{for } \ y \in \Omega.$$

(b) Consider ball $B(0, R)$ and domain $\Omega_R = \Omega \cap B(0, R)$. Its boundary is $\Sigma_R := (\Sigma \cap B(0, R)) \cup (\Omega \cap S(0, R))$. Then as $R \geq |y|$

$$\min_{\Sigma_R} u \leq u(y) \leq \max_{\Sigma_R} u$$

but

$$\min_{\Sigma_R} u = \min\Big(\min_{\Sigma\cap B(0,R)} u, \min_{\Omega\cap S(0,R)} u\Big) \tag{7.2.15}$$

if $\Sigma \cap B(0,R) \neq \emptyset$, otherwise the we get just $\min_{\Omega\cap S(0,R)} u$. However $\min_{\Omega\cap S(0,R)} u \to 0$ as $R \to \infty$ due to condition to infinity. Also $\min_{\Sigma\cap B(0,R)} u = 0$ and therefore the left-hand expression in (7.2.15) tends to 0 as $R \to \infty$.

Similarly the right-hand expression in (7.2.15) tends to 0 as $R \to \infty$ and we have $0 \le u(y) \le 0$. □

7.3 Green function

7.3.1 Newton shell theorem

Let $n \ge 3$. Consider spherically symmetric density f (thus depending on $r = (x_1^2 + x_2^2 + \ldots + x_n^2)^{\frac{1}{2}}$ only). Then it creates a density which is also spherically symmetric.

Assume first that $f = 0$ in $B(0,R)$ and consider u in $B(0,R)$. Here u must be harmonic and then due to mean-value theorem $u = u(0)$ is constant (in particular $\nabla u = 0$). More precisely

$$u = -\frac{1}{n-2}\int_R^\infty \rho f(\rho)\, d\rho \tag{7.3.1}$$

where we replaced lower limit 0 by R since $f(\rho) = 0$ as $r < R$.

Assume now that $f = 0$ in $\{r \ge R\}$ and consider u there. Then $u = r^{2-n}A + B$.

Exercise 7.3.1. Prove that if f is spherically symmetric and $f = 0$ in $\{r \ge R\}$ then $u = r^{2-n}A + B$ there.

To have potential at infinity equal to 0 we must take $B = 0$. Further plug it into (**??**) with $\Omega = B(0,R)$. The left-hand expression becomes the total charge $\int_{B(0,R)} f(x)\, dV$ while the right-hand expression becomes $-(n-2)\sigma_n A$ and therefore

$$u = -\frac{1}{(n-2)\sigma_n} r^{2-n}\int_{B(0,R)} f\, dx = -\frac{1}{n-2} r^{2-n}\int_0^R \rho^{n-1} f(\rho)\, d\rho. \tag{7.3.2}$$

Then in the general case we to calculate u we break $f = f_1 + f_2$ where $f_1 = 0$ for $r \geq R$ and $f_2 = 0$ for $r \leq R$ and then calculate u as $r = R$ and finally set $R = r$:

$$u = -\frac{1}{n-2}\int_r^\infty \rho f(\rho)\, d\rho - \frac{1}{n-2} r^{2-n} \int_0^r f(\rho)\, d\rho. \tag{7.3.3}$$

In particular, if

Theorem 7.3.1. *Let f be spherically symmetric. Then*

(a) If $f = 0$ as $r \geq R$ then u coincides with a potential created by the same mass was concentrated in the origin.

(b) If $f = 0$ as $r \leq R$ then $u =$ const there.

Remark 7.3.1. Statement (b) is often expressed by a catch phrase "There is no gravity in the cavity". In particular if $f(r) =$ const as $R_1 \leq r \leq R_2$ and $f(r) = 0$ as $r \leq R_1$ or $r \geq R_2$, then all calculations are easy. Amazingly, the same is true for a shell between two proportional ellipsoids (Ivory theorem).

7.3.2 Green function. I

Recall (7.2.7)–(7.2.8)

$$u(y) = \int_\Omega G^0(x,y)\Delta u(x)\, dV + \int_\Sigma \bigl(-u(x)\frac{\partial G^0}{\partial \nu_x}(x,y) + G^0(x,y)\frac{\partial u}{\partial \nu}(x)\bigr)\, dS \tag{7.2.7}$$

with

$$G^0(x,y) = \begin{cases} -\dfrac{1}{(n-2)\sigma_n}|x-y|^{2-n} & n \neq 2, \\ -\dfrac{1}{4\pi}|x-y|^{-1} & n = 3, \\ \dfrac{1}{2\pi}\log|x-y| & n = 2, \\ \dfrac{1}{2}|x-y| & n = 1 \end{cases} \tag{7.2.8}$$

where we changed notation G^0 instead of G as we will redefine G later.

Let $g(x,y)$ be solution to problem

$$\Delta_x g(x,y) = 0 \qquad \text{in } \Omega, \tag{7.3.4}$$

$$g(x,y) = -G^0(x,y) \qquad \text{as } x \in \Sigma \tag{7.3.5}$$

and condition $u \to 0$ as $|x| \to \infty$ if Ω is unbounded. In virtue of (7.3.4) and (7.2.5)

$$0 = \int_\Omega g(x,y)\Delta u(x)\,dV + \int_\Sigma \bigl(-u(x)\frac{\partial g}{\partial \nu_x}(x,y) + g(x,y)\frac{\partial u}{\partial \nu}(x)\bigr)\,dS. \tag{7.3.6}$$

Adding to (7.2.7) we get

$$u(y) = \int_\Omega G(x,y)\Delta u(x)\,dV + \int_\Sigma \bigl(-u(x)\frac{\partial G}{\partial \nu_x}(x,y) + G(x,y)\frac{\partial u}{\partial \nu}(x)\bigr)\,dS \tag{7.3.7}$$

with

$$G(x,y) := G^0(x,y) + g(x,y). \tag{7.3.8}$$

So far we have not used (7.3.5) which is equivalent to $G(x,y) = 0$ as $x \in \Sigma$. But then

$$u(y) = \int_\Omega G(x,y)\Delta u(x)\,dV - \int_\Sigma \frac{\partial G}{\partial \nu_x}(x,y)u(x)\,dS$$

and therefore

$$u(y) = \int_\Omega G(x,y)f(x)\,dV - \int_\Sigma \frac{\partial G}{\partial \nu_x}(x,y)\phi(x)\,dS \tag{7.3.9}$$

is a solution to

$$\Delta u = f \qquad \text{in } \Omega, \tag{7.3.10}$$

$$u = \phi \qquad \text{on } \Sigma. \tag{7.3.11}$$

Similarly if we replace (7.3.5) by Robin boundary condition

$$\big(\frac{\partial g}{\partial \nu_x} - \alpha g\big)(x,y) = -\big(\frac{\partial G^0}{\partial \nu_x} - \alpha G^0\big)(x,y) \qquad \text{as} \quad x \in \Sigma \tag{7.3.12}$$

we get $\big(\frac{\partial G}{\partial \nu_x} - \alpha G\big)(x,y) = 0$ as $x \in \Sigma$ and rewriting (7.3.7) as

$$u(y) = \int_\Omega G(x,y)\Delta u(x)\,dV + \int_\Sigma \big(-u(x)\big[\frac{\partial G}{\partial \nu_x} - \alpha G\big](x,y) + G(x,y)\big[\frac{\partial u}{\partial \nu} - \alpha u\big]\big)\,dS \tag{7.3.13}$$

we get

$$u(y) = \int_\Omega G(x,y)f(x)\,dV + \int_\Sigma G(x,y)\psi(x)\,dS \tag{7.3.14}$$

for solution of the (7.3.10) with the boundary condition

$$\frac{\partial u}{\partial \nu} - \alpha u = \psi \qquad \text{on} \quad \Sigma. \tag{7.3.15}$$

Finally, if we take g satisfying mixed boundary condition

$$g(x,y) = -G^0(x,y) \qquad \text{as} \quad x \in \Sigma', \tag{7.3.16}$$

$$\big(\frac{\partial g}{\partial \nu_x} - \alpha g\big)(x,y) = -\big(\frac{\partial G^0}{\partial \nu_x} - \alpha G^0\big)(x,y) \qquad \text{as} \quad x \in \Sigma'' \tag{7.3.17}$$

we get

$$u(y) = \int_\Omega G(x,y)f(x)\,dV + \int_{\Sigma'} G(x,y)\psi(x)\,dS - \int_{\Sigma''} \frac{\partial G(x,y)}{\partial \nu_x}\phi(x)\,dS \tag{7.3.18}$$

for solution of the (7.3.10) with the boundary condition

$$u = \phi \qquad \text{on} \quad \Sigma', \tag{7.3.19}$$

$$\frac{\partial u}{\partial \nu} - \alpha u = \psi \qquad \text{on} \quad \Sigma'', \tag{7.3.20}$$

where $\Sigma = \Sigma' \cup \Sigma''$ and $\Sigma' \cap \Sigma'' = \emptyset$.

Definition 7.3.1. $G(x,y)$ (defined for corresponding boundary problem) is called *Green's function*.

Remark 7.3.2. It is similar (by usage) to heat kernel $\frac{1}{2\sqrt{kt}}e^{-\frac{|x-y|^2}{4kt}}$.

One can prove

Theorem 7.3.2.

$$G(x,y) = G(y,x). \tag{7.3.21}$$

7.3.3 Green function. II

Consider now purely Neumann problem in the connected domain. We cannot solve $\Delta g = 0$ with the boundary condition $\frac{\partial g}{\partial \nu_x} = -\frac{\partial G^0}{\partial \nu_x}$ as such problem requires one solvability condition to the right-hand expression and boundary value

$$\int_\Omega f\,dV + \int_\Sigma \psi\,dS = 0 \tag{7.3.22}$$

and this condition fails as $\int_\Sigma \frac{\partial G^0}{\partial \nu_x}\,dS \neq 0$ (one can prove it).

To fix it we consider g satisfying

$$\Delta_x g(x,y) = c \qquad \text{in } \Omega, \tag{7.3.23}$$

$$\frac{\partial g}{\partial \nu_x}(x,y) = -\frac{\partial G^0}{\partial \nu_x}(x,y) \qquad \text{as } x \in \Sigma \tag{7.3.24}$$

with unknown constant c; chosing it correctly one can satisfy solvability condition. Then

$$u(y) = \int_\Omega G(x,y)f(x)\,dV + \int_\Sigma G(x,y)\psi(x)\,dS + c\int_V u\,dx$$

and therefore

$$u(y) = \int_\Omega G(x,y)f(x)\,dV + \int_\Sigma G(x,y)\psi(x)\,dS + C \tag{7.3.25}$$

gives us solution if it exists (and it exists under solvability condition (7.3.24)). This solution is defined up to a constant.

7.3.4 Problems to Chapter 7

Problem 7.3.1. (a) Find the solutions that depend only on r of the equation

$$\Delta u := u_{xx} + u_{yy} = 0.$$

(b) Find the solutions that depend only on ρ of the equation

$$\Delta u := u_{xx} + u_{yy} + u_{zz} = 0.$$

(c) (bonus) In n-dimensional case prove that if $u = u(r)$ with $r = (x_1^2 + x_2^2 + \ldots + x_n^2)^{\frac{1}{2}}$ then

$$\Delta u = u_{rr} + \frac{n-1}{r} u_r = 0. \tag{7.3.26}$$

(d) (bonus) In n-dimensional case prove ($n \neq 2$) that $u = u(r)$ satisfies Laplace equation as $x \neq 0$ iff $u = Ar^{2-n} + B$.

Problem 7.3.2. Using the proof of mean value theorem (see Subsection **??**) prove that if $\Delta u \geq 0$ in $B(y, r)$ then

(a) $u(y)$ does not exceed the mean value of u over the sphere $S(y, r)$ bounding this ball:

$$u(y) \leq \frac{1}{\sigma_n r^{n-1}} \int_{S(y,r)} u \, dS. \tag{7.3.27}$$

(b) $u(y)$ does not exceed the mean value of u over this ball $B(y, r)$:

$$u(y) \leq \frac{1}{\omega_n r^n} \int_{B(y,r)} u \, dV. \tag{7.3.28}$$

(c) Formulate similar statements for functions satisfying $\Delta u \leq 0$ (in the next problem we refer to them as (a)' and (b)').

Problem 7.3.3. (a) Functions having property (a) (or (b) does not matter) of the previous problem are called *subharmonic.*

(b) Functions having property (a)'(or (b)' does not matter) are called *superharmonic.*

Problem 7.3.4. (a) Using the proof of maximum principle prove the maximum principle for subharmonic functions and minimum principle for superharmonic functions.

(b) Show that minimum principle for subharmonic functions and maximum principle for superharmonic functions do not hold (*Hint*: construct counterexamples with $f = f(r)$).

(c) Prove that if u, v, w are respectively harmonic, subharmonic and superharmonic functions in the bounded domain Ω, coinciding on its boundary ($u|_\Sigma = v|_\Sigma = w|_\Sigma$) then in $w \geq u \geq v$ in Ω.

Problem 7.3.5 (bonus). Using Newton shell theorem (see Subsection 7.3.1) prove that if Earth was a homogeneous solid ball then the gravity pull inside of it would be proportional to the distance to the center.

Problem 7.3.6. Find function u harmonic in $\{x^2+y^2+z^2 \leq 1\}$ and coinciding with $g = z^3$ as $x^2 + y^2 + z^2 = 1$.

Hint. According to Section 8.1 solution must be a harmonic polynomial of degree 3 and it should depend only on $x^2 + y^2 + z^2$ and z (Explain why). The only way to achive it (and still coincide with g on $\{x^2 + y^2 + z^2 = 1\}$) is to find

$$u = z^3 + az(1 - x^2 - y^2 - z^2)$$

with unknown coefficient A.

Problem 7.3.7. Apply method of descent described in Subsection 9.1.4 but to Laplace equation in $\mathbb{R}^2$ and starting from Coulomb potential in $3D$

$$U_3(x, y, z) = -\frac{1}{4\pi}\big(x^2 + y^2 + z^2\big)^{-\frac{1}{2}}, \tag{7.3.29}$$

derive logarithmic potential in $2D$

$$U_2(x, y, z) = \frac{1}{2\pi}\log\big(x^2 + y^2\big)^{\frac{1}{2}}, \tag{7.3.30}$$

Hint. You will need to calculate diverging integral $\int_0^\infty U_3(x, y, z)$. Instead consider $\int_0^N U_3(x, y, z)$, subtract constant (f.e. $\int_0^N U_3(1, 0, z)$) and then tend $N \to \infty$.

Problem 7.3.8. Using method of reflection (studied earlier for different equations) construct Green function for

(a) Dirichlet problem

(b) Neumann problem

for Laplace equation in

a. half-plane
b. half-space

as we know that in the whole plane and space they are just potentials

$$\frac{1}{2\pi}\log\bigl((x_1-y_1)^2+(x_2-y_2)^2\bigr)^{\frac{1}{2}}, \tag{7.3.31}$$

$$-\frac{1}{4\pi}\bigl((x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2\bigr)^{-\frac{1}{2}} \tag{7.3.32}$$

respectively.

Problem 7.3.9. Apply method of descent but now looking for stationary solution of $-\Delta u = f(x_1,x_2,x_3)$ instead of non-stationary solution of

$$\begin{aligned} &u_{tt}-\Delta u = f(x_1,x_2,x_3),\\ &u|_{t=0} = g(x_1,x_2,x_3),\\ &u_t|_{t=0} = h(x_1,x_2,x_3) \end{aligned}$$

start from Kirchhoff formula (9.1.12) and derive for $n=3$ (7.2.10) with $G(x,y)$ equal to (7.3.32) here.

Chapter 8

Separation of variables

8.1 Separation of variable in spherical coordinates

8.1.1 Laplace equation in the ball

Consider Laplace equation in spherical coordinates defined by (6.3.7)–(6.3.86.3.8)

$$\Delta = \partial_\rho^2 + \frac{2}{\rho}\partial_\rho + \frac{1}{\rho^2}\Lambda \tag{8.1.1}$$

with

$$\Lambda := \left(\partial_\phi^2 + \cot(\phi)\partial_\phi\right) + \frac{1}{\sin^2(\phi)}\partial_\theta^2. \tag{8.1.2}$$

Let us plug $u = P(\rho)Y(\phi,\theta)$ into $\Delta u = 0$:

$$P''(\rho)Y(\phi,\theta) + \frac{2}{\rho}P'(\rho)Y(\phi,\theta) + \frac{1}{\rho^2}P(\rho)\Lambda Y(\phi,\theta) = 0$$

which could be rewritten as

$$\frac{\rho^2 P''(\rho) + \rho P'(\rho)}{P(\rho)} + \frac{\Lambda Y(\phi,\theta)}{Y(\phi,\theta)} = 0$$

and since the first term depends only on ρ and the second only on ϕ,θ we conclude that both are constant:

$$\rho^2 P'' + 2\rho P' = \lambda P, \tag{8.1.3}$$

$$\Lambda Y(\phi,\theta) = -\lambda Y(\phi,\theta). \tag{8.1.4}$$

The first equation is of Euler type and it has solutions $P := \rho^l$ iff $\lambda = l(l+1)$. However if we are considering ball, solution *must* be infinitely smooth in its center due to some general properties of Laplace equation and this is possible iff $l = 0, 1, 2, \dots$ and in this case u must be a polynomial of (x, y, z).

Definition 8.1.1. Such polynomials are called *harmonic polynomials*.

One can prove

Theorem 8.1.1. *Harmonic polynomials of degree l form $(2l+1)$-dimensional space.*

l	Basis in the space of harmonic polynomials
0	1
1	x, y, z
2	xy, xz, yz, x^2-y^2, x^2-z^2
3	x^3-3xz^2, y^3-3yz^2, xz^2-xy^2, yz^2-yx^2, xyz, x^2z-y^2z, $2z^3-3x^2z-3y^2z$

Table 8.1

Then

$$\Lambda Y(\phi,\theta) = -l(l+1)Y(\phi,\theta). \tag{8.1.5}$$

Definition 8.1.2. Solutions of $\Lambda v = 0$ are called *spherical harmonics*.

To find spherical harmonics we apply method of separation of variables again: $Y(\phi,\theta) = \Phi(\phi)\Theta(\theta)$. Recalling (8.1.2) we see that

$$\underbrace{\frac{\sin^2(\phi)\big(\Phi'' + \cot(\phi)\Phi'\big)}{\Phi} + l(l+1)\sin^2(\phi)} + \underbrace{\frac{\Theta''}{\Theta}} = 0. \tag{8.1.6}$$

Therefore again both terms in the left-hand expression must be constant:

$$\sin^2(\phi)\big(\Phi'' + \cot(\phi)\Phi'\big) = -\big(l(l+1)\sin^2(\phi) - \mu\big)\Phi, \tag{8.1.7}$$

$$\Theta'' = -\mu\Theta. \tag{8.1.8}$$

The second equation is easy, and keeping in mind 2π-periodicity of Θ we get $\mu = m^2$ and $\Theta = e^{-im\phi}$ with $m = -l, 1-l, \dots, l-1, l$ (for $|m| > l$ we would not get a polynomial).

Therefore (8.1.7) becomes

$$\sin^2(\phi)\Phi'' + 2\sin(\phi)\cos(\phi)\Phi' = -\big(l(l+1)\sin^2(\phi) - m^2\big)\Phi, \tag{8.1.9}$$

One can prove that Φ is a polynomial of $\cos(\phi)$:

Theorem 8.1.2. $\Phi(\phi) = L(\cos(\phi))$.

Such polynomials are called Legendre polynomials as $m = 0$ and Associated Legendre polynomials as $m \neq 0$.

Therefore we number spherical harmonics by l, m: we have Y_{lm} with $l = 0, 1, \dots$ and $m = -l, 1-l, \dots, l-1, m$.

Remark 8.1.1. (a) We are talking now about spherical harmonics with separated ϕ, θ; linear combination of spherical harmonics with the same l but different m is again a spherical harmonic albeit without separated ϕ, θ.

(b) Such harmonics for a basis in the linear space of spherical harmonics with fixed l;

(c) Choice of the polar axis z matters here: selecting other direction bring us a different basis.

8.1.2 Laplace equation outside of the ball

Consider solutions of the Laplace equation for $\rho > 0$ decaying as $\rho \to \infty$. Since spherical harmonics are already defined we have $\lambda = -l(l+1)$ and then $P = \rho^k$ with $k < 0$ satisfying $k(k+1) = l(l+1)$ which implies that $k = -1 - l$. In particular we get from Table 8.1

l	Basis in the space of homogeneous harmonic functions
0	1
1	x/ρ^3, y/ρ^3, z/ρ^3
2	xy/ρ^5, xz/ρ^5, yz/ρ^5, $(x^2-y^2)/\rho^5$, $(x^2-z^2)/\rho^5$

Table 8.2

with $\rho = (x^2 + y^2 + z^2)^{1/2}$.

8.1.3 Applications to the theory of Hydrogen atom

Spherical harmonics play crucial role in the mathematical theory of Hydrogen-like atoms (with 1-electron):

$$-\frac{\hbar^2}{2\mu}\Delta\Psi - \frac{Ze^2}{\rho}\Psi = E\Psi. \tag{8.1.10}$$

Here $\hbar$ is a Planck constant, $-Ze$ is the charge of the nucleus, e is the charge of electron, μ is its mass, $E<0$ is an energy level.

After separation of variables we get $\Psi = P(\rho)Y_{lm}(\phi,\theta)$ with P satisfying

$$-P'' - \frac{2}{\rho}P' - \frac{\eta}{\rho}P + \frac{l(l+1)}{\rho^2}P = -\alpha^2 P \tag{8.1.11}$$

with $\eta = 2\mu Ze^2\hbar^{-2}$, $\alpha = (-2E\mu)^{\frac{1}{2}}\hbar^{-1}$.

Solutions are found in the form of $e^{-\alpha\rho}\rho^l Q(\rho)$ where $Q(\rho)$ is a polynomial satisfying

$$\rho Q'' + (2l+2-2\alpha\rho) + (\eta - 2\alpha)\rho - 2\alpha l)Q = 0 \tag{8.1.12}$$

It is known that such solution (polynomial of degree exactly $n-l-1$, $n = l+1, l+2, \dots$) exists and is unique (up to a multiplication by a constant) iff $2\alpha(n-1) + 2\alpha - \eta = 0$ i.e. $\alpha = \frac{\eta}{2n}$ and also $l \le n-1$. Such polynomials are called Laguerre polynomials.

Therefore $E_n = -\frac{\kappa}{n^2}$ (one can calculate κ) and has multiplicity $\sum_{l=0}^{n-1}\sum_{m=-l}^{l} 1 = \sum_{l=0}^{n-1}(2l+1) = \frac{1}{2}n(n+1)$.

Remark 8.1.2. We see that E_n are very degenerate. Different perturbations decrease or remove degenerations splitting these eigenvalues into clusters of less degenerate or non-degenerate eigenvalues.

8.1.4 Applications to wave equation in the ball

Consider now 3D-wave equation in the ball

$$u_{tt} - c^2\Delta u = 0 \qquad \rho \le a \tag{8.1.13}$$

with Dirichlet or Neumann boundary conditions. Separating t and the spatial variables $u = T(t)v(x,y,z)$ we get *Helmholtz equation*

$$\Delta v = -\lambda v \qquad \rho \le a \tag{8.1.14}$$

with the same boundary condition and

$$T'' = -c^2\lambda T \tag{8.1.15}$$

Separating ρ from spherical variables ϕ, θ we get

$$\underbrace{\frac{\rho^2 P'' + 2\rho P' + \lambda P}{\rho^2 P}} + \underbrace{\frac{\Lambda Y}{Y}} = 0$$

and therefore both selected expressions must be μ and $-\mu$ respectively. So $Y(\phi, \theta)$ is a spherical harmonic and $\mu = l(l+1)$. Then

$$\rho^2 P'' + 2\rho P' + (\lambda\rho^2 - l(l+1))P = 0. \tag{8.1.16}$$

As $\lambda = 1$ Ssolutions are *spherical Bessel functions* j_l and y_l which are called spherical Bessel functions of the 1st kind and of the 2nd kind, respectively, and the former are regular at 0.

So $P = j_l(\rho\sqrt{\lambda})$ and for u to satisfy Dirichlet or Neumann boundary conditions we need to impose the same conditions to P resulting in

$$j_l(a\sqrt{\lambda}) = 0, \tag{8.1.17}$$

$$j_l'(a\sqrt{\lambda}) = 0, \tag{8.1.18}$$

and then $\lambda = z_{l,n}^2 a^{-2}$ and $\lambda = w_{l,n}^2 a^{-2}$ respectively where $z_{l,n}$ and $w_{l,n}$ are n-th zero of j_l or j_l' respectively.

8.2 Separation of variable in polar and cylindrical coordinates

8.2.1 Helmholtz equation in the disk

Consider Helmholtz equation in the disk (recall that such equation is obtained from wave equation after separation of t from spatial variables):

$$v_{rr} + r^{-1}v_r - r^{-2}v_{\theta\theta} = -\lambda v \qquad r \le a. \tag{8.2.1}$$

Separating variables $v = R(r)\Phi(\phi)$ we arrive to

$$\frac{r^2 R'' + rR' + \lambda r^2 R}{R} + \frac{Phi''}{|Phi} = 0$$

and therefore

$$\Phi'' = -\mu Phi, \tag{8.2.2}$$

$$r^2R'' + rR' + (\lambda r^2 - \mu)R = 0 \tag{8.2.3}$$

and $\mu = -l^2$, $\Phi = e^{\pm in\theta}$ and

$$r^2R'' + rR' + (\lambda r^2 - l^2)R = 0. \tag{8.2.4}$$

As $\lambda = 1$ it is Bessel equation and solutions are Bessel functions J_l and Y_l which which are called Bessel functions of the 1st kind and of the 2nd kind, respectively, and the former are regular at 0. Therefore $R = J_l(r\sqrt{\lambda})$ and plugging into Dirichlet or Neiumann boundary conditions we get respectively

$$J_l(a\sqrt{\lambda}) = 0, \tag{8.2.5}$$

$$J_l'(a\sqrt{\lambda}) = 0, \tag{8.2.6}$$

and then $\lambda = z_{l,n}^2 a^{-2}$ and $\lambda = w_{l,n}^2 a^{-2}$ respectively where $z_{l,n}$ and $w_{l,n}$ are n-th zero of J_l or J_l' respectively.

Remark 8.2.1. Bessel functions are elementary only for half-integer $l = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$ when they are related to spherical Bessel functions.

8.2.2 Helmholtz equation in the cylinder

Consider Laplace equation in the cylinder $\{r \le a, 0 \le z \le b\}$ with homogeneous Dirichlet (or Neumann, etc) boundary conditions:

$$u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} + u_{zz} = -\omega^2 u, \tag{8.2.7}$$

$$u|_{z=0} = u|_{z=b} = 0, \tag{8.2.8}$$

$$u|_{r=a} = 0. \tag{8.2.9}$$

Separating Z from r, θ $u = Z(z)v(r,\theta)$ we get

$$\frac{\Lambda v}{v} + \frac{Z''}{Z} = -\omega^2 v$$

and then $Z'' = -\beta Z$, and $\Lambda v := v_{rr} + r^{-1}v_r + r^{-2}v_{\theta\theta} = -\lambda v$ with $\lambda = \omega^2 - \beta$ and from boundary conditions to Z we have $\beta = \pi^2 m^2 b^{-2}$ and separating r, ϕ: $v = R(r)\Phi(\phi)$ we arrive like in the previous Subsection to (8.2.4). One can prove that there are no nontrivial solutions as $\lambda \le 0$ and therefore $\lambda > 0$ and everything is basically reduced to the previous Subsection.

Exercise 8.2.1. Do it in detail.

8.3 Laplace equation in the cylinder

Consider Laplace equation in the cylinder $\{r \le a, 0 \le z \le b\}$ with homogeneous Dirichlet (or Neumann, etc) boundary conditions on the top and bottom leads and non-homogeneous condition on the lateral boundary:

$$u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} + u_{zz} = -\omega^2 u, \tag{8.3.1}$$
$$u|_{z=0} = u|_{z=b} = 0, \tag{8.3.2}$$
$$u|_{r=a} = g(z,\theta). \tag{8.3.3}$$

Separating Z from r,θ Separating Z from r,θ $u = Z(z)v(r,\theta)$ we get

$$\frac{\Lambda v}{v} + \frac{Z''}{Z} = 0$$

and then $Z'' = -\beta Z$, and $\Lambda v := v_{rr} + r^{-1}v_r + (-\beta + r^{-2}v_{\theta\theta}) = 0$. and from boundary conditions to Z we have $\beta = \pi^2 m^2 b^{-2}$ and separating r,ϕ: $v = R(r)\Phi(\phi)$ we arrive like in the previous Subsection to

$$r^2R'' + rR' + (-\beta r^2 - l^2)R = 0. \tag{8.3.4}$$

However now $\beta > 0$ and we do not need to satisfy homogeneous condition as $r = a$ (on the contrary, we do not want it to have non-trivial solutions.

Then we use *modified Bessel functions* I_l and K_l and $R = CI_l(r\sqrt{\beta})$.

8.A Separation of variable in elliptic and parabolic coordinates

Recall that elliptic and parabolic coordinates, and also elliptic cylindrical and parabolic cylindrical coordinates are described in Subsection 6.3.4.

8.A.1 Laplace equation in the ellipse

Consider Laplace equation in the elliptic coordinates (μ,ν):

$$\Delta u = \frac{1}{c^2\big(\sinh^2(\mu) + \sin^2(\nu)\big)}(\partial_\mu^2 + \partial_\nu^2)u = 0 \tag{8.A.1}$$

which is obviously equivalent to

$$(\partial_\mu^2 + \partial_\nu^2)u = 0; \tag{8.A.2}$$

separating variables $u = M(\mu)N(\nu)$ we arrive to $M'' = \alpha M$, $N'' = -\alpha N$ with periodic boundary conditions for N; so $N = \cos(n\nu), \sin(n\nu)$, $\alpha = n^2$ and $N = A\mathit{cosh}(n\mu) + B\sinh(n\mu)$. So

$$\begin{aligned} u_n = A\cosh(n\mu)\cos(n\nu) + B\cosh(n\mu)\sin(n\nu)+ \\ C\sinh(n\mu)\cos(n\nu) + D\sinh(n\mu)\sin(n\nu) \end{aligned} \tag{8.A.3}$$

as $n = 1, 2, \dots$ and similarly

$$u_0 = A + Bu. \tag{8.A.4}$$

8.A.2 Laplace equation in the parabolic annulus

Consider Laplace equation in the parabolic coordinates (σ, τ):

$$\Delta u = \frac{1}{\sigma^2 + \tau^2}(\partial_\sigma^2 + \partial_\tau^2) = 0. \tag{8.A.5}$$

Then again formulae (8.A.3) and (8.A.4) work but with (μ, ν) replaced by (σ, τ).

8.A.3 Helmholtz equation in the ellipse

Consider Helmholtz equation in the elliptic coordinates (μ, ν):

$$\Delta u = \frac{1}{c^2\big(\sinh^2(\mu) + \sin^2(\nu)\big)}(\partial_\mu^2 + \partial_\nu^2)u = -k^2u \tag{8.A.6}$$

which can be rewritten as

$$\Big(\partial_\mu^2 + k^2c^2\sinh^2(\mu) + \partial_\nu^2 + \sin^2(\nu)\Big)u = 0 \tag{8.A.7}$$

and separating variables we get

$$M'' + k^2c^2\big(\sinh^2(\mu) + \lambda\big)M = 0, \tag{8.A.8}$$

$$N'' + k^2c^2\big(\sin^2(\nu) - \lambda\big)N = 0. \tag{8.A.9}$$

8.A.4 Helmholtz equation in the parabolic annulus

Consider Helmholtz equation in the parabolic coordinates (σ, τ):

$$\Delta u = \frac{1}{\sigma^2+\tau^2}(\partial_\sigma^2 + \partial_\tau^2) = -k^2 u \tag{8.A.10}$$

which can be rewritten as

$$\Big(\partial_\sigma^2 + k^2\sigma^2 + \partial_\tau^2 + k^2\tau^2\Big)u = 0 \tag{8.A.11}$$

and separating variables we get

$$S'' + k^2\big(\sigma^2 + \lambda\big)S = 0, \tag{8.A.12}$$

$$N'' + k^2\big(\tau^2 - \lambda\big)T = 0. \tag{8.A.13}$$

Exercise 8.A.1. Consider Laplace and Helmholtz equations in elliptic cylindrical and parabolic cylindrical coordinates.

Chapter 9

Wave equation

9.1 Wave equation in dimensions 3 and 2

9.1.1 3D-Wave equation: special case

Consider Cauchy problem for 3-dimensional wave equation

$$u_{tt} - c^2\Delta u = f, \tag{9.1.1}$$

$$u|_{t=0} = g, \tag{9.1.2}$$

$$u_t|_{t=0} = h. \tag{9.1.3}$$

Assume first that $f = g = 0$. We claim that in this case as $t > 0$

$$u(x,t) = \frac{1}{4\pi c^2 t}\iint_{S(x,ct)} h(y)\,d\sigma \tag{9.1.4}$$

where we integrate along sphere $S(x,ct)$ with a center at x and radius ct; $d\sigma$ is an area element.

Let us prove (9.1.4) first as $h(x) = e^{ix\cdot\xi}$ with $\xi \in \mathbb{R}^3 \setminus 0$; we use the standard notation $x\cdot\xi = x_1\xi_1 + x_2\xi_2 + x_3\xi_3$. In this case

$$u(x,t) = e^{ix\cdot\xi}c^{-1}|\xi|^{-1}\sin(ct|\xi|) \tag{9.1.5}$$

is obviously a solution to Cauchy problem (9.1.1)–(9.1.3).

On the other hand, the right-hand expression of (9.1.4) becomes

$$\frac{1}{4\pi c^2 t}\iint_{S(x,ct)} e^{iy\cdot\xi}\,d\sigma = \frac{1}{4\pi c^2 t}e^{ix\cdot\xi}\iint_{S(0,ct)} e^{iz\cdot\xi}\,d\sigma$$

where we changed variable $y = x + z$ with z running $S(0, ct)$ (sphere with a center at 0) and we need to calculate integral in the right-hand expression. Let us select coordinate system in which $\xi = (0, 0, \omega)$ with $\omega = |\xi|$ and introduce corresponding spherical coordinates (ρ, ϕ, θ); then on $S(0, ct)$ $\rho = ct$, $z \cdot \xi = z_3\omega = ct\omega\cos(\phi)$ and $d\sigma = c^2t^2\sin(\phi)d\phi d\theta$; so integral becomes

$$c^2t^2 \int_0^{\pi} e^{ict\omega\cos(\phi)}\sin(\phi)\,d\phi \int_0^{2\pi} d\theta = \\ -2\pi c^2t^2 \int_0^{\pi} e^{ict\omega\cos(\phi)}\,d\cos(\phi) = 2\pi c^2t^2 \frac{1}{ict\omega}\Big(e^{ict\omega} - e^{-ict\omega}\Big) = \\ 4\pi ct\omega^{-1}\sin(ct\omega)$$

and multiplying by $e^{ix\cdot\xi}$ and dividing by $4\pi c^2t$ we get $e^{ix\cdot\xi}c^{-1}|\xi|^{-1}\sin(ct|\xi|)$ which is the right-hand expression of (9.1.5).

So, for $h(x) = e^{ix\cdot\xi}$ (9.1.4) has been proven. However the general function $h(x)$ could be decomposed into such special functions using *multidimensional Fourier transform* and *multidimensional Fourier integral* which is nothing but repeated 1-dimensional Fourier transform and Fourier integral:

$$h(x) = \iiint \hat{h}(\xi)e^{ix\cdot\xi}\,d\xi, \tag{9.1.6}$$

$$\hat{h}(\xi) = (2\pi)^{-n}\iiint h(x)e^{-ix\cdot\xi}\,dx \tag{9.1.7}$$

and therefore (9.1.4) extends to general functions as well.

Remark 9.1.1. We should deal with the fact that only decaying functions could be decomposed into Fourier integral, but this is easy due to the fact that integral in (9.1.4) is taken over bounded domain.

9.1.2 3D-Wave equation: general case

To cover $t < 0$ we replace (9.1.4) by

$$u(x,t) = \frac{1}{4\pi ct}\iint_{S(x,c|t|)} h(y)\,d\sigma \tag{9.1.8}$$

which is obvious as u must be odd with respect to t.

Consider now $g \neq 0$. Let v be given by (9.1.8) with h replaced by h; then

$$
\begin{aligned}
&v_{tt} - c^2\Delta v = 0,\\
&v|_{t=0} = 0,\\
&v_t|_{t=0} = g.
\end{aligned}
$$

Then $\Delta v|_{t=0} = 0$ and therefore $v_{tt}|_{t=0} = 0$ and therefore differentiating equation with respect to t we conclude that $u := v_t$ solves

$$
\begin{aligned}
&u_{tt} - c^2\Delta u = 0,\\
&u|_{t=0} = g,\\
&u_t|_{t=0} = 0.
\end{aligned}
$$

Now

$$
u(x,t) = \frac{\partial}{\partial t}\Big(\frac{1}{4\pi c^2 t}\iint_{S(x,c|t|)} g(y)\,d\sigma\Big). \tag{9.1.9}
$$

Therefore solving separately (9.1.1)–(9.1.3) for $f = g = 0$ (given by (9.1.8)) and for $f = h = 0$ (given by (9.1.9) and adding solutions due to linearity we arrive to

$$
u(x,t) = \frac{\partial}{\partial t}\Big(\frac{1}{4\pi c^2 t}\iint_{S(x,c|t|)} g(y)\,d\sigma\Big) + \frac{1}{4\pi c^2 t}\iint_{S(x,c|t|)} h(y)\,d\sigma \tag{9.1.10}
$$

covering case $f = 0$.

To cover case of arbitrary f but $g = h = 0$ we apply Duhanel integral formula (see Subsection 2.5.1). Consider problem

$$
\begin{aligned}
&U_{tt} - c^2\Delta U = 0,\\
&U|_{t=\tau} = 0,\\
&U_t|_{t=\tau} = f(x,\tau).
\end{aligned}
$$

Its solution is given by

$$
U(x,t,\tau) = \frac{1}{4\pi c^2(t-\tau)}\iint_{S(x,c|t-\tau|)} f(y,\tau)\,d\sigma
$$

and therefore

$$u(x,t) = \int_0^t \frac{1}{4\pi c^2(t-\tau)} \iint_{S(x,c|t-\tau|)} f(y,\tau)\, d\sigma d\tau. \tag{9.1.11}$$

Assembling (9.1.10) and (9.1.11) together we arrive to *Kirchhoff formula*

$$u(x,t) = \frac{\partial}{\partial t}\Big(\frac{1}{4\pi c^2 t} \iint_{S(x,c|t|)} g(y)\, d\sigma\Big) + \frac{1}{4\pi c^2 t} \iint_{S(x,c|t|)} h(y)\, d\sigma + \int_0^t \frac{1}{4\pi c^2(t-\tau)} \iint_{S(x,c|t-\tau|)} f(y,\tau)\, d\sigma d\tau. \tag{9.1.12}$$

providing solution to (9.1.1)–(9.1.3).

Remark 9.1.2. As $t > 0$ one can rewrite the right-hand expression in (9.1.11) as

$$\iiint_{B(x,ct)} \frac{1}{4\pi c^2|x-y|} f(y, t - c^{-1}|x-y|)\, dy \tag{9.1.13}$$

where we we integrate over ball $B(x,ct)$ of radius ct with the center at x.

9.1.3 Spherical means

Definition 9.1.1. $M_r(h,x) = \frac{1}{4\pi r^2}\int_{S(x,r)} h(y)\, d\sigma$ is a *spherical mean* of h. Recall that $4\pi r^2$ is an area of $S(x,r)$.

Therefore (9.1.8) is exactly $u(x,t) = tM_{\{cjtj\}}(h,x)$ and all other formulae (9.1.9)–(9.1.13) could be modified similarly.

9.1.4 2D-wave equation: method of descent

Consider now the same problem (9.1.1)–(9.1.3) but in dimension 2. To apply (9.1.12) we introduce a third spatial variable x_3 and take f, g, h not depending on it; then u also does not depend on x_3 and solves original $2D$-problem.

So, the right-hand expression in (9.1.8) becomes for $\pm t > 0$

$$\frac{1}{4\pi c^2 t} \iint_{S(x,c|t|)} h(y)\, d\sigma = \pm \frac{1}{2\pi c} \iint_{B(x,c|t|)} \frac{h(y)}{\sqrt{c^2t^2 - |x-y|^2}}\, dy \tag{9.1.14}$$

where $y = (y_1, y_2)$ and we took into account that $S(x, ct)$ covers disk $B(x, c|t|)$ twice (so factor 2 appears) and $d\sigma = \pm\frac{ct}{\sqrt{c^2t^2-|x-y|^2}}\,dy$, $dy = dy_1dy_2$.

Thus (9.1.12) implies that for $\pm t > 0$

$$\begin{aligned} u(x,t) = \pm\frac{\partial}{\partial t}\Big(\frac{1}{2\pi c}\iint_{B(x,c|t|)} \frac{g(y)}{\sqrt{c^2t^2-|x-y|^2}}\,dy\Big) \\ \pm \frac{1}{2\pi c}\iint_{B(x,c|t|)} \frac{h(y)}{\sqrt{c^2t^2-|x-y|^2}}\,dy \\ \pm \int_0^t \frac{1}{4\pi c}\iint_{B(x,c|t-\tau|)} \frac{f(y,\tau)}{\sqrt{c^2(t-\tau)^2-|x-y|^2}}\,dyd\tau. \end{aligned} \tag{9.1.15}$$

9.1.5 Limiting amplitude principle

Let $n = 3$. Consider solution to inhomogeneous wave equation with a special right-hand expression

$$\Delta u - c^{-2}u_{tt} = f(x)e^{i\omega t} \tag{9.1.16}$$

where $\omega \neq 0$ and $f(x)$ does not depend on t and fast decays as $|x| \to \infty$. Assume that $g(x) = u(x, 0)$ and $h(x) = u_t(x, 0)$ also fast decay as $|x| \to \infty$. Plugging all these functions into Kirchhoff formula (9.1.12) and considering $|t| \gg 1$ and fixed x we see that

$$|u(x,t) - v_\omega^\pm(x)e^{i\omega t}| \to 0 \qquad \text{as } \ t \to \pm\infty \tag{9.1.17}$$

with

$$v_\omega^\pm = -\frac{1}{4\pi}\iiint |x-y|^{-1}e^{\mp i\omega c^{-1}|x-y|}\,dy. \tag{9.1.18}$$

One can check easily that $v = v_\omega^\pm$ satisfies *Helmholtz equation*

$$\big(\Delta + \frac{\omega^2}{c^2}\big)v = f(x) \tag{9.1.19}$$

with *Sommerfeld radiating conditions*

$$v = o(1) \qquad \text{as } r \to \infty, \tag{9.1.20}$$

$$(\partial_r \mp ic^{-1}\omega)v = o(r^{-1}) \qquad \text{as } r \to \infty \tag{9.1.21}$$

where $r := |x|$ and $\partial_r := |x|^{-1}x \cdot \nabla$.

This is called *Limiting amplitude principle*

Remark 9.1.3. (a) Formula (9.1.18) gives us Green function for problem (9.1.19)-(9.1.21)

$$G_\omega^\pm(x,y) = -\frac{1}{4\pi}|x-y|^{-1}e^{\mp i\omega c^{-1}|x-y|} \tag{9.1.22}$$

which coincides as $\omega = 0$ with a Green function $G(x,y) = -\frac{1}{4\pi}|x-y|^{-1}$;

(b) However now we have two Green functions as (9.1.22) distinguishes and between them and $u(x,t)$ has different amplitudes $v_\omega^\pm$ as $tto\pm\infty$.

(c) For fast–decaying f one can replace in (9.1.20) and (9.1.21) $o(1)$ and $o(r^{-1})$ by $O(r^{-1})$ and $O(r^{-2})$ respectively.

9.1.6 Remarks

Remark 9.1.4. Formulae (9.1.13) and (9.1.15) could be generalized to the case of odd $n \geq 3$ and even $n \geq 2$ respectively. These formulae imply that $u(x,t)$ does not depend on $g(y), h(y)$ with $|y-x| > ct$ and on $f(y,\tau)$ with $|y-x| > c|t-\tau|$. This could be interpreted as "nothing propagates with a speed exceeding c". We will prove it again by completely different method in the next Section 9.2

Remark 9.1.5. (a) As $n \geq 3$ is odd $u(x,t)$ is given by the following formula

$$\begin{aligned} u(x,t) = c^{1-n}\kappa_n \frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} \left(t^{-1}\iint_{S(x,c|t|)} g(y)\,d\sigma\right) \\ + c^{1-n}\kappa_n \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} \left(t^{-1}\iint_{S(x,c|t|)} h(y)\,d\sigma\right) \end{aligned} \tag{9.1.23}$$

provided $f = 0$ (which could be generalized to $f \neq 0$ using Duhamel principle).

(b) As $n \geq 2$ is even $u(x,t)$ is given by the following formula obtained by

the method of descent

$$\begin{aligned} & u(x,t) \\ &= c^{1-n}\kappa_n \frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} \left(\iiint_{B(x,c|t|)} \frac{g(y)}{(c^2t^2-|x-y|^2)^{\frac{1}{2}}}\,dy\right) \\ &\quad + c^{1-n}\kappa_n \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} \left(\iiint_{B(x,c|t|)} \frac{g(y)}{(c^2t^2-|x-y|^2)^{\frac{1}{2}}}\,dy\right) \end{aligned} \tag{9.1.24}$$

provided $f = 0$ (which could be generalized to $f \neq 0$ using Duhamel principle).

(c) Here κ_n is a numerical coefficient which could be easily calculated from $g \equiv 1,\ h \equiv 0,\ f \equiv 0 \implies u \equiv 1$.

(d) In particular, for odd $n \geq 3$ solution $u(x,t)$ does not depend on $g(y), h(y)$ with $|y-x| < ct$ and on $f(y,\tau)$ with $|y-x| < c|t-\tau|$. This could be interpreted as "no leftovers after front passed with a speed c". In mathematical literature this is called *Huygens principle* (there is another Huygens principle aka *Huygens-Fresnel principle*). This property is a rare commodity: adding lower-order terms to the equation breaks it.

9.2 Wave equation: energy method

9.2.1 Energy method: local form

Consider wave equation

$$u_{tt} - c^2\Delta u = 0. \tag{9.2.1}$$

Multiplying by u_t we arrive to

$$0 = u_t u_{tt} - c^2 u_t \Delta u = \frac{1}{2}\partial_t(u_t^2) - c^2\nabla\cdot(u_t\nabla u) + c^2\nabla u_t\cdot\nabla u = \frac{1}{2}\partial_t\big(u_t^2 + c^2|\nabla u|^2\big) - c^2\nabla\cdot(u_t\nabla u)$$

that is

$$\frac{1}{2}\partial_t\big(u_t^2 + c^2|\nabla u|^2\big) - c^2\nabla\cdot(u_t\nabla u) = 0. \tag{9.2.2}$$

This is an *energy conservation law in the local form.*

If we integrate over domain $\Omega \subset \mathbb{R}_t \times \mathbb{R}^n_x$ we arrive to

$$\iint_{\Sigma} \Big(\big(u_t^2 + |\nabla u|^2\big)\nu_t - c^2 u_t \nabla u \cdot \nu_x \Big)\, d\sigma = 0 \tag{9.2.3}$$

where Σ is a boundary of Ω, ν is an external normal and $d\sigma$ is an alement of "area"; ν_t and ν_x are its t and x components.

9.2.2 Classification of hypersurfaces

Consider a quadratic form

$$Q(U_0, \mathbf{U}) = U_0^2 + |\mathbf{U}|^2 - 2U_0 c \nu_t^{-1} \nu_x \cdot \mathbf{U}. \tag{9.2.4}$$

Proposition 9.2.1. *(a) If $c|\nu_x| < |\nu_t|$ then Q is positive definite (i.e. $Q(U_0, \mathbf{U}) \geq 0$ and $Q(U_0, \mathbf{U}) = 0$ iff $U_0 = \mathbf{U} = 0$);*

(b) If $c|\nu_x| = |\nu_t|$ then Q is non-negative definite (i.e. $Q(U_0, \mathbf{U}) \geq 0$);

(c) If $c|\nu_x| > |\nu_t|$ then Q is not non-negative definite.

Proof. is obvious. □

Definition 9.2.1. (a) If $c|\nu_x| < |\nu_t|$ then Σ is a *space-like surface* (in the given point).

(b) If $c|\nu_x| = |\nu_t|$ then Σ is a *characteristic* (in the given point).

(c) If $c|\nu_x| > |\nu_t|$ then Σ is a *time-like surface* (in the given point).

Remark 9.2.1. Those who studied special relativity can explain (a), (c).

9.2.3 Application to Cauchy problem

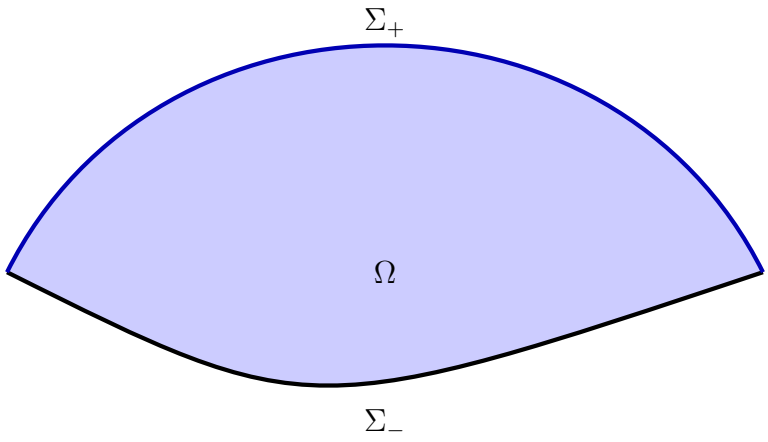
Consider now bounded domain Ω bounded by $\Sigma = \Sigma_+ \cup \Sigma_-$ where $c|\nu_x| \leq -\nu_t$ at each point of Σ_- and $c|\nu_x| \leq \nu_t$ at each point of Σ_+. Assume that u satisfies (9.2.1)

$$u = u_t = 0 \qquad \text{on } \Sigma_-. \tag{9.2.5}$$

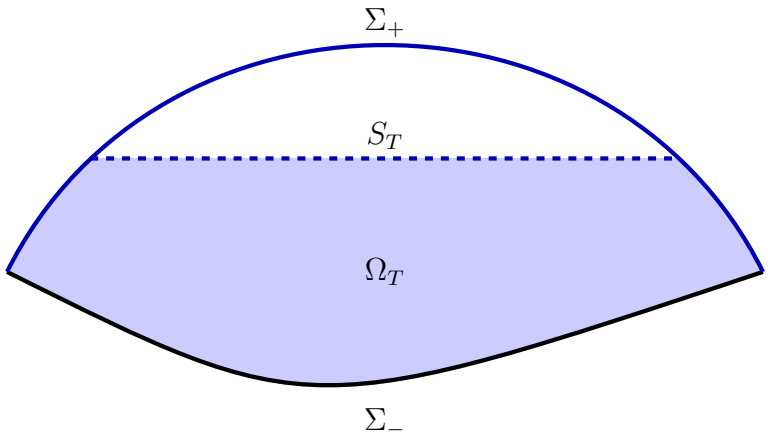
Then (9.2.3) implies that

$$\iint_{\Sigma_+} \Big(\big(u_t^2 + |\nabla u|^2\big)\nu_t - c^2 u_t \nabla u \cdot \nu_x \Big)\, d\sigma = 0$$

which due to assumption about Σ_+ implies that integrand is 0 and therefore $u_t = \nabla u = 0$ in each point where $c|\nu_x| < \nu_t$.



We can apply the same arguments to $\Omega_T := \Omega \cap \{t < T\}$ with the boundary $\Sigma_T = \Sigma \cap \{t < T\} \cup S_T$, $S_T := \Omega \cap \{t = T\}$; note that on S_T $\nu_t = 1$, $\nu_x = 0$.



Therefore $u_t = \nabla u = 0$ on S_T and since we can select T arbitrarily we conclude that this is true everywhere in Ω. Since $u = 0$ on Σ_- we conclude that $u = 0$ in Ω. So we proved:

Theorem 9.2.1. *Consider a bounded domain Ω bounded by $\Sigma = \Sigma_+ \cup \Sigma_-$ where $c|\nu_x| \le -\nu_t$ at each point of Σ_- and $c|\nu_x| \le \nu_t$ at each point of Σ_+. Assume that u satisfies (9.2.1), (9.2.5). Then $u = 0$ in Ω.*

It allows us to prove

Theorem 9.2.2. *Consider (y,τ) with $\tau > 0$ and let $K^-(y,\tau) = \{(x,t) : t \le \tau, |y-x| < c(\tau - t)\}$ be a* backward light cone *issued from (y,τ). Let*

(a) u satisfy (9.2.1) in $K^-(y,\tau) \cap \{t > 0\}$,

(b) $u = u_t = 0$ at $K^-(y,\tau) \cap \{t = 0\}$.

Then $u = 0$ in $K^-(x,t) \cap \{t > 0\}$.

Proof is obvious: we can use $\Omega = K^-(x,t) \cap \{t > 0\}$. Note that the border of $K^-(x,t)$ is characteristic at each point and $\nu_t > 0$.

9.2.4 Application to IBVP

Consider domain $\mathcal{D} \subset \mathbb{R}^n$ with a boundary Γ.

Theorem 9.2.3. *Consider (y,τ) with $\tau > 0$ and let $K^-(y,\tau) = \{(x,t) : t \le \tau, |y-x| < c(\tau - t)\}$ be a* backward light cone *issued from (y,τ). Let*

(a) u satisfy (9.2.1) in $K^-(y,\tau) \cap \{t > 0\} \cap \{x \in \mathcal{D}\}$,

(b) $u = u_t = 0$ at $K^-(y,\tau) \cap \{t = 0\} \cap \{x \in \mathcal{D}\}$,

(c) At each point of $K^-(y,\tau) \cap \{t > 0\} \cap \{x \in \Gamma\}$ either $u = 0$ or $\frac{\partial u}{\partial n} = 0$ where n is a normal to Γ.

Then $u = 0$ in $K^-(y,\tau) \cap \{t > 0\} \cap \{x \in \mathcal{D}\}$.

Proof. Proof uses the same energy approach but now we have also integral over part of the surface $K^-(y,\tau) \cap \{t > 0\} \cap \{x \in \Gamma\}$ (which is time-like) but this integral is 0 due to (c). □

9.2.5 Remarks

Remark 9.2.2. The energy approach works in a very general framework and is used not only to prove unicity but also an existence and stability of solutions.

Chapter 10

Variational methods

10.1 Functionals, extremums and variations

10.1.1 Functionals: definitions

Definition 10.1.1. *Functional* is a map from some space of functions (or subset in the space of functions) H to $\mathbb{R}$ (or $\mathbb{C}$):

$$\Phi : \mathsf{H} \ni u \to \Phi[u] \in \mathbb{R}. \tag{10.1.1}$$

Remark 10.1.1. Important that we consider a whole function as an argument, not its value at some particular point!

Example 10.1.1. (a) On the space $C(I)$ of continuos functions on the closed interval I consider functional $\Phi[u] = u(a)$ where $a \in I$ (value at the point);

(b) On $C(I)$ consider functionals $\Phi[u] = \max_{x\in I} u(x)$, $\Phi[u] = \min_{x\in I} u(x)$ and $\Phi[u] = \max_{x\in I} |u(x)|$, $\Phi[u] = \min_{x\in I} |u(x)|$;

(c) Consider $\Phi[u] = \int_I f(x)u(x)\,dx$ where $f(x)$ is some fixed function.

(d) On the space $C^1(I)$ of continuos and continuously differentiable functions on the closed interval I consider functional $\Phi[u] = u'(a)$.

Definition 10.1.2. (a) Sum of functionals $\Phi_1 + \Phi_2$ is defined as $(\Phi_1 + \Phi_2)[u] = \Phi_1[u] + \Phi_2[u]$;

(b) Product of functional by a number: $\lambda\Phi$ is defined as $(\lambda\Phi)[u] = \lambda(\Phi[u])$;

(c) Function of functionals: $F(\Phi_1,\dots,\Phi_s)$ is defined as $F(\Phi_1,\dots,\Phi_s)[u] = F(\Phi_[u],\dots,\Phi_s[u])$.

Definition 10.1.3. Functional $\Phi[u]$ is called *linear* if

$$\Phi[u+v] = \Phi[u] + \Phi[v], \tag{10.1.2}$$
$$\Phi[\lambda u] = \lambda\Phi[u] \tag{10.1.3}$$

for all functions u and scalars λ.

Remark 10.1.2. Linear functionals will be crucial in the definition of *distributions* later.

Exercise 10.1.1. Which functionals of 10.1.1 are linear?

10.1.2 Variations of functionals

Let us consider functional

$$\Phi[u] = \iiint_\Omega L(x,u,\nabla u)\,dx \tag{10.1.4}$$

where Ω is n-dimensional domain and L is some function of $n+2$ variables. Let us consider $u+\delta u$ where δu is a "small" function. We do not formalize this notion, just $\varepsilon\phi$ with fixed ϕ and $\varepsilon\to 0$ is considered to be small. We call δu *variation* of u and important is that we change a function as a whole object. Let us consider

$$\begin{aligned}\Phi[u+\delta u]-\Phi[u] &= \iiint_\Omega \Big(L(x,u+\delta u,\nabla u+\nabla\delta u) - L(x,u,\nabla u)\Big)\,dx\\ &\approx \iiint_\Omega \Big(\frac{\partial L}{\partial u}\delta u + \sum_{1\le j\le n}\frac{\partial L}{\partial u_{x_j}}\delta u_{x_j}\Big)\,dx\end{aligned} \tag{10.1.5}$$

where we calculated the linear part of expression in the parenthesis; if $\delta u = \varepsilon\phi$ and all functions are sufficiently smooth then $\approx$ would mean "equal modulo $o(\varepsilon)$ as $\varepsilon\to 0$".

Definition 10.1.4. (a) Function L we call *Lagrangian*.

(b) The right-hand expression of (10.1.5) which is a linear functional with respect to δu we call *variation of functional* Φ and denote by $\delta\Phi$.

Assumption 10.1.1. All functions are sufficiently smooth.

Under this assumption, we can integrate the right-hand expression of (10.1.5) by parts:

$$\begin{aligned}\delta\Phi &:= \iiint_\Omega \Big(\frac{\partial L}{\partial u}\delta u + \sum_{1\le j\le n}\frac{\partial L}{\partial u_{x_j}}\delta u_{x_j}\Big)\,dx\\ &= \iiint_\Omega \Big(\frac{\partial L}{\partial u} - \sum_{1\le j\le n}\frac{\partial}{\partial x_j}\frac{\partial L}{\partial u_{x_j}}u\Big)\delta u\,dx - \iint_{\partial\Omega}\Big(\sum_{1\le j\le n}\frac{\partial L}{\partial u_{x_j}}\nu_j\Big)\delta u\,d\sigma\end{aligned} \tag{10.1.6}$$

where $d\sigma$ is an area element and ν is a unit interior normal to $\partial\Omega$.

10.1.3 Stationary points of functionals

Definition 10.1.5. If $\delta\Phi = 0$ for all *admissible variations* δu we call u a *stationary point* or *extremal* of functional Φ.

Remark 10.1.3. (a) We consider u as a point in the functional space;

(b) In this definition we did not specify which variations are admissible. Let us consider as admissible all variations which are 0 at the boundary:

$$\delta u|_{\partial\Omega} = 0. \tag{10.1.7}$$

We will consider different admissible variations later.

In this framework

$$\delta\Phi = \iiint_\Omega \Big(\frac{\partial L}{\partial u} - \sum_{1\le j\le n}\frac{\partial}{\partial x_j}\frac{\partial L}{\partial u_{x_j}}\Big)\delta u\,dx. \tag{10.1.8}$$

Lemma 10.1.1. *Let f be a continuos function in Ω. If $\iiint_\Omega f(x)\phi(x)\,dx = 0$ for all ϕ such that $\phi|_{\partial\Omega} = 0$ then $f = 0$ in Ω.*

Proof. Indeed, let us assume that $f(\bar x) > 0$ at some point $\bar x \in \Omega$ (case $f(\bar x) < 0$ is analyzed in the same way). Then $f(x) > 0$ in some vicinity $\mathcal{V}$ of $\bar x$. Consider function $\phi(x)$ which is 0 outside of $\mathcal{V}$, $\phi \ge 0$ in $\mathcal{V}$ and $\phi(\bar x) > 0$. Then $f(x)\phi(x)$ has the same properties and $\iiint_\Omega f(x)\phi(x)\,dx > 0$. Contradiction! □

As a corollary we arrive to

Theorem 10.1.1. *Let us consider a functional (10.1.4) and consider as admissible all δu satisfying (10.1.7). Then u is a stationary point of Φ if and only if it satisfies* Euler-Lagrange equation

$$\frac{\delta \Phi}{\delta u} := \frac{\partial L}{\partial u} - \sum_{1\le j\le n} \frac{\partial}{\partial x_j}\left(\frac{\partial L}{\partial u_{x_j}}\right) = 0. \tag{10.1.9}$$

10.1.4 Extremums of functionals

Definition 10.1.6. If $\Phi[u] \ge \Phi[u+\delta u]$ for all small admissible variations δu we call u a *local maximum* of functional Φ. If $\Phi[u] \le \Phi[u+\delta u]$ for all small admissible variations δu we call u a *local minimum* of functional Φ.

Here again we do not specify what is *small admissible variation*.

Theorem 10.1.2. *If u is a* local extremum *(that means either local minimum or maximum) of Φ and variation exits, then u is a stationary point.*

Proof. Consider case of minimum. Let $\delta u = \varepsilon\phi$. Then $\Phi[u+\delta u] - \Phi[u] = \varepsilon(\delta\Phi)(\phi) + o(\varepsilon)$. If $\pm\delta\Phi > 0$ then choosing $\mp\varepsilon < 0$ we make $\varepsilon(\delta\Phi)(\phi) \le -2\epsilon_0\varepsilon$ with some $\epsilon_0 > 0$. Meanwhile for sufficiently small ε "$o(\varepsilon)$" is much smaller and $\Phi[u+\delta u]-\Phi[u] \le -2\epsilon_0\varepsilon < 0$ and u is not a local minimum. □

Remark 10.1.4. We consider neither sufficient conditions of extremums nor *second variations* (similar to second differentials). In some cases they will be obvious.

Example 10.1.2. (a) Consider a surface $\Sigma = \{(x,y,z) : (x,y)\in\Omega, z = u(x,y)\}$ which has (x,y)-projection Ω. Then the surface area of Σ is

$$A(\Sigma) = \iint_\Omega \bigl(1+u_x^2+u_y^2\bigr)^{\frac12}\,dxdy. \tag{10.1.10}$$

We are interested in such surface of minimal area (aka *minimal surface*) under restriction $u = g$ at points $\partial\Omega$. It is a famous *minimal surface problem* (under the assumption that it projects nicely on (x,y)-plane (which is not necessarily the case). One can formulate it: find the shape of the soap film on the wire. Then Euler-Lagrange equation is

$$-\frac{\partial}{\partial x}\Bigl(u_x\bigl(1+u_x^2+u_y^2\bigr)^{-\frac12}\Bigr) - \frac{\partial}{\partial y}\Bigl(u_y\bigl(1+u_x^2+u_y^2\bigr)^{-\frac12}\Bigr) = 0. \tag{10.1.11}$$

(b) Assuming that $u_x, u_y \ll 1$ one can approximate $A(\Sigma) - A(\Omega)$ by

$$\frac{1}{2}\iint_{\Omega}\bigl(u_x^2+u_y^2\bigr)\,dxdy \tag{10.1.12}$$

and for this functional Euler-Lagrange equation is

$$-\Delta u = 0. \tag{10.1.13}$$

(c) Both (a) and (b) could be generalized to higher dimensions.

Remark 10.1.5. Both equations (10.1.11) and (10.1.12) come with the boundary condition $u|_{\partial\Omega} = g$. In the next section we analyse the case when such condition is done in the original variational problem only on the part of the boundary.

10.2 Functionals, extremums and variations

10.2.1 Boundary conditions

Let us consider functional (10.1.4) but now instead of constrain (10.1.7) we put a less restrictive

$$\delta u|_{\Sigma} = 0 \tag{10.2.1}$$

where $\Sigma \subset \partial\Omega$ (may be even empty). Then Euler-Lagrange equation (10.1.9) must still be fulfilled but it does not guarantee that $\delta\Phi = 0$ for all admissible variations as according to (10.1.6)

$$\delta\Phi = \iint_{\partial\Sigma'}\Bigl(-\sum_{1\le j\le n}\frac{\partial L}{\partial u_{x_j}}\nu_j\Bigr)\delta u\,d\sigma \tag{10.2.2}$$

where $\Sigma' = \partial\Omega \setminus \Sigma$: the part of $\partial\Omega$ complemental to Σ.

Now we need to have it be 0 as well an since δu is arbitrary there according to Lemma 10.1.1 we need to have expression in parenthesis vanish:

$$\Bigl(-\sum_{1\le j\le n}\frac{\partial L}{\partial u_{x_j}}\nu_j\Bigr)\Big|_{\Sigma'} = 0. \tag{10.2.3}$$

However under assumption (10.2.1) it makes sense to consider more general functional than (10.1.4):

$$\Phi[u] = \iiint_{\Omega} L(x,u,\nabla u)\,dx + \underbrace{\iint_{\Sigma'} M(x,u)\,d\sigma} \tag{10.2.4}$$

which includes a *boundary term*. One can see easily that variation of the boundary term is $\iint_{\Sigma'} \frac{\partial M}{\partial u}\delta\, d\sigma$ which should be added to (10.2.2) which becomes

$$\delta\Phi = \iint_{\partial\Sigma'} \Big(-\sum_{1\le j\le n} \frac{\partial L}{\partial u_{x_j}}\nu_j + \frac{\partial M}{\partial u}\Big)\delta u\, d\sigma; \tag{10.2.5}$$

then (10.2.3) becomes

$$\Big(-\sum_{1\le j\le n} \frac{\partial L}{\partial u_{x_j}}\nu_j + \frac{\partial M}{\partial u}\Big)\Big|_{\Sigma'} = 0. \tag{10.2.6}$$

Then we arrive to the following generalization of Theorem 10.1.1:

Theorem 10.2.1. *Let us consider a functional* (10.2.4) *and consider as admissible all δu satisfying* (10.2.1). *Then u is a stationary point of Φ if and only if it satisfies Euler-Lagrange equation* (10.1.9) *and a boundary condition* (10.2.6).

Example 10.2.1. Consider

$$\iiint_{\Omega} \Big(\frac{1}{2}|\nabla u|^2 - f(x)u\Big)\, d^n x + \iint_{\Sigma'} \Big(\frac{1}{2}\alpha(x)|u|^2 - h(x)u\Big)\, d\sigma \tag{10.2.7}$$

under assumption

$$u|_{\Sigma} = g. \tag{10.2.8}$$

Then we have equation

$$\Delta u = -f \tag{10.2.9}$$

with the boundary condition (10.2.8) on Σ and

$$\big(\frac{\partial u}{\partial \boldsymbol{\nu}} - \alpha u\big)\big|_{\Sigma'} = -h \tag{10.2.10}$$

on Σ'. So at each point of the boundary we have exactly one condition.

Observe that (10.2.10) is Robin condition (Neumann condition as $\alpha = 0$).

10.2.2 Vector and complex valued functions

If our function $u(x)$ is vector–valued: $u = (u_1, \dots, u_m)$ then we can consider variations with respect to different components and derive corresponding equations

$$\frac{\partial L}{\partial u_k} - \sum_{1\le j\le n} \frac{\partial}{\partial x_j}\frac{\partial L}{\partial u_{k,x_j}} = 0 \qquad k = 1, \dots, m. \tag{10.2.11}$$

We also get boundary conditions

Example 10.2.2. (a) Consider functional

$$\Phi[\mathbf{u}] = \frac{1}{2}\iiint_\Omega \Big(\alpha|\nabla \otimes \mathbf{u}|^2 + \beta|\nabla\cdot\mathbf{u}|^2\Big)\,dx \tag{10.2.12}$$

with $\mathbf{u} = (u_1, \dots, u_n)$, $|\nabla\otimes\mathbf{u}|^2 = \sum_{j,k} |u_{k,x_j}|^2$, $\nabla\cdot\mathbf{u} = \sum_j u_{j,x_j}$. Then as $\delta\mathbf{u} = 0$ on $\partial\Omega$

$$\delta\Phi = \iiint_\Omega \Big(-\alpha(\Delta\mathbf{u}) - \beta\nabla(\nabla\cdot\mathbf{u})\Big)\cdot\delta\mathbf{u}\,dx \tag{10.2.13}$$

where for simplicity we assume that α and β are constant and we have a system

$$-\alpha\Delta\mathbf{u} - \beta\nabla(\nabla\cdot\mathbf{u}) = 0. \tag{10.2.14}$$

(b) Then without for this functional

$$\delta\Phi = -\iint_{\Sigma'} \Big(\alpha\frac{\partial\mathbf{u}}{\partial\boldsymbol{\nu}} + \beta(\nabla\cdot\mathbf{u})\boldsymbol{\nu}\Big)\cdot\delta\mathbf{u}\,d\sigma = 0 \tag{10.2.15}$$

with $\frac{\partial\mathbf{u}}{\partial\boldsymbol{\nu}} := \sum_j \nu_j \frac{\partial\mathbf{u}}{\partial x_j}$. Then if we assume that $\delta\mathbf{u}$ on Σ' can be arbitrary, we arrive to boundary condition

$$\alpha\frac{\partial\mathbf{u}}{\partial\boldsymbol{\nu}} + \beta(\nabla\cdot\mathbf{u})\boldsymbol{\nu} = 0. \tag{10.2.16}$$

(c) However there could be other "natural" conditions on Σ'. F.e. if we assume that $\delta\mathbf{u} \parallel \boldsymbol{\nu}$ on Σ' we get

$$\Big(\alpha\frac{\partial\mathbf{u}}{\partial\boldsymbol{\nu}} + \beta(\nabla\cdot\mathbf{u})\boldsymbol{\nu}\Big)\cdot\boldsymbol{\nu} = 0; \tag{10.2.17}$$

if we assume instead that $\delta\mathbf{u}\cdot\boldsymbol{\nu}=0$ we get

$$\frac{\partial\mathbf{u}}{\partial\boldsymbol{\nu}} \parallel \boldsymbol{\nu}=0. \tag{10.2.18}$$

Here and everywhere $\parallel$ means parallel (proportional).

Remark 10.2.1. Complex-valued functions u could be considered as vector-valued functions $\mathbf{u}=(\operatorname{Re} u \operatorname{Im} u)$. Similarly we can treat functions which are vector–valued with complex components: we just double m.

10.2.3 Extremals under constrains. I

We can consider extremals of functionals under constrains

$$\Psi_1[u]=\Psi_2[u]=\ldots=\Psi_s[u]=0 \tag{10.2.19}$$

where Ψ_j are other functionals. This is done in the same way as for extremums of functions of several variables: instead of $\Phi[u]$ we consider *Lagrange functional*

$$\Phi^*[u]:=\Phi[u]-\lambda_1\Psi_1[u]-\lambda_2\Psi_2[u]-\ldots-\lambda_s\Psi_s[u] \tag{10.2.20}$$

and look for it extremals without constrains; factors $\lambda_1,\ldots,\lambda_s$ are *Lagrange multipliers*.

Remark 10.2.2. This works provided $\delta\Psi_1,\ldots,\delta\Psi_s$ are *linearly independent* which means that if $\alpha_1\delta\Psi_1[u]+\alpha_2\delta\Psi_2[u]+\ldots+\alpha_s\delta\Psi_s[u]=0$ for all admissible δu then $\alpha_=\ldots=\alpha_s=0$.

Example 10.2.3. (a) Let us consider

$$\Phi[u]:=\frac{1}{2}\int|\nabla u|^2\,dx \tag{10.2.21}$$

under constrains

$$\Psi[u]:=\frac{1}{2}\int|u|^2\,dx \tag{10.2.22}$$

and $u|_{\partial\Omega}=0$. Then $\Phi^*[u]=\frac{1}{2}\int\big(|\nabla u|^2-\lambda|u|^2\big)\,dx$ and Euler-Lagrange equation is

$$-\Delta u=\lambda u; \tag{10.2.23}$$

so λ and u must be eigenvalue and eigenfunction of $-\Delta$ (with Dirichlet boundary conditions. However only the lowest (base) eigenvalue λ_1 delivers minimum.

(b) To deal with the the next eigenvalues and eigenfunctions let us assume that we got $\lambda_1, \dots, \lambda_{s-1}$ and orthogonal $u_1, \dots, u_{s-1}$ and let us consider constrains (10.2.22) and

$$(u, u_1) = (u, u_2) = \dots = (u, u_{s-1}) = 0 \tag{10.2.24}$$

where $(u, v) = \iint_\Omega uv\, dx$ is an inner product (we consider real-valued functions). Then $\Phi^*[u] = \int \bigl(\frac{1}{2}|\nabla u|^2 - \frac{\lambda}{2}|u|^2 - \mu_1 u_1 u - \dots \mu_{s-1}u_{s-1}u\bigr)\, dx$ and we arrive to equation

$$-\Delta u - \lambda u - \mu_1 u_1 - \dots - \mu_{s-1}u_{s-1} = 0. \tag{10.2.25}$$

Taking an inner product with u_k we arrive to

$$-((\Delta + \lambda)u, u_k) - \mu_k \|u_k\|^2 = 0$$

because we know that $u_1, \dots, u_{s-1}$ are orthogonal. Further, $((\Delta+\lambda)u, u_k) = (u, (\Delta + \lambda)u_k) = (u, (-\lambda_k + \lambda)u_k) = 0$ and we conclude that $\mu_k = 0$. Then (10.2.24) implies

$$-\Delta u - \lambda u = 0 \tag{10.2.26}$$

and we got the next eigenvalue and eigenfunction.

(c) The same analysis works for Neumann and Robin boundary conditions (but we need to assume that $\alpha \geq 0$ or at least is not "too negative".

10.2.4 Extremals under constrains. II

However constrains could be different from those described in the previous subsection. They can be not in the form of functionals but in the form of functions. In this case we have continuum conditions and Lagrange multipliers became functions as well. Let us consider this on examples.

Example 10.2.4. Consider $\mathbf{u}$ as in 10.2.2 and functional

$$\Phi[u] = \frac{1}{2}\iiint_\Omega |\nabla \otimes \mathbf{u}|^2\, dx \tag{10.2.27}$$

under constrain

$$\nabla \cdot \mathbf{u} = 0. \tag{10.2.28}$$

Then we consider functional

$$\Phi[u] = \iiint_\Omega \big(\frac{1}{2}|\nabla \otimes \mathbf{u}|^2 - \lambda(x)\nabla\cdot\mathbf{u}\big)\,dx = \iiint_\Omega \big(\frac{1}{2}|\nabla \otimes \mathbf{u}|^2 + \nabla\lambda(x)\cdot\mathbf{u}\big)\,dx$$

where we integrated by parts and ignored boundary term since here we are interested only in equation rather than boundary conditions. Then Euler-Lagrange equation is

$$\Delta\mathbf{u} = \nabla\lambda \tag{10.2.29}$$

with unknown function λ. However the right-hand expression is not an arbitrary vector-valued function but a gradient of some scalar function. Applying $\nabla\cdot$ to (10.2.29) and using (10.2.28) we conclude that $\Delta\lambda = 0$ so λ is a harmonic function.

Example 10.2.5. 10.2.2(c) could be considered in the same way. Indeed, let us consider constrain $\mathbf{u}\cdot\boldsymbol{\nu} = 0$ at Σ'. Then we need to consider functional $\Phi^*[u] = \Phi[u] - \iint_\Sigma \lambda(x)\mathbf{u}\cdot\boldsymbol{\nu}\,d\sigma$ where $\Phi[u]$ is defined by (10.2.12) and λ is unknown function on Σ'. We arrive to the same equation (10.2.14) but now (10.2.15) becomes

$$\delta\Phi = -\iint_{\Sigma'}\Big(\alpha\frac{\partial\mathbf{u}}{\partial\boldsymbol{\nu}} + \underbrace{(\lambda+\beta\nabla\cdot\mathbf{u})}\boldsymbol{\nu}\Big)\cdot\delta\mathbf{u}\,d\sigma = 0$$

and we have no constrains to $\delta\mathbf{u}$ on Σ' and we arrive to condition that expression in the parenthesis is 0 at Σ'. Since λ and thus highlighted expression are arbitrary functions on Σ' we arrive exactly to (10.2.18).

Remark 10.2.3. Obviously we can combine different types of constrains and different types of Lagrange multipliers.

10.2.5 Higher order functionals

We could include into functional higher-order derivatives. Let us consider functional

$$\Phi[u] = \iiint_\Omega L(x,u,\nabla u,\nabla^{(2)}u)\,dx \tag{10.2.30}$$

where $\nabla^{(2)}u$ is a set of all derivatives $u_{x_ix_j}$ with $i\le j$ (Think why). Then the same arguments as before lead us to Euler-Lagrange equation

$$\frac{\delta\Phi}{\delta u} := \frac{\partial L}{\partial u} - \sum_{1\le j\le n}\frac{\partial}{\partial x_j}\left(\frac{\partial L}{\partial u_{x_j}}\right) + \sum_{1\le i\le j\le n}\frac{\partial^2}{\partial x_i\partial x_j}\left(\frac{\partial L}{\partial u_{x_ix_j}}\right) = 0. \tag{10.2.31}$$

But what about boundary conditions? We need to have now two of them at each point. Obviously we have them if we are looking for solution satisfying

$$u\big|_{\partial\Omega} = g(x),\ \frac{\partial u}{\partial \boldsymbol{\nu}}\big|_{\partial\Omega} = h(x). \tag{10.2.32}$$

Otherwise we need to consider $\delta\Phi$. We consider this only on example.

Example 10.2.6. Let

$$\Phi[u] = \frac{1}{2}\iiint_{\Omega}\Big(\sum_{i,j}|u_{x_ix_j}|^2\Big)\,dx. \tag{10.2.33}$$

where we sum with respect to all pairs i,j (so $|u_{x_ix_j}|^2$ with $i\neq j$ is added twice. Then equation is

$$\Delta^2 u = 0 \tag{10.2.34}$$

(so u is *biharmonic function*) and

$$\delta\Phi = \iint_{\Sigma'}\Big(-\sum_{i,j}u_{x_ix_j}\nu_i\delta u_{x_j} + \sum_{i,j}(\Delta u_{x_j})\nu_j\delta u\Big)\,d\sigma \tag{10.2.35}$$

(a) If we have have both constrains (10.2.32) then we are done.

(b) Let us have only the first of constrains (10.2.32). Then n Σ' we have $\delta u = 0$ and $\nabla\delta u$ is parallel to $\boldsymbol{\nu}$: $\nabla\delta u = \lambda\boldsymbol{\nu}$ with arbitrary function ϕ (think why) and the second term in the parenthesis is 0 and the first term becomes $-\sum_{i,j}u_{x_ix_j}\nu_i\nu_j\phi$ and we arrive to the missing condition

$$\sum_{i,j}u_{x_ix_j}\nu_i\nu_j\big|_{\Sigma'} = 0. \tag{10.2.36}$$

(c) Let us have no constrains on Σ'. Then we can recover this condition (10.2.36) but we need one more. Let us assume for simplicity that Σ' is flat; then without any loss of the generality Σ' locally coincides with $\{x_n = 0\}$, $\boldsymbol{\nu} = (0,\dots,0,1)$ and condition (10.2.36) means that $u_{x_nx_n} = 0$ on Σ'; further, one can prove easily that then $\delta\Phi = \iint_{\Sigma'}\frac{\partial\Delta u}{\partial\boldsymbol{\nu}}\delta u\,d\sigma$ with arbitrary δu and it implies the second condition

$$\frac{\partial\Delta u}{\partial\boldsymbol{\nu}}\big|_{\Sigma'} = 0. \tag{10.2.37}$$

10.A Variational methods in physics

In Theoretical Physics equations of movement are frequently derived as Euler-Lagrange equations for a functional called *action* and traditionally denoted by S.

10.A.1 Classical dynamic

In the *classical dynamics* important role is played by *Lagrange formalism*. Positions of the system are described by *generalized coordinates* $\mathbf{q} = (q^1, q^2, \dots, q^N)$ which are functions of *time* t: $\mathbf{q} = \mathbf{q}(t)$. Then their derivatives $\dot{\mathbf{q}} := \mathbf{q}_t$ are called *generalized velocities* (in physics upper dot traditionally is used for derivative with respect to t).

Lagrangian then is a function of $\mathbf{q}, \dot{\mathbf{q}}, t$: $L = L(\mathbf{q}, \dot{\mathbf{q}}, t)$ and usually $L = T - U$ where T is a kinetic energy and U is a potential energy of the system.

Finally, *action* S is defined as

$$S = \int_{t_0}^{t_1} L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)\, dt \tag{10.A.1}$$

and we are looking for extremals of S as $\mathbf{q}(t_0)$ (initial state) and $\mathbf{q}(t_1)$ (final state) are fixed.

Then *Lagrange equations* are

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0 \qquad k = 1, \dots, N. \tag{10.A.2}$$

Next $\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}}$ that means

$$p_k = \frac{\partial L}{\partial \dot{q}_k} \qquad = 1, \dots, N \tag{10.A.3}$$

are *generalized momenta* and

$$H := \mathbf{p} \cdot \mathbf{q} - L = \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k - L \tag{10.A.4}$$

is considered as an *energy* and if expressed through $\mathbf{q}, \mathbf{p}, t$ is called *Hamiltionian* $H = H(\mathbf{q}, \mathbf{p}, t)$.

Transition from generalized velocities to generalized momenta are called *Legendre transformation* and remarkable fact is that in $(\mathbf{q},\mathbf{p})$ movement equations are

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \tag{10.A.5}$$

$$\dot{p}_k = -\frac{\partial H}{\partial q_k}, \qquad k=1,\dots,N. \tag{10.A.6}$$

This is a *Hamiltonian formalism* and $-\frac{\partial H}{\partial q_k}$ are called *generalized forces*. Another remarkable equality is

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \tag{10.A.7}$$

where in the left-hand expression $(\mathbf{q},\mathbf{p})$ are considered as functions of t.

We will not pursue this road, just mention that if we fix $\mathbf{q}(t_0)$ and calculate action S defined by (10.A.1) along extremals we get $S = S(\mathbf{q},t)$. Then it satisfies *Hamilton-Jacobi equation*

$$\frac{\partial S}{\partial t} + H(\mathbf{q}, \nabla S, t) = 0 \tag{10.A.8}$$

which is a *first order nonlinear PDE* mentioned in Subsection 2.2.2.

10.A.2 Continuum dynamics

Now the state of the system is described by $u(\mathbf{x};t)$ where $\mathbf{x} = (x_1,\dots,x_n)$ are *spatial variables* and the initial state $u(x;t_0)$ and the final state $u(x;t_1)$ are fixed and action is defined by

$$S = \int_{t_0}^{t_1} \mathcal{L}(u,u_t,t)\,dt := \int_{t_0}^{t_1} \iiint L(u,u_x,\dots,,u_t,\dots,t)\,d^n x\,dt \tag{10.A.9}$$

and *Lagrangian* L in fact depends on u, u_x and may be higher derivatives of u with respect to spatial variables and on u_t and may be its derivatives (including higher order) with respect to spatial variables.

Deriving Lagrange equations we treat t as just one of the coordinates (so we have $x = (x_0,\mathbf{x}) = (x_0,x]_1,\dots,x_n)$ but defining generalized momenta

and forces and defining Hamiltonian t "sticks out":

$$\pi = \frac{\delta \mathcal{L}}{\delta u_t}, \tag{10.A.10}$$

$$\varphi = -\frac{\delta \mathcal{L}}{\delta u}, \tag{10.A.11}$$

$$H = \iiint \pi u_t \, dx - \mathcal{L}. \tag{10.A.12}$$

Example 10.A.1. Let

$$S = \frac{1}{2} \int \iiint \Big(\rho u_t^2 - K|\nabla u|^2 + 2fu \Big) \, d^n x dt; \tag{10.A.13}$$

Here $f = f(\mathbf{x}, t)$ is a density of external force.

Then corresponding Lagrange equation is

$$-(\rho u_t)_t + \nabla \cdot (K \nabla u) - f = 0 \tag{10.A.14}$$

which for constant ρ, K becomes a standard *wave equation*. Meanwhile as

$$\mathcal{L} = \frac{1}{2} \iiint \Big(\rho u_t^2 - K|\nabla u|^2 + 2fu \Big) \, d^n x \tag{10.A.15}$$

we have according to (10.A.10)–(10.A.12)

$$\begin{aligned} &\pi(x) = \rho u_t, \\ &\varphi(x) = \nabla \cdot (K \nabla u) + f, \\ &H = \frac{1}{2} \iiint \Big(\rho u_t^2 + K|\nabla u|^2 - 2fu \Big) \, d^n x \end{aligned} \tag{10.A.16}$$

and H is preserved as long as ρ, K, f do not depend on t.

Example 10.A.2. Similarly,

$$S = \frac{1}{2} \int \iiint \Big(\rho |\mathbf{u}_t|^2 - \lambda |\nabla \otimes \mathbf{u}|^2 - \mu |\nabla \cdot \mathbf{u}|^2 + 2\mathbf{f} \cdot \mathbf{u} \Big) \, d^n x dt \tag{10.A.17}$$

with constant ρ, λ, μ leads to *elasticity equations*

$$-\rho \mathbf{u}_{tt} + \lambda \Delta \mathbf{u} + \mu \nabla (\nabla \cdot \mathbf{u}) + \mathbf{f} = 0 \tag{10.A.18}$$

and

$$H = \frac{1}{2} \iiint \Big(\rho |\mathbf{u}_t|^2 + \lambda |\nabla \otimes \mathbf{u}|^2 + \mu |\nabla \cdot \mathbf{u}|^2 - 2\mathbf{f} \cdot \mathbf{u} \Big) \, d^n. \tag{10.A.19}$$

Example 10.A.3. Let $n = 3$, then $|\nabla \otimes \mathbf{u}|^2 = |\nabla \times \mathbf{u}|^2 + |\nabla \cdot \mathbf{u}|^2$. Taking in Example 2 $\rho = 1$, $\mu = -\lambda = -c^2$ and $\mathbf{f} = 0$ we have

$$S = \frac{1}{2}\int \iiint \Big(|\mathbf{u}_t|^2 - c^2|\nabla \times \mathbf{u}|^2\Big)\, d^3x dt, \tag{10.A.20}$$

$$-\mathbf{u}_{tt} - c^2 \nabla \times (\nabla \times \mathbf{u}) = 0 \tag{10.A.21}$$

which is Maxwell equations without charges and currents for a *vector potential* $\mathbf{u}$, taking $\mathbf{E} = \mathbf{u}_t$, $\mathbf{H} = \nabla \times \mathbf{u}$ we arrive to Maxwell equations in more standard form Section 14.3.

Example 10.A.4. Let

$$S = \frac{1}{2}\int \iiint \Big(u_t^2 - \sum_{i,j} K u^2_{x_i x_j} + 2fu\Big)\, d^n x dt. \tag{10.A.22}$$

Then we arrive to

$$-u_{tt} - K\Delta^2 u + f = 0 \tag{10.A.23}$$

which is *vibrating beam equation* as $n = 1$ and *vibrating plate equation* as $n = 2$; further

$$H = \frac{1}{2}\iiint \Big(\rho u_t^2 + \sum_{i,j} K u^2_{x_i x_j} - 2fu\Big)\, d^n. \tag{10.A.24}$$

Example 10.A.5. Let u be complex-valued function, $\bar{u}$ its complex-conjugate and

$$S = \int \iiint \Big(-i\hbar u_t \bar{u} - \big(\frac{\hbar^2}{2m}|\nabla u|^2 + V(x)|u|^2\big)\Big)\, d^n x dt. \tag{10.A.25}$$

It does not seem to be real-valued but it is in its essential part: integrating by parts by t the first term, we get $i\hbar u \bar{u}_t$ which is complex-conjugate to $-i\hbar u_t \bar{u}$ plus terms without integration by t (and with $t = t_0, t_1$).

Then Lagrange equation is $-i\hbar \bar{u}_t + (\frac{\hbar^2}{2m}\Delta - V)\bar{u} = 0$ which is equivalent to the standard Schrdinger equation

$$i\hbar u_t = -\frac{\hbar^2}{2m}\Delta u + V(x)u; \tag{10.A.26}$$

further $\pi(x) = -i\hbar\bar{u}$ and

$$H = \iiint \Big(\frac{\hbar^2}{2m}|\nabla u|^2 + V(x)|u|^2\Big)\, d^n x = \iiint \Big(-\frac{\hbar^2}{2m}\Delta u + V(x)u\Big)\bar{u}\, d^n x. \tag{10.A.27}$$

Hint. Write $u = u_1 + iu_2$ where $u_1 = \operatorname{Re} u, u_2 = \operatorname{Im} u$ and use

$$\frac{\delta}{\delta u} = \frac{1}{2}\left(\frac{\delta}{\delta u_1} - i\frac{\delta}{\delta u_2}\right) \tag{10.A.28}$$

which corresponds conventions of CV.

10.A.3 Equilibria

Selecting $u = u(\mathbf{x})$ in the dynamical equations (and then $u_t = 0$) we get equations of equilibria. Equilibria which delivers to potential energy U local minimum is stable; otherwise it is unstable.

Exercise 10.A.1. In Examples 10.A.1, 10.A.2 and 10.A.4 write equations of equilibria.

10.3 Problems to Chapter 10

There are several classical problems

Problem 10.3.1. The heavy flexible but unstretchable wire (chain) has a length and an energy respectively

$$L = \int_0^a \sqrt{1 + u'^2}\, dx, \tag{10.3.1}$$

$$U = \rho g \int_0^a u\sqrt{1 + u'^2}\, dx \tag{10.3.2}$$

where ρ is a linear density.

(a) Write down an equation minimizing energy U as length L is fixed.

(b) Find solution satisfying $u(0) = h_0$, $u(a) = h_1$.

Problem 10.3.2. We need to construct the fastest slide from point $(0,0)$ to $(a,-h)$. If $u(x)$ describes its shape then time is

$$T = \int_0^a \frac{1}{\sqrt{2gu}}\sqrt{1+u'^2}\,dx. \tag{10.3.3}$$

(a) Write down an equation minimizing energy U as length L is fixed.

(b) Find solution satisfying $u(0)=0$, $u(a)=-h$.

Chapter 11

Distributions and weak solutions

In this Chapter we extend notion of function. These new "functions" (actually most of them are not functions at all) are called *distributions* and are very useful for PDE (and not only). We define them as linear forms on the *test functions* which are some nice functions. For usual function f such form is

$$f(\varphi) = \int f(x)\varphi(x), dx.$$

We also extend the notion of solution.

11.1 Distributions

11.1.1 Test functions

We introduce three main spaces of test functions:

Definition 11.1.1. Let

(a) $\mathcal{D} = C_0^\infty$ is a space of infinitely smooth functions with compact support. It means that for each function φ exists a such that $\varphi(x) = 0$ as $|x| \geq a$;

(b) $\mathcal{E} = C^\infty$ is a space of infinitely smooth functions;

(c) $\mathcal{S}$ is a space of infinitely smooth functions which decay at infinity (with all their derivatives faster than any power:

$$|\partial^m \varphi|(1+|x|)^k \le M_{mk} \qquad \forall x\, \forall m,k. \tag{11.1.1}$$

Loran Schwartz who provided the first systematic theory of distributions used these notations and they became traditional. However we need to explain what does it mean *convergence of test function*:

Definition 11.1.2. (a) $\varphi_n \to \varphi$ in $\mathcal{D}$ iff $\max |\partial^m(\varphi_n - \varphi)| \to 0$ as $n \to \infty$ for all m and also there exist a such that $\varphi_n(x) = 0$ as $|x| \ge a$ for all n;

(b) $\varphi_n \to \varphi$ in $\mathcal{E}$ iff $\max_{|x|\le a} |\partial^m(\varphi_n - \varphi)| \to 0$ as $n \to \infty$ for all m, a;

(c) $\varphi_n \to \varphi$ in $\mathcal{S}$ iff

$$|\partial^m(\varphi_n - \varphi)|(1+|x|)^k \to 0 \qquad \forall x\, \forall m,k. \tag{11.1.2}$$

Theorem 11.1.1.

$$\mathcal{D} \subset \mathcal{S} \subset \mathcal{E} \tag{11.1.3}$$

where $\mathcal{K}_1 \subset \mathcal{K}_2$ means not only that all elements of $\mathcal{K}_1$ are also elements of $\mathcal{K}_2$ but also that $\varphi_n \overset{\mathcal{K}_1}{\to} \varphi$ implies that $\varphi_n \overset{\mathcal{K}_2}{\to} \varphi$. Also in (11.1.3) *each smaller space $\mathcal{K}_1$ is* dense *in the larger one $\mathcal{K}_2$: for each $\varphi \in \mathcal{K}_2$ there exists a sequence $\varphi_n \in \mathcal{K}_1$ converging to φ in $\mathcal{K}_2$.*

Remark 11.1.1. Those who studies Real Analysis heard about *Topological Vector Spaces* but we are not going to introduce topology (which is ridiculously complicated on $\mathcal{D}$), just convergence is sufficient for all needs. The same approach is also used in the very advanced cources.

11.1.2 Distributions

Definition 11.1.3. (a) *Distribution* f (over $\mathcal{K}$) is a continuous linear form on $\mathcal{K}$: $f : \mathcal{K} \to \mathbb{C}$ such that

$$f(\alpha_1\varphi_1 + \alpha_2\varphi_2) = \alpha_1 f(\varphi_1) + \alpha_2 f(\varphi_2) \qquad \forall \varphi_1,\varphi_2 \in \mathcal{K}\ \forall \alpha_1,\alpha_2 \in \mathbb{C};$$
$$\varphi_n \overset{\mathcal{K}}{\to} \varphi \implies f(\varphi_n) \to f(\varphi).$$

(b) The space of such linear forms is denoted by $\mathcal{K}'$.

11.1.1 immediately implies

Theorem 11.1.2.

$$\mathcal{D}' \supset \mathcal{S}' \supset \mathcal{E}' \tag{11.1.4}$$

where $\mathcal{K}_1' \supset \mathcal{K}_2'$ *means not only that all elements of* $\mathcal{K}_2$ *are also elements of* $\mathcal{K}_1$ *but also that* $f_n \overset{\mathcal{K}_2'}{\to} f$ *implies that* $f_n \overset{\mathcal{K}_1'}{\to} f$. *Also in (11.1.4) each smaller space* $\mathcal{K}_2'$ *is* dense *in the larger one* $\mathcal{K}_1'$*: for each* $f \in \mathcal{K}_1'$ *there exists a sequence* $f_n \in \mathcal{K}_2'$ *converging to* f *in* $\mathcal{K}_1'$.

So far we have not introduced the convergence of distributions, so we do it right now:

Definition 11.1.4. $f_n \overset{\mathcal{K}'}{\to} f$ iff $f_n(\varphi) \to f(\varphi)$ for all $\varphi \in \mathcal{K}$.

Remark 11.1.2. (a) $\mathcal{E}'$ consists of distributions with compact support: $f \in \mathcal{D}'$ belongs to $\mathcal{E}'$ iff there exists a such that $f(\varphi) = 0$ for all φ such that $\varphi(x) = 0$ as $|x| \le a$.

(b) $\mathcal{S}'$ consists off *temperate distributions*.

(c) For $f \in L^1_{\mathrm{loc}}$ we can define action $f(\varpi)$ on $\mathcal{D}$

$$f(\varphi) = \int f(x)\varphi(x)\,dx \tag{11.1.5}$$

where integral is always understood as integral over the whole line $\mathbb{R}$ (or a whole space $\mathbb{R}^d$) and L^1_{loc} consists of *locally integrable functions* (notion from the Real Analysis which means that $\int_{|x|\le a} |f(x)|\,dx < \infty$ for all a but integral is a *Lebesgue integral* which is a natural extension of Riemann integral). One can prove that this form is continuous and thus $f \in \mathcal{D}'$. Due to this we sometimes non–rigorously will write (11.1.5) even for distributions which are not ordinary functions.

Example 11.1.1. $\delta := \delta(x)$ is an element of $\mathcal{E}'$ defined as $\delta(\varphi) = \varphi(0)$. It is traditionally called δ*-function* or *Dirac* δ*-function* despite not being a function but a distribution.

11.1.3 Operations on distributions

We introduce operations on distributions as an extension of operations on ordinary functions as long as they make sense.

Definition 11.1.5. *Linear operations*:

$$(\alpha_1 f_1 + \alpha_2 f_2)(\varphi) = \alpha_1 f_1(\varphi) + \alpha_2 f_2(\varphi) \tag{11.1.6}$$

as $\alpha_1, \alpha_2 \in \mathbb{C}$.

Exercise 11.1.1. Check that for ordinary functions f_1, f_2 we get a standard definition of $\alpha_1 f_1 + \alpha_2 f_2$ (in the framework of (11.1.5)).

Definition 11.1.6. *Shift.* Let T_a denote a shift of φ: $(T_a\varphi)(x) = \varphi(x-a)$. Then

$$(T_a f)(\varphi) = f(T_{-a}\varphi). \tag{11.1.7}$$

We will write $T_a f$ as $f(x-a)$.

Exercise 11.1.2. (a) Check that for ordinary function f we get a standard definition of $f(x-a)$ (in the framework of (11.1.5)).

(b) Check that for δ we $\delta_a(x) := \delta(x-a)$ is defined as $\delta_a(\varphi) = \varphi(a)$.

Definition 11.1.7. *Linear change of variables.* Let R_A denote a linear change of variables: $(R_A\varphi)(x) = \varphi(Ax)$ where A is a non-degenerate linear transformation. Then

$$(R_A f)(\varphi) = |\det A|^{-1} f(R_{A^{-1}}\varphi) \tag{11.1.8}$$

We will write $R_A f$ as $f(Ax)$.

Exercise 11.1.3. (a) Check that for ordinary function f we get a standard definition of $R_A f$ (in the framework of (11.1.5)).

(b) Check that for δ we get $\delta(Ax) = j \det A j\{-1\}\delta(x)$. In particular as $|\det A| = 1$ we have $\delta(Ax) = \delta(x)$ and as $Ax = \lambda x$ (uniform dilatation) $\delta(\lambda x) = \lambda^{-d}\delta(x)$ where d is a dimension. Therefore δ is *spherically symmetric* and *positively homogeneous of degree* $-d$.

Definition 11.1.8. *Derivative.* Then

$$(\partial f)(\varphi) = -f(\partial\varphi) \tag{11.1.9}$$

where ∂ is a first order derivative.

Exercise 11.1.4. (a) Check that for ordinary function f we get a standard definition of ∂f (in the framework of (11.1.5)). Use integration by parts.

(b) Check that for δ we get δ': $\delta'_a(\varphi) = -\varphi'(a)$ (in one dimension and similarly in higher dimensions).

(c) Check that if $\theta(x)$ is a *Heaviside function*: $\theta(x) = 1$ as $x > 0$ and $\theta(x) = 0$ as $x \le 0$ then $\theta'(x) = \delta(x)$.

(d) Check that if $f(x)$ is a smooth function as $x < a$ and as $x > 0$ but with a jump at a then $f' = \overset{\circ}{f}' + (f(a+0) - f(a-0))\delta(x-a)$ where f' is understood in the sense of distributions and $\overset{\circ}{f}'(x)$ is an ordinary function equal to derivative of f as $x \ne a$.

(e) Prove that if $f = \ln|x|$ then $f'(\varphi) = pv\int x\{-1\}\varphi(x), dx$ where integral is understood as a principal value integral.

Let $g \in C^\infty$. Observe that for $g\varphi \in \mathcal{D}$ and $g\varphi \in \mathcal{E}$ for $\varphi \in \mathcal{D}$ and $\varphi \in \mathcal{E}$ respectively. Therefore the following definition makes sense:

Definition 11.1.9. *Multiplication by a function.* Let either $f \in \mathcal{D}'$ or $f \in \mathcal{E}'$. Then $gf \in \mathcal{D}'$ or $gf \in \mathcal{E}'$ respectively is defined as

$$(gf)(\varphi) = f(g\varphi). \tag{11.1.10}$$

Exercise 11.1.5. (a) Check that for ordinary function f we get a standard definition of gf (in the framework of (11.1.5)).

(b) Prove that $g\delta_a = g(a)\delta_a$ (use definitions);

(c) Calculate $g\delta'_a$, $g\delta''_a$ (use definitions!).

We cannot define in general the product of two distributions. However in some cases it is possible, f.e. when distributions are of different arguments.

Definition 11.1.10. *Direct product.* Let f, g be distributions. Then $f(x)g(y)$ (also denoted as $f \otimes g$) is defined as

$$(fg)(\varphi) = f(g(\varphi)) \tag{11.1.11}$$

where $\varphi = \varphi(x, y)$, then applying g to it we get $\psi(x) := g(\varphi)$ a test function, and then applying f we get a number. Similarly we get the same fg if we apply first f and then g.

Exercise 11.1.6. (a) Check that for ordinary functions f, g we get a standard definition of fg (in the framework of (11.1.5)).

(b) Prove that $\delta_{\{a_1\}}(x_1) \cdots \delta_{\{a_d\}}(x_d) = \delta_a(x)$ with $a = (a_1, \dots, a_d)$, $x = (x_1, \dots, x_d)$ and we have on the left product of 1-dimensional δ-functions and on the right n-dimensional.

11.2 Distributions: more

11.2.1 Supports

Definition 11.2.1. Let us consider ordinary function f. Observe that if $f = 0$ on open sets Ω_ι (where ι runs any set of indices—-finite, infinite or even non-enumerable) then $f = 0$ on $\bigcup_\iota \Omega_\iota$. Therefore there exists a largest open set Ω such that $f = 0$ on Ω. Complement to this set is called *support of* f and denoted as $\operatorname{supp}(f)$.

Definition 11.2.2. (a) Let us consider distribution f. We say that $f = 0$ on open set *Omega* if $f(\varphi) = 0$ for any test function φ such that $\operatorname{supp} \varphi \subset \Omega$.

(b) Then the same observation as in (a) holds and therefore there exists a largest open set Ω such that $f = 0$ on Ω. Complement to this set is called *support of* f and denoted as $\operatorname{supp}(f)$.

Definition 11.2.3. Observe that $\operatorname{supp}(f)$ is always a closed set. If it is also bounded we say that f has a compact support.

Exercise 11.2.1. (a) Prove that for two functions f, g and for $f \in \mathcal{D}'$, $g \in \mathcal{E}$

$$\operatorname{supp}(gf) \subset \operatorname{supp}(f) \cap \operatorname{supp}(g), \tag{11.2.1}$$
$$\operatorname{supp}(\partial f) \subset \operatorname{supp}(f) \tag{11.2.2}$$

where ∂ is a differentiation;

(b) Prove that $\operatorname{supp}(f) = \emptyset$ iff $f = 0$ identiacally;

(c) Prove that $\operatorname{supp}(\delta_a) = \{a\}$. Prove that the same is true for any of its derivatives.

Remark 11.2.1. In fact, the converse to Exercise 1(c) is also true: if $\operatorname{supp}(f) = \{a\}$ then f is a linear combination of $\delta(x-a)$ and its derivatives (up to some order).

Remark 11.2.2. In the previous section we introduced spaces of test functions $\mathcal{D}$ and $\mathcal{E}$ and the corresponding spaces of distributions $\mathcal{D}'$ and $\mathcal{E}'$. However for domain $\Omega \subset \mathbb{R}^d$ one can introduce $\mathcal{D}(\Omega) := \{\varphi \in \mathcal{D} : \operatorname{supp}\varphi \subset \Omega\}$ and $\mathcal{E} = C^\infty(\Omega)$. Therefore one can introduce corresponding spaces of distributions $\mathcal{D}'(\Omega)$ and $\mathcal{E}'(\Omega) = \{f \in \mathcal{E} : \operatorname{supp} f \subset \Omega\}$. As $\Omega = \mathbb{R}^d$ we get our "old spaces".

11.2.2 Non-linear change of variables

Definition 11.2.4. Let f be a distribution with $\operatorname{supp} f \subset \Omega_1$ and let $\Phi : \Omega_1 \to \Omega_2$ be one-to-one correspondence, infinitely smooth and with non-vanishing Jacobian $\det \Phi'$. Then $\Phi_* f$ is a distribution:

$$(\Phi_* f)(\varphi) = f(|\det \Phi'| \cdot \Phi^* \varphi) \tag{11.2.3}$$

where $(\Phi^*\varphi)(x) = \varphi(\Phi(x))$.

Remark 11.2.3. (a) This definition generalizes Definitions 11.1.6 and 11.1.7

(b) Mathematicians call $\Phi^*\varphi$ *pullback of* φ and $\Phi_* f$ *pushforward* of f.

Exercise 11.2.2. Check that for ordinary function f we get $(\Phi_* f)(x) = f(\Phi^{-1}(x))$.

11.2.3 Fourier transform

Definition 11.2.5. Let $f \in \mathcal{S}'$. Then Fourier transform $\hat{f} \in \mathcal{S}'$ is defined as

$$\hat{f}(\varphi) = f(\hat{\varphi}) \tag{11.2.4}$$

for $\varphi \in \mathcal{S}$. Similarly, inverse Fourier transform $\check{f} \in \mathcal{S}'$ is defined as

$$\check{f}(\varphi) = f(\check{\varphi}) \tag{11.2.5}$$

Exercise 11.2.3. (a) Check that for ordinary function f we get a standard definition of $\hat{f}$ and $\check{f}$.

(b) To justify Definition 5 one need to prove that $f \in \mathcal{S} \iff \hat{f} \in \mathcal{S}$. Do it!

(c) Prove that for $f \in \mathcal{E}'$ both $\hat{f}$ and $\check{f}$ are ordinary smooth functions

$$\hat{f}(k) = (2\pi)^{-d} f(e^{-ix\cdot k}), \tag{11.2.6}$$

$$\check{f}(k) = f(e^{ix\cdot k}). \tag{11.2.7}$$

(d) Check that all properties of Fourier transform (excluding with norms and inner products which may not exist are preserved.

Exercise 11.2.4. (a) Prove that Fourier transforms of $\delta(x-a)$ is $(2\pi)^{-d}e^{-ix\cdot a}$.

(b) Prove that Fourier transforms of $e^{ix\cdot a}$ is $\delta(x-a)$.

Exercise 11.2.5. In dimension 1

(a) Prove that Fourier transforms of $\theta(x-a)$ and $\theta(-x+a)$ are respectively $(2\pi i)^{-1}(k-a-i0)^{-1}$ and $-(2\pi i)^{-1}(k-a+i0)^{-1}$ which are understood as limits in the sense of distributions of $(2\pi i)^{-1}(k-a\mp i\varepsilon)^{-1}$ as $\varepsilon \to +0$. Recall that $\theta(x)$ is a Heaviside function.

(b) As a corollary conclude that Fourier transform of $\operatorname{sgn}(x) := \theta(x) - \theta(-x) = x/|x|$ is $(2\pi i)^{-1}\big((k-a-i0)^{-1}+(k-a+i0)\big)^{-1} = \pi^{-1}(k-a)^{-1}$ with the latter understood in as principal value (see 11.1.4(f)).

(c) As another corollary conclude that Fourier transform of $\theta(x)+\theta(-x) = 1$ is $(2\pi i)^{-1}\big((k-a-i0)^{-1} - (k-a+i0)\big)^{-1}$ and therefore

$$(2\pi i)^{-1}\big((k-a-i0)^{-1} - (k-a+i0)\big)^{-1} = \delta(x-a). \tag{11.2.8}$$

11.2.4 Convolution

Recall convolution (see Definition 5.2.1) and its connection to Fourier transform.

Definition 11.2.6. Let $f, g \in \mathcal{D}'$ (or other way around), $\varphi \in \mathcal{D}$ Then we can introduce $h(y) \in \mathcal{E}$ as

$$h(y) = g(T_y\varphi), \qquad T_y\varphi(x) := \varphi(x-y).$$

Observe that $h \in \mathcal{D}$ provided $g \in \mathcal{E}'$. In this case we can introduce $h \in \mathcal{E}$ for $\varphi \in \mathcal{E}$.

Therefore if either $f \in \mathcal{E}'$ or $g \in \mathcal{E}'$ we introduce $f * g$ as

$$(f * g)(\varphi) = f(h).$$

Exercise 11.2.6. (a) Check that for ordinary function f we get a standard definition of the convolution;

(b) Prove that convolution convolution has the same properties as multiplication;

(c) Prove that Theorem 5.2.4 holds;

(d) Prove that $f * \delta = \delta * f = f$;

(e) Prove that $\partial(f * g) = (\partial f) * g = f * (\partial g)$;

(f) Prove that $T_a(f * g) = (T_a f) * g = f * (T_a g)$ for operator of shift T_a;

(g) Prove that $\operatorname{supp}(f * g) \subset \operatorname{supp}(f) + \operatorname{supp}(g)$ where *arithmetic sum* of two sets is defined as $A + B := \{x + y : x \in A, y \in B\}$.

Remark 11.2.4. (a) One can prove that if a linear map $L : \mathcal{E}' \to \mathcal{D}'$ commutes with all shifts: $T_a(Lf) = L(T_a f)$ for all $f \in \mathcal{E}'$ then there exists $g \in \mathcal{D}'$ such that L is an operator of convolution: $Lf = g * f$;

(b) One can extend convolution if none of f, g has a compact support but some other assumption is fulfilled. For example, in one–dimensional case we can assume that either $\operatorname{supp}(f) \subset [a, \infty)$, $\operatorname{supp}(g) \subset [a, \infty)$ or that $\operatorname{supp}(f) \subset (-\infty, a]$, $\operatorname{supp}(g) \subset (-\infty, a]$.

Similarly in multidimensional case we can assume that $\operatorname{supp}(f) \subset C$, $\operatorname{supp}(g) \subset C$ where C is a cone with angle $< -\pi$ at its vertex a.

11.2.5 Fourier series

Definition 11.2.7. (a) We call one-dimensional distribution f *periodic with period* L if $f(x-L)=f(x)$.

(b) More generally, let Γ be a lattice of periods (see Definition 4.B.1). We call distribution f Γ-*periodic* if $f(x-n)=f(x)$ for all $n\in\Gamma$.

Periodic distributions could be decomposed into Fourier series: in one-dimensional case we have

$$f=\sum_{-\infty<m<\infty} c_n e^{\frac{2\pi imx}{L}} \tag{11.2.9}$$

and in multidimensional case

$$f=\sum_{m\in\Gamma^*} c_m e^{im\cdot} \tag{11.2.10}$$

where Γ^* is a dual lattice (see Definition 4.B.3).

To define coefficients c_m we cannot use ordinary formulae since integral over period (or elementary cell, again see the same definition) is not defined properly. Instead we claim that there exists $\varphi\in\mathcal{D}$ such that

$$\sum_{n\in\Gamma}\varphi(x-n)=1. \tag{11.2.11}$$

Indeed, let $\psi\in\mathcal{D}$ be non-negative and equal 1 in some elementary cell. Then $\varphi(x)=\psi(x)/\bigl(\sum_{n\in\Gamma}\psi(x-n)\bigr)$ is an appropriate function.

Then

$$c_m=|\Omega|^{-1}(\varphi f)(e^{-im\cdot x} \tag{11.2.12}$$

where $|\Omega|$ is a volume of the elementary cell.

Exercise 11.2.7. (a) Find decomposition in Fourier series of one-dimensional distribution $f=\sum_{-\infty<n<\infty}\delta(x-nL)$;

(b) Find Fourier transform of f defined in (a);

(c) Find the connection to Poisson summation formula (see Theorem 5.2.5).

(d) Find decomposition in Fourier series of d-dimensional distribution $f=\sum_{n\in\Gamma}\delta(x-n)$;

(e) Find Fourier transform of f defined in (d);

(f) Find the connection to multidimensional Poisson summation formula (see Remark 5.2.4).

11.3 Applications of distributions

11.3.1 Linear operators and their Schwartz kernels

During this course we considered many linear operators: differential operators like $u \to Lu := au_{xx} + 2bu_{xy} + cu$ and integral operators

$$u(x) \to (Lu)(x) := \int K(x,y)u(y)\,dy \tag{11.3.1}$$

(recall that solutions of IVP, BVP and IBVP for PDEs was often given in such form). For integral operators $K(x,y)$ is called *(integral) kernel of operator* L (not to be confused with the *kernel of operator* L which is $N(L) = \{u : Lu = 0\}$.

L.Schwartz proved a remarkable theorem showing that each linear operator is in some sense integral operator:

Theorem 11.3.1. *(a) Let $L : \mathcal{D}(\Omega_1) \mapsto \mathcal{D}'(\Omega_2)$ be a linear continuous operator (the latter means that if $u_n \to u$ in $\mathcal{D}(\Omega_1)$ then $Lu_n \to Lu$ in $\mathcal{D}'(\Omega_2)$). Then there exists a unique $K \in \mathcal{D}'(\Omega_2 \times \Omega_1)$ such that for any $u \in \mathcal{D}(\Omega_1)$ and $v \in \mathcal{D}(\Omega_2)$*

$$(Lu)(v) = K(v \otimes u) \tag{11.3.2}$$

with $(v \otimes u)(x,y) = v(x)u(y)$ (then $(v \otimes u) \in \mathcal{D}(\Omega_2 \times \Omega_1)$).

(b) Conversely, if $K \in \mathcal{D}'(\Omega_2 \times \Omega_1)$ then there exists a unique linear continuous operator $L : \mathcal{D}(\Omega_1) \mapsto \mathcal{D}'(\Omega_2)$ such that (11.3.2) holds.

Definition 11.3.1. Such K is called *Schwartz kernel of operator* L.

Example 11.3.1. (a) For integral operator (11.3.1) Schwartz kernel is $K(x,y)$;

(b) For $I : u \mapsto u$ Schwartz kernel is $\delta(x-y)$;

(c) In 1-dimensional case for operator of differentiation $u \mapsto u'$ Schwartz kernel is $\delta(x-y)$ (dimension is 1); similarly in multi-dimensional case $\partial_{x_j} : u \mapsto u_{x_j}$ Schwartz kernel is $\delta_{x_j}(x-y) = -\delta_{y_j}(x-y)$.

(d) For higher order derivatives we have the same picture: In 1-dimensional case for operator of second differentiation $u \mapsto u'$ Schwartz kernel is $\delta(x-y)$ (dimension is 1); similarly in multi-dimensional case $\partial_{x_j}\partial_{x_k} : u \mapsto u_{x_j x_k}$ Schwartz kernel is $\delta_{x_j x_k}(x-y) = -\delta_{x_j y_k}(x-y) = \delta_{y_j y_k}(x-y)$.

More examples are coming.

11.3.2 Densities

In Physics you considered masses or charges at several points (*atom* masses or charges) and continuously distributed. In the Probability Theory the same was for probabilities. Continuously distributed masses, charges or probabilities had densities while atom masses, charges or probabilities had not.

Remark 11.3.1. For those who took Real Analysis: in fact densities were assigned only to *absolutely continuous*, not to *singular continuous* distributions. But it does not matter here.

The theory of distributions allows us a unified approach. Consider 1-dimensional case first. Let $Q(x)$ be a charge of $(-\infty, x)$. Similarly, in probability theory let $P(x)$ be a probability to fall into $(-\infty, x)$. Both $Q(x)$ and $P(x)$ are functions of distribution (of charge or probability). If they were smooth functions then their derivatives would be densities. But even if they are not we can differentiate them in the sense of distributions!

Similarly, in d-dimensional case for $x = (x_1, x_2, \dots, x_d)$ we can introduce d-dimensional functions of distributions: let $Q(x)$ be a charge of $(-\infty, x_1) \times (-\infty, x_2) \times \dots \times (-\infty, x_d)$ or probability to fall into this set. Then

$$p(x) = \partial_{x_1}\partial_{x_2}\cdots\partial_{x_d}P(x) \tag{11.3.3}$$

is a density.

Example 11.3.2. (a) For charge or probability q concentrated in a single point a the density is $q\delta(x-a)$. If charges $q_1, \dots, q_N$ are concentrated in the points $a_1, \dots, a_N$ then a density is $\sum_{1\le n\le N} q_n\delta(x-a_N)$.

(b) If the charge or probability is concentrated on the plane $\{x_1 = a_1\}$ but on this plane it is distributed with a density $p(x_2, \dots, x_d)$ then d-dimensional density is $\delta(x_1 - a_1)p(x_2, \dots, x_d)$.

(c) We can "mix and match" continuously distributed and concentrated at points, lines and surfaces charges or probabilities which allows us in the equations involving densities not to distinguish between cases.

11.3.3 Laplace equation

We know that function $g(x) = \begin{cases} |x|^{2-d} & d \neq 2, \\ \ln|x| & d = 2 \end{cases}$ satisfies Laplace equation as $x \neq 0$. Now we need to check what happens at $x = 0$, Namely, we need to calculate Δg in the sense of distributions and to do this we consider

$$g(\Delta\varphi) = \int g(x)\Delta\varphi(x)\,dx = \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} g(x)\Delta\varphi(x)\,dx$$

where $\Omega_\varepsilon = \mathbb{R}^d \setminus B(0,\varepsilon)$. The same calculations as earlier in Subsection 7.2.2 show that it is equal to (we skip limit temporarily)

$$\int_{\Omega_\varepsilon} (\Delta g(x))\varphi(x)\,dx + \int_{S(0,\varepsilon)} \Big[(-\partial_r\varphi(x))g(x) + \varphi(x)\partial_r g(x)\Big]\,d\sigma$$

and since $\Delta g = 0$ in Ω_ε the first term is 0 and the second one is

$$\int_{S(0,\varepsilon)} \Big[(-\varepsilon^{2-d}\partial_r\varphi(x)) + \underbrace{(2-d)\varepsilon^{1-d}}\,\varphi(x)\Big]\,d\sigma$$

where for $d = 2$ selected expression should be replaced by ε^{-1}. Taking the limit we see that since the area of $S(0,\varepsilon)$ is $\sigma_d\varepsilon^{-1}$ the first term disappears and the second becomes $(2-d)\sigma_d\varphi(0)$ as $d \neq 2$ and $\sigma_2\varphi(0)$ where σ_d is an "area" of $S(0,1)$.

Therefore

$$\Delta G(x) = \delta(x) \tag{11.3.4}$$

where

$$G(x) = \begin{cases} -(d-2)^{-1}\sigma_d^{-1}|x|^{2-d} & d \neq 2, \\ (2\pi)^{-1}\ln|x| & d = 2 \end{cases}. \tag{11.3.5}$$

But then $u = G * f$ solves $\Delta u = f$ in $\mathbb{R}^d$. Indeed, as Δ has constant coefficients and derivatives could be applied to any "factor" in the convolution $\Delta(G * f) = (\Delta G) * f = \delta * f = f$.

Remark 11.3.2. (a) We get $G(x) = \frac{1}{2}|x|$ as $d = 1$, $G(x) = \frac{1}{2\pi} \ln |x|$ as $d = 2$ and $-\frac{1}{4\pi}|x|^{-1}$ as $d = 3$;

(b) As $d \geq 3$ $G(x)$ is a unique solution to (11.3.4) under additional condition $G(x) = o(1)$ as $|x| \to \infty$ (or $G(x) = O(|x|^{2-d})$ as $|x| \to \infty$;

(c) Up to a numerical factor one can guess $G(x)$ (as $d \geq 3$) from the following arguments: $\delta(x)$ is spherically symmetric positive homogeneous of degree $-d$, Δ is spherically symmetric and it decreases degree of homogeneity by 2 so G must be spherically symmetric positive homogeneous of degree $2 - d$. These arguments fail as $d = 2$ as because there is no unique natural solution (solutions differ by a constant).

Remark 11.3.3. As $d = 3$ from Subsection 9.1.5 we can guess that $G(x, \omega) = -\frac{1}{4\pi}|x|^{-1} e^{\mp i\omega|x|}$ solves Helmholtz equation with Sommerfeld radiating conditions

$$\big(\Delta + \omega^2\big)G = \delta(x), \tag{11.3.6}$$

$$G = o(1) \qquad \text{as } |x| \to \infty, \tag{11.3.7}$$

$$(\partial_r \pm i\omega)G = o(r^{-1}) \qquad \text{as } |x| \to \infty. \tag{11.3.8}$$

Remark 11.3.4. Similarly Green function $G(x, y)$ for Dirichlet boundary problem in Ω satisfies

$$\Delta_x G = \delta(x - y), \tag{11.3.9}$$

$$G|_{x \in \partial\Omega} = 0. \tag{11.3.10}$$

and Green function $G(x, y)$ for Robin boundary problem in Ω satisfies the same equation but with Robin boundary condition

$$(\partial_{\nu_x} + \alpha)G|_{x \in \partial\Omega} = 0. \tag{11.3.11}$$

Here $x, y \in \Omega$.

11.3.4 Diffusion equation

Consider Green function for diffusion equation im $\mathbb{R}^d$:

$$G(x,t) = (4\pi kt)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4kt}}. \tag{11.3.12}$$

It satisfies diffusion equation $G_t = k\Delta G$ as $t > 0$ and as $t \to 0$ it tends to $\delta(x)$.

Further, if we define

$$H^+(x,t) = \begin{cases} G(x,t) & t > 0, \\ 0 & t \le 0 \end{cases} \tag{11.3.13}$$

we see that

$$(\partial_t - k\Delta)H^+(x,t) = \delta(x)\delta(t). \tag{11.3.14}$$

Similarly

$$G(x,t) = (4\pi k|t|)^{-\frac{d}{2}} e^{\mp\frac{id\pi}{4}} e^{-\frac{|x|^2}{4kit}} \qquad \text{as } \pm t > 0 \tag{11.3.15}$$

satisfies Schrdinger equation $G_t = ik\Delta G$ (even as $t = 0$) and as $t \to 0$ it tends to $\delta(x)$. Further, if we define

$$H^\pm(x,t) = \begin{cases} \pm G(x,t) & \pm t > 0, \\ 0 & \pm t \le 0 \end{cases} \tag{11.3.16}$$

we see that

$$(\partial_t - ik\Delta)H^\pm(x,t) = \delta(x)\delta(t). \tag{11.3.17}$$

11.3.5 Wave equation

It follows from Kirchhoff formula (9.1.12) that

$$G(x,t) = \frac{1}{4\pi c^2|t|}\big(\delta(|x| - ct) - \delta(|x| + ct)\big) \tag{11.3.18}$$

satisfies 3-dimensional wave equation $G_{tt} - c^2\Delta G = 0$ and also

$$G(x,t)\big|_{t=0} = 0, \qquad G(x,t)\big|_{t=0} = \delta(x). \tag{11.3.19}$$

It follows from (9.1.15) that

$$G(x,t) = \pm\frac{1}{2\pi c}(c^2t^2 - |x-y|^2)_+^{-\frac{1}{2}} \tag{11.3.20}$$

satisfies 2-dimensional wave equation $G_{tt} - c^2\Delta G = 0$ and also (11.3.19); here $z_+^\lambda = z^\lambda\theta(z)$; recall that $\theta(z)$ is a Heaviside function.

Formulae (9.1.23) and (9.1.24) allow us to construct $G(x,t)$ for odd $d \geq 3$ and even $d \geq 2$.

D'Alembert formula allows us to construct as $d = 1$:

$$G(x,t) = \frac{1}{2c}\big(\theta(x+ct) - \theta(x-ct)\big). \tag{11.3.21}$$

Defining $H^\pm(x,t)$ by (11.3.16) we get

$$(\partial_t^2 - c^2\Delta)H^\pm(x,t) = \delta(x)\delta(t). \tag{11.3.22}$$

Definition 11.3.2. If H satisfies $LH = \delta(x-y)$ it is called *fundamental solution* to L^*.

11.4 11.4. Weak solutions

11.4.1 Examples

Weak solutions occupy a place between ordinary regular solutions and solutions which are distributions, usually when the existence of the former is not proven (or proven yet, or does not have place) and the second do not make sense.

Weak solutions must satisfy certain integral identities which in some sense are equivalent to original equations.

Example 11.4.1. (a) Consider equation

$$\sum_{j,k}(a_{jk}u)_{x_jx_k} + \sum_j (b_ju)_{x_j} + cu = f \tag{11.4.1}$$

in domain Ω. In the smooth case this equation is equivalent to

$$\iint_\Omega u\Big(\sum_{j,k} a_{jk}\varphi_{x_jx_k} - \sum_j b_j\varphi_{x_j} + c\varphi\Big)\,dx = \iint_\Omega f\varphi\,dx \tag{11.4.2}$$

for all $\varphi \in C_0^2(\Omega)$ and we call $u \in C(\Omega)$ (or even worse) *weak solution* to equation (11.4.1) in this case. Note that coefficients may be even discontinuous so we cannot plug into (11.4.1) distributions.

(b) Consider equation in the *divergent form*

$$\sum_{j,k}(a_{jk}u_{x_j})_{x_k} + \frac{1}{2}\sum_j \big[(b_j u)_{x_j} + b_j u_{x_j} + cu = f \tag{11.4.3}$$

in domain Ω. In the smooth case this equation is equivalent to

$$\iint_\Omega \Big(\sum_{j,k} -a_{jk}u_{x_j}\varphi_{x_k} + \frac{1}{2}\sum_j \big[-b_j u\varphi_{x_j} + b_j u_{x_j}\varphi\big] + cu\varphi\Big)\,dx = \iint_\Omega f\varphi\,dx \tag{11.4.4}$$

for all $\varphi \in C_0^1(\Omega)$ and we call $u \in C^1(\Omega)$ (or even worse) *weak solution* to equation (11.4.3) in this case. Note that coefficients may be even discontinuous so we cannot plug into (11.4.3) distributions.

Example 11.4.2. Consider equation (11.4.3) again but now we add condition

$$\big[-\sum_{jk} a_{jk}u_{x_j}\nu_k - \frac{1}{2}\sum_j b_j u\nu_j\big]\big|_\Gamma = h \tag{11.4.5}$$

where ν_k are components of the unit inner normal to Γ. Then in the smooth case (11.4.3) and (11.4.5) together are equivalent to

$$Q[u,\varphi] = \iint_\Omega f\varphi\,dx + \int_\Gamma h\varphi\,dS \tag{11.4.6}$$

which should hold for all $\varphi \in C^1(\mathbb{R}^d)$ (now φ is not necessarily 0 near Γ!) and we call $u \in C^1(\Omega)$ (or even worse) *weak solution* to problem (11.4.3), (11.4.5) in this case. Here $Q[u,\varphi]$ is the left-hand expression in (11.4.4).

These examples could be extended to quasilinear or even nonlinear equation. See Burgers equation in Section 12.1.

Chapter 12

Nonlinear equations

12.1 Burgers equation

12.1.1 Two problems

Consider equation

$$u_t + f(u)u_x = 0, \qquad t > 0 \tag{12.1.1}$$

Then we consider a problem

$$u(x,0) = \begin{cases} u_- & x < 0, \\ u_+ & x > 0 \end{cases} \tag{12.1.2}$$

There are two cases:

Case 1. $f(u_-) < f(u_+)$. In this case characteristics

$$\frac{dt}{1} = \frac{dx}{f(u)} = \frac{du}{0} \tag{12.1.3}$$

originated at $\{(x,0) : 0 < x < \infty\}$ fill $\{x < f(u_-)t,\, t > 0\}$ where $u = u_-$ and $\{x > f(u_+)t,\, t > 0\}$ where $u = u_+$ and leave sector $\{f(u_-)t < x < f(u_+)t\, t > 0\}$ empty. In this sector we can construct continuous self-similar solution $u = g(x/t)$ and this construction is unique provided f is monotone function (say increasing)

$$f'(u) > 0 \tag{12.1.4}$$

(and then $f(u_-) < f(u_+)$ is equivalent to $u_< u_+$). Namely

$$u(x,t) = \begin{cases} u_- & x < f(u_-)t, \\ g\left(\dfrac{x}{t}\right) & f(u_-)t < x < f(u_+)t, \\ u_+ & x > f(u_+)t \end{cases} \tag{12.1.5}$$

provides solution for (12.1.1)-(12.1.2) where g is an inverse function to f.

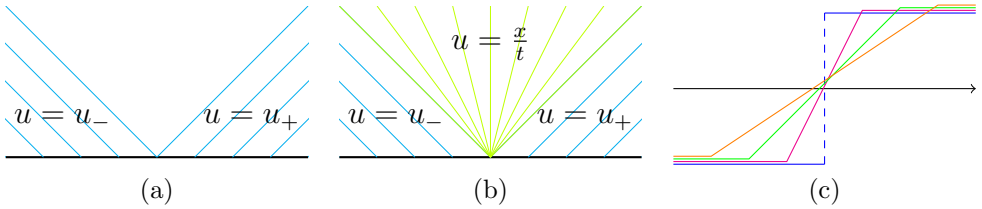


Figure 12.1: For $f(u) = u$ as $u_- < u_+$ characteristics and solution consecutive profiles (slightly shifted up)

Case 2. $f(u_-) > f(u_+)$. In this case characteristics collide

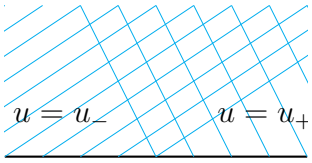


Figure 12.2: For $f(u) = u$ as $u_- > u_+$ characteristics collide

and to provide solution we need to reformulate our equation (12.1.1) as

$$u_t + (F(u))_x = 0, \qquad t > 0 \tag{12.1.6}$$

where $F(u)$ is a primitive of f: $F'(u) = f(u)$. Now we can understand equation in a weak sense and allow discontinuous (albeit bounded) solutions. So, let us look for

$$u = \begin{cases} u_- & x < st, \\ u_+ & x > st \end{cases} \tag{12.1.7}$$

where so far s is unknown. Then $u_t = -s[u_+ - u_-]\delta(x-st)$, $u_x = [F(u_+) - F(u_-)]\delta(x-st)$ where brackets contain jumps of u and $F(u)$ respectively and (12.1.6) means exactly that

$$s = \frac{[F(u_+) - F(u_-)]}{u_+ - u_-} \tag{12.1.8}$$

which is equal to $F'(v) = f(v)$ at some $v \in (u_-, u_+)$ and due to (12.1.4) $s \in (f(u_-), f(u_+))$:

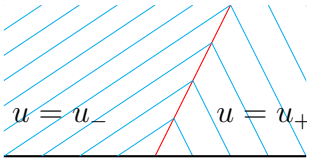
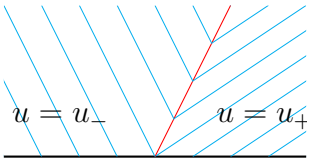


Figure 12.3: For $f(u) = u$ as $u_- < u_+$ characteristics and a line of jump (profiles are just shifted steps)

12.1.2 Shock waves

Huston, we have a problem!!! However allowing discontinuous solutions to (12.1.6) we opened a floodgate to many discontinuous solution and broke unicity. Indeed, let us return to Case 1 and construct solution in the same manner as in Case 2. Then we get (12.1.8)–(12.1.7) solution with $s = F'(v)$ at some $v \in (u_+, u_-)$ and due to (12.1.4) $s \in (f(u_+), f(u_-))$: So, we got two



solutions: new discontinuous and old continuous (12.1.5). In fact, situation is much worse since there are many hybrid solutions in the form (12.1.8) albeit with discontinuous g. To provide a uniqueness we need to weed out all such solutions.

Remark 12.1.1. Equation (12.1.6) is considered to be a *toy-model* for gas dynamics. Discontinuous solutions are interpreted as *shock waves* and solution

in Case 1 are considered *rarefaction waves*. Because of this was formulated principle *there are no shock rarefaction waves* which mathematically means $u(x-0,t) \geq u(x+0,t)$ where $u(x \pm 0,t)$ are limits from the right and left respectively. However it is not a good condition in the long run: for more general solutions these limits may not exist. To do it we multiply equation (12.1.1) by u and write this new equation

$$uu_t + f(u)uu_x = 0, \qquad t > 0 \tag{12.1.9}$$

in the divergent form

$$(\frac{1}{2}u^2)_t + (\Phi(u))_x = 0, \qquad t > 0 \tag{12.1.10}$$

where $\Phi(u)$ is a primitive of $uf(u)$.

Remark 12.1.2. Let us observe that while equations (12.1.1) and (12.1.9) are equivalent for continuous solutions, equations (12.1.6) and (12.1.10) are not equivalent for discontinuous ones. Indeed, for solution (12.1.7) the left-hand expression in (12.1.10) is

$$K\delta(x-st) \tag{12.1.11}$$

with

$$\begin{aligned} K(u_-,u_+) = -s\frac{1}{2}[u_+^2 - u_-^2] + [\Phi(u_+) - \Phi(u_-)] = \\ -\frac{1}{2}(u_+ + u_-)[F(u_+) - F(u_-)] + [\Phi(u_+) - \Phi(u_-)] \end{aligned} \tag{12.1.12}$$

Exercise 12.1.1. Prove that $K(u_-,u_+) \gtrless 0$ as $u_+ \gtrless u_-$. To do it consider $K(u-v,u+v)$, observe that it is 0 as $v=0$ and $\partial_v K(u-v,u+v) = v(f(u+v) - f(u-v)) > 0$ for $v \neq 0$ due to (12.1.4).

So for "good" solutions

$$(\frac{1}{2}u^2)_t + (\Phi(u))_x \geq 0, \qquad t > 0 \tag{12.1.13}$$

where we use the following

Definition 12.1.1. Distribution $U \geq 0$ if for all non-negative test functions φ $U(\varphi) \geq 0$.

It was proven

Theorem 12.1.1. *Solution to the problem* (12.1.6), (12.1.2) *with additional restriction* (12.1.13) *exists and is unique as* ϕ *is* bounded total variation function.

Remark 12.1.3. Restriction (12.1.13) is interpreted as *entropy cannot decrease.*

12.1.3 Examples

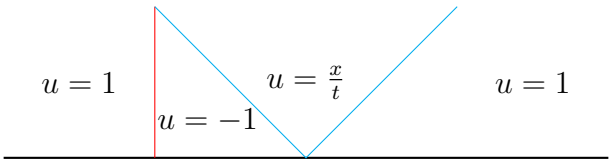
Example 12.1.1. The truly interesting example is Burgers equation ($f(u) = u$) with initial conditions which are not monotone. So we take

$$u(x,0) = \phi(x) := \begin{cases} 1 & x < -1, \\ -1 & -1 < x < 0, \\ 1 & x > 1. \end{cases} \tag{12.1.14}$$

Then obviously the solution is first provided by a combination of Case 1 and Case 2:

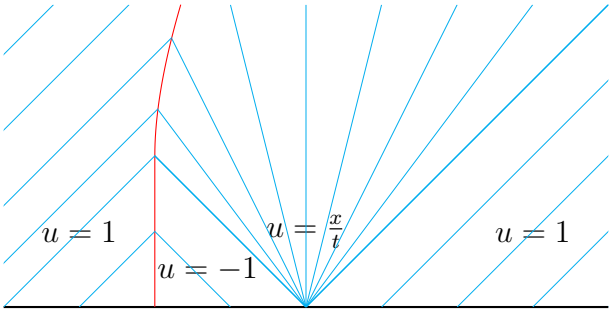
$$u(x,t) = \begin{cases} 1 & x < -1, \\ -1 & -1 < x < -t, \\ \dfrac{x}{t} & -t < x < t, \\ 1 & x > t. \end{cases} \tag{12.1.15}$$

This holds as $0 < t < 1$ because at $t > 1$ rarefaction and shock waves collide.



Now there will be a shock wave at $x = \xi(t)$, $t > 1$. On its left $u = 1$, on its right $u = \xi t^{-1}$ and therefore slope is a half-sum of those:

$$\frac{d\xi}{dt} = \frac{1}{2} + \frac{\xi}{2t}; \tag{12.1.16}$$



this ODE should be solved with initial condition $\xi(1) = -1$ and we have $\xi(t) = t - 2t^{\frac{1}{2}}$; Observe that while $\max_{-\infty<x<\infty} u(x,t) = 1$, $\min_{-\infty<x<\infty} u(x,t) = \xi(t)t^{-1} = 1 - 2t^{-\frac{1}{2}}$ and

$$\big(\max_{-\infty<x<\infty} u(x,t) - \min_{-\infty<x<\infty} u(x,t)\big) \to 0 \qquad \text{as } t \to +\infty.$$

We can consider example with $u(x,0) = -\phi(x,0)$ by changing $x \mapsto -x$ and $u \mapsto -u$.

Example 12.1.2. Consider now

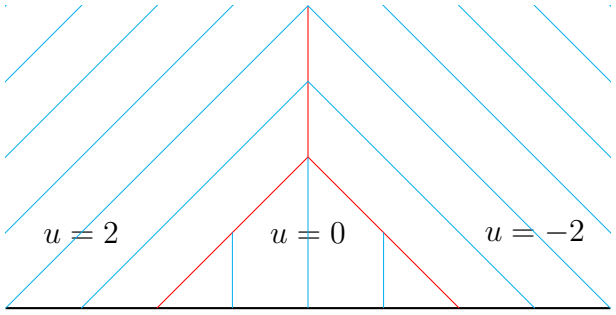
$$u(x,0) = \phi(x) := \begin{cases} 2 & x < -1, \\ 0 & -1 < x < 1, \\ -2 & x > 1. \end{cases} \tag{12.1.17}$$

Then for $0 < t < 1$, we have two shock waves:

$$u(x,t) = \begin{cases} u_- & x < -1 + t, \\ u_0 & -1 + t < x < 1 - t, \\ u_+ & x > 1 - t \end{cases} \tag{12.1.18}$$

and for $t = 1$ both shock waves collide at $x = 0$ and then for $t > 1$

$$u(x,t) = \begin{cases} 2 & x < 0, \\ -2 & x > 0. \end{cases} \tag{12.1.19}$$


$u = 2$
$u = 0$
$u = -2$

Chapter 13

Eigenvalues and eigenfunctions

13.1 Variational theory

13.1.1 Introduction

Consider quadratic forms

$$Q_0(u) = \|u\|^2 = \iiint_\Omega |u|^2 \, dx; \tag{13.1.1}$$

$$Q(u) = \iiint_\Omega \|\nabla u\|^2 \, dx + \iint_\Sigma \alpha |u|^2 \, d\sigma \tag{13.1.2}$$

where Ω is a bounded domain with a smooth boundary $\Sigma = \partial\Omega$.

Let us consider a variational problem for $Q(u)$ under restriction $\|u\| = 1$ and

$$u\big|_{\Sigma^-} = 0 \tag{13.1.3}$$

where $\Sigma^- \subset \Sigma$.

Constructing functional $Q_\lambda(u) = Q(u) - \lambda Q(u)$ and taking it variation we arrive to Euler-Lagrange equation

$$-\Delta u = \lambda u \tag{13.1.4}$$

with the boundary conditions

$$u\big|_{\Sigma^-} = 0, \qquad (\partial_{\boldsymbol{\nu}} u - \alpha u)\big|_{\Sigma^+} = 0 \qquad \Sigma^+ = \Sigma \setminus \Sigma^- \tag{13.1.5}$$

where $\boldsymbol{\nu}$ is a unit inner normal.

So we arrived to the eigenvalue problem. We need the following theorems from the Real Analysis:

Theorem 13.1.1. *Let* H *be a Hilbert space. Let* $\|v_n\| \le M$*; then there exists a subsequence* v_{n_k} *which* converges weakly *in* H*:* $(v_{n_k} - v, \phi) \to 0$ *for all* $\phi \in \mathsf{H}$*.*

Theorem 13.1.2. *Let* Ω *be a domain of a finite volume with a boundary* Σ*. Then any sequence* v_n *such that* $\|\nabla v_n\|| \le M$ *and* $v_n|_\Sigma = 0$*, contains a subsequence* v_{n_k} *which converges in* $L^2(\Omega)$*:* $\|v_{n_k} - v\| \to 0$ *for some* $v \in L^2(\Omega)$*,* $\nabla v \in L^2(\Omega)$*.*

Theorem 13.1.3. *Let* Ω *be a bounded domain with a smooth boundary. Then*

(a) $v \in L^2(\Omega)$*,* $\nabla v \in L^2(\Omega)$ *imply that* $v \in L^2(\Sigma)$*;*

(b) any sequence v_n *such that* $\|v_n\| + \|\nabla v_n\|| \le M$ *contains a subsequence* v_{n_k} *which converges in* $L^2(\Omega)$ *and also converges in* $L^2(\Sigma)$*:* $\|v_{n_k} - v\|_{L^2(\Sigma)} \to 0$ *to some* $v \in L^2(\Omega) \cap L^2(\Sigma)$*,* $\nabla v \in L^2(\Omega)$*;*

(c) For any $\epsilon > 0$ *there exists* C_ϵ *such that*

$$\|v\|_{L^2(\Sigma)} \le \epsilon \|\nabla v\| + C_\epsilon \|v\|. \tag{13.1.6}$$

Remark 13.1.1. Actually one need only assume that Σ^+ is bounded and smooth.

13.1.2 Main variational principles

Theorem 13.1.4. *Let* Ω *be a bounded domain with the smooth boundary. Then*

(a) there exists a sequence of eigenvalues $\lambda_1 \le \lambda_2 \le \lambda_3 \le \dots$, $\lambda_k \to \infty$ and a sequence of corresponding eigenfunctions $u_1, u_2, \dots$ which forms a basis in $L^2(\Omega)$;

(b) Variational problem of minimizing $Q(u)$ under constrains $\|u\| = 1$ and (13.1.3) has solution u_1 and this minimum is λ_1.

(c) Variational problem of minimizing $Q(u)$ under constrains $\|u\| = 1$, (13.1.3) and $(u, u_1) = (u, u_2) = \dots = (u, u_{n-1}) = 0$ has solution u_n and this minimum is λ_n.

Proof. Let us prove first that u_1 exists. Let us consider minimizing sequence v_m for $Q(u)$ under constrain $\|v_m\| = 1$. Observe that due to Theorem 13.1.3(c)

$$\|\nabla v\|^2_{L^2(\Sigma)} \le (1-\epsilon)Q(v) + C_\epsilon \|v\|^2. \tag{13.1.7}$$

Then we have $\|v_m\| + \|\nabla v_m\| \le M$ and in virtue of Theorem 13.1.1 and either Theorem 13.1.2 or Theorem 13.1.3 there is a subsequence v_{m_k} converging to some v both in $L^2(\Omega)$ and weakly in $\mathsf{K} = \{v : \|v\|_{\mathsf{K}} < \infty,\ u|_{\Sigma^-} = 0\}$ with $\|v\|_{\mathsf{K}} = \big(C_0\|v\|^2 + Q(v)\big)^{\frac{1}{2}}$. Then $\|v\| = 1$ and $Q(v)$ is minimal. We skip the proof.

Similarly we prove existence λ_n, u_n by induction. Now we claim that $\lambda_n \to \infty$. Indeed if it is not the case then $\|u_n\| = 1$ and $\|\nabla u_n\| \le M$ for some M; then in virtue of either Theorem 13.1.2 or Theorem 13.1.3 there exists a subsequence u_{n_k} converging in $L^2(\Omega)$ which is impossible since u_{n_k} are mutually orthogonal and $\|u_{n_k} - u_{n_l}\|^2 = 2$ as $k \ne l$.

(i) (ii) Further, observe that u_n are orthogonal in K. Let us prove that system u_n is complete in $\mathsf{K} =$. If it is not the case then there exists $u \in \mathsf{K}$ which is orthogonal to all u_n. But then $(u_n, u)_{\mathsf{K}} = (-\Delta u_n, u) + C_0(u_n, u) = (\lambda_n + C_0)(u_n, u)$ and therefore u is orthogonal to all u_n in $L^2(\Omega)$ as well. But then since $\lambda_k \to \infty$ and $Q(u) < \infty$ we conclude that u must appear as a minimizing element in the sequence which is the contradiction.

(iii) Finally, u_n is complete in $L^2(\Omega)$ as well. It follows from completeness in K and the fact that K is dense in $L^2(\Omega)$. The latter proof is elementary but we skip it as well.

□

Remark 13.1.2. If $\Sigma = \Sigma^-$ we do not need to assume that Σ is smooth or even Ω is bounded; it is sufficient to assume that it has a finite volume (but even this is not necessary!)

Corollary 13.1.1. *Consider n-dimensional subspace $L \subset \mathsf{H} = \{u \in C^1(u), u|_{\Sigma^-} = 0\}$. Let $\mu_n(L) = \max_{u \in L:\, \|u\|=1} Q(u)$. Then*

$$\lambda_n = \min_{L \subset \mathsf{H}:\, \dim(L)=n} \mu_n(L) = \min_{L \subset \mathsf{H}:\, \dim(L)=n} \ \max_{u \in L:\, \|u\|=1} Q(u) \tag{13.1.8}$$

Proof. If L is a span of $u_1, \dots, u_n$ then $\lambda_n(L) = \lambda_n$ so the right-hand expression is not greater than the left=hand one. On the other hand, if L is not a span of $u_1, \dots, u_n$ then there exists $u \in L$, $u \ne 0$ which is orthogonal to $u_1, \dots, u_n$ and therefore $Q(u) \ge \lambda_{n+1}\|u\|^2$. □

13.1.3 Dependence on parameters

Theorem 13.1.5. *(a) Eigenvalues $\lambda_n = \lambda_n(\alpha)$ does not decrease as α increases;*

(b) Eigenvalues $\lambda_n = \lambda_n(\Sigma^-)$ does not decrease as Σ^- increases;

(c) As $\alpha \to +\infty$ these eigenvalues $\lambda_n = \lambda_n(\alpha)$ tend to $\lambda_{D,n}$ which are eigenvalues of Dirichlet problem (i.e. when $\Sigma^- = \Sigma$).

Proof. (a) As α increases then $Q(u)$ also increases and then the right-hand expression in (13.1.6) cannot decrease.

(b) As we increase Σ^- we make space H only smaller and thus make smaller the choice of n-dimensional subspaces L so the right-hand expression in (13.1.6) cannot decrease.

(c) Therefore $\lambda_n \le \lambda_{D,n}$. Then as $\alpha \to +0$ expressions $\alpha \int\!\!\int_\Sigma |u_n(\alpha)|^2\, d\sigma$ are bounded by $\lambda_{D,n}$ and therefore $u|_\Sigma \to +0$. Assume that $\lambda_n(\alpha) \le \mu_n$ for all α. Then by means of Real Analysis one can prove that there is a sequence $\alpha_k \to +\infty$ such that $u_n(\alpha_k) \to v_n$ and $\|\nabla v_n\|^2 \le \mu_n$, $v_n|_\Sigma = 0$ and $\|v_n\| = 1$. But then $\mu_n \ge \lambda_n$. Therefore $\lambda_n = \mu_n$ and $\lambda_n(\alpha) \to \lambda_{D,n}$. □

13.1.4 Dependence on domain

Consider now *only* Dirichlet boundary problem.

Theorem 13.1.6. *Eigenvalues $\lambda_n = \lambda_{D,n}(\Omega)$ does not increase as Ω increases.*

Proof. If $\Sigma^- = \Sigma$ we can identify $\mathsf{H} = \mathsf{H}(\Omega)$ with $\{u : \nabla u \in L^2(\mathbb{R}^d),\ u = 0 \text{ on } \mathbb{R}^d \setminus \bar\Omega\}$ where $\bar\Omega$ is a closure of Ω. Then

$$Q(u) = \int |\nabla u|^2\, dx, \qquad Q_0(u) = \int |u|^2\, dx$$

with integrals taken over $\mathbb{R}^d$. Therefore as Ω increases then $\mathsf{H}(\Omega)$ increases and the choice of L increases, so the right-hand expression in (13.1.6) cannot increase. □

Remark 13.1.3. This statement holds only for Dirichlet eigenvalues.

Theorem 13.1.7. *Let vary Ω by moving by h into direction of ν (where h is a small function on Σ. Then for a simple eignevalue λ_n*

$$\delta\lambda_{D,n} = \frac{1}{\|u_n\|^2}\int_\Sigma h|\partial_{\boldsymbol{\nu}} u_n|^2\,d\sigma. \tag{13.1.9}$$

Proof. Consider $u_n+\delta u_n$ matching to a new domain. Since it must be 0 on the new boundary, modulo smaller terms we conclude that it is $u_n+\delta u_n = -h\partial_{\{\boldsymbol{\nu}\}}u_n$ on the old boundary Σ and since $u_n=0$ there we conclude that

$$\delta u_n = -h\partial_{\boldsymbol{\nu}} u_n \qquad \text{on } \Sigma. \tag{13.1.10}$$

On the other hand $(\Delta+\lambda_n+\delta\lambda_n)(u_n+\delta u_n)=0$ and then modulo smaller terms we conclude that

$$(\Delta+\lambda_n)\delta u_n = -\delta\lambda_n u_n. \tag{13.1.11}$$

Let us multiply by u_n and integrate over Ω; the left-hand expression becomes

$$\begin{aligned}\int_\Omega (\Delta+\lambda_n)\delta u_n\cdot u_n\,dx = \\ \int_\Omega \delta u_n\cdot(\Delta+\lambda_n)u_n\,dx - \int_\Sigma \partial_{\boldsymbol{\nu}}\delta u_n\cdot u_n\,d\sigma + \int_\Sigma \delta u_n\cdot\partial_{\boldsymbol{\nu}}u_n\,d\sigma = \\ -\int_\Sigma h|\partial_{\boldsymbol{\nu}}u_n|^2\,d\sigma\end{aligned} \tag{13.1.12}$$

where we used that $(\Delta+\lambda_n)u_n=0$, $u_n|_\Sigma=0$ and (13.1.10).

Meanwhile the right-hand expression becomes $-\delta\lambda_n\|u_n\|^2$. Since it i must be equal to the right-hand expression of (13.1.12) we arrive to (13.1.9). □

13.2 Asymptotic distribution of eigenvalues

13.2.1 Introduction

We are interested how eignevalues are distributed. We introduce eigenvalue distribution function $N(\lambda)$ which is the number of eigenvalues (of operator $-\Delta$) which are less than λ. In other words,

$$N(\lambda) = \max_{\lambda_n<b} n \tag{13.2.1}$$

where λ_n are eigenvalues, or, using Corollary 13.1.1 we reformulate it as

Definition 13.2.1. Eignevalue distribution function is

$$N(\lambda) = \max_{L\subset \mathsf{H}:\ Q(v)<\lambda\|v\|^2\ \forall 0\neq v\in L} \dim L \tag{13.2.2}$$

Remark 13.2.1. While these two definitions coincide in our particular case there is a huge difference: formula (13.2.1) counts only eigenvalues. However formula (13.2.2) takes into account also a *continuous spectrum*: this way defined $N(\lambda)$ is $+\infty$ if $(-\infty,\lambda)$ contains (some) points of continuous spectrum. This definition is one widely used.

In the same way as Theorem 13.1.5 and Theorem 13.1.6 one can prove

Theorem 13.2.1. *(a) $N(\lambda;\alpha)$ does not increase as α increases;*

(b) $N(\lambda;\Sigma^-)$ does not increase as Σ^- increases;

(c) As $\alpha\to+\infty$ $N(\lambda;\alpha)\to N_D(\lambda)$ which is an eigenvalue counting function for Dirichlet problem (i.e. when $\Sigma^-=\Sigma$).

(d) $N_D(\lambda;\Omega)$ does not decrease as Ω increases.

13.2.2 Rectangular box

Consider rectangular box in $\mathbb{R}^d$: $\Omega=\{x:\ 0<x_1<a_1, 0<x_2<a_2,\dots,0<x_d<a_d\}$ with Dirichlet or Neumann boundary conditions. Then separation of variables brings us eigenfunctions and eigenvalues

$$X_m = \sin\Big(\frac{\pi m_1 x_1}{a_1}\Big)\sin\Big(\frac{\pi m_2 x_2}{a_2}\Big)\cdots\sin\Big(\frac{\pi m_d x_d}{a_d}\Big), \tag{13.2.3}$$

$$\lambda_m = \pi^2\Big(\frac{m_1^2}{a_1^2}+\frac{m_2^2}{a_2^2}+\ldots+\frac{m_d^2}{a_d^2}\Big)\qquad m_1\ge 1,\dots,m_d\ge 1 \tag{13.2.4}$$

for Dirichlet problem and

$$X_m = \cos\Big(\frac{\pi m_1 x_1}{a_1}\Big)\cos\Big(\frac{\pi m_2 x_2}{a_2}\Big)\cdots\cos\Big(\frac{\pi m_d x_d}{a_d}\Big), \tag{13.2.5}$$

$$\lambda_m = \pi^2\Big(\frac{m_1^2}{a_1^2}+\frac{m_2^2}{a_2^2}+\ldots+\frac{m_d^2}{a_d^2}\Big)\qquad m_1\ge 0,\dots,m_d\ge 0 \tag{13.2.6}$$

for Neumann problem where $m=(m_1,\dots,m_d)\in\mathbb{Z}^d$ in both cases.

Exercise 13.2.1. Prove it by separation of variables.

Therefore

Theorem 13.2.2. *For rectangular box* $\Omega = \{x : 0 < x_1 < a_1, 0 < x_2 < a_2, \dots, 0 < x_d < a_d\}$

(a) $N_D(\lambda)$ *equals to the number of integer points (points with all integer coordinates) in the domain*

$$E_-(\lambda) = \Big\{m = (m_1, \dots, m_d) : m_1 > 0, m_2 > 0, \dots, m_d > 0, \\ \frac{m_1^2}{a_1^2} + \frac{m_2^2}{a_2^2} + \dots + \frac{m_d^2}{a_d^2} < \pi^{-2}\lambda\Big\};$$

(b) $N_N(\lambda)$ *equals to the number of integer points (points with all integer coordinates) in the domain*

$$E_+(\lambda) = \Big\{m = (m_1, \dots, m_d) : m_1 \geq 0, m_2 > 0, \dots, m_d \geq 0, \\ \frac{m_1^2}{a_1^2} + \frac{m_2^2}{a_2^2} + \dots + \frac{m_d^2}{a_d^2} < \pi^{-2}\lambda\Big\}.$$

Calculation of the number of integer points in the "large" domains is an important problem of the *Number Theory*. For us the answer "the number of integer points inside of the large domain approximately equals to its volume" is completely sufficient but let us use the following

Theorem 13.2.3. *As* $d \geq 2$ *the number of integer points inside ellipsoid*

$$\mathcal{E}(\lambda) = \Big\{m = (m_1, \dots, m_d) : \frac{m_1^2}{a_1^2} + \frac{m_2^2}{a_2^2} + \dots + \frac{m_d^2}{a_d^2} < \pi^{-2}\lambda\Big\}$$

equals to the volume of $\mathcal{E}(\lambda)$ *plus* $o(\lambda^{(d-1)/2})$.

Remark 13.2.2. Actually much more precise results are known.

Observe that the volume of this ellipsoid $\mathcal{E}(\lambda)$ is $\pi^{-d}\omega_d a_1 a_2 \cdots a_d \lambda^{d/2}$ where ω_d is a volume of the unit ball in $\mathbb{R}^d$.

Exercise 13.2.2. Prove it observing that ellipsoid $\mathcal{E}(\lambda)$ is obtained by stretching of the unit ball.

Observe also that both $E_+(\lambda)$ and $E_-(\lambda)$ constitute $1/2^d$ part of $\mathcal{E}(\lambda)$ and therefore their volumes are $(2\pi)^{-d}\omega_d a_1\cdots a_d\lambda^{d/2}$. However if we consider integer points we observe that

(a) the number of integer points in $E_-(\lambda)$ multiplied by 2^d equals to the number of integer points in $\mathcal{E}(\lambda)$ minus the number of integer points belonging to the intersection of $\mathcal{E}(\lambda)$ with one of the planes $\Pi_j = \{m = (m_1,\dots,m_d) : m_j = 0\}$ while

(b) the number of integer points in $E_+(\lambda)$ multiplied by 2^d equals to the number of integer points in $\mathcal{E}(\lambda)$ plus the number of integer points belonging to the intersection of $\mathcal{E}(\lambda)$ with one of the planes $\Pi_j = \{m = (m_1,\dots,m_d) : m_j = 0\}$.

Actually, it is not completely correct as one needs to take into account intersections of $\mathcal{E}(\lambda)$ with two different planes Π_j and Π_k but the number of integer points there is $O(\lambda^{(d-2)/2})$.

Since intersections of $\mathcal{E}(\lambda)$ with Π_j is an ellipsoid we conclude that the number of integer points there is $\pi^{1-d}\omega_{d-1}a_1a_2\cdots a_d/a_j\lambda^{(d-1)/2}$.

Therefore the number of integer points in $E_\pm(\lambda)$ is equal to

$$(2\pi)^{-d}\omega_d a_1a_2\cdots a_d\lambda^{d/2}\pm \frac{1}{2}(2\pi)^{1-d}\omega_{d-1}\underbrace{a_1a_2\cdots a_d\big(a_1^{-1}+a_2^{-1}+\dots a_d^{-1}\big)}\lambda^{(d-1)/2}+o\big(\lambda^{(d-1)/2}\big).$$

Observe that $a_1a_2\cdots a_d$ is a volume of Ω and $a_1a_2\cdots a_d\big(a_1^{-1}+a_2^{-1}+\dots a_d^{-1}\big)$ is a half of area of its boundary $\partial\Omega$.

So we arrive to

Theorem 13.2.4. *For rectangular box $\Omega = \{x : 0 < x_1 < a_1, 0 < x_2 < a_2, \dots, 0 < x_d < a_d\}$, $d \ge 2$*

$$N(\lambda) = (2\pi)^{-d}\mathrm{mes}_d(\Omega)\lambda^{d/2} + O\big(\lambda^{(d-1)/2}\big) \tag{13.2.7}$$

and more precisely

$$N(\lambda) = (2\pi)^{-d}\mathrm{mes}_d(\Omega)\lambda^{d/2} \pm \frac{1}{4}(2\pi)^{1-d}\mathrm{mes}_{d-1}(\partial\Omega)\lambda^{(d-1)/2} + o\big(\lambda^{(d-1)/2}\big) \tag{13.2.8}$$

where "+" corresponds to Neumann and "−" to Dirichlet boundary conditions and mes_k *means k-dimensional volume.*

Remark 13.2.3. (a) Asymptotics (13.2.7) holds for $d = 1$ as well;

(b) Asymptotics (13.2.7) and (13.2.8) with the sign "+" hold for Robin boundary condition.

13.2.3 Weyl formula

Now we want to generalize these results for an arbitrary domain Ω. To do so let us consider domain Ω_ε^+ which includes Ω and all points on the distance $\le C\varepsilon$ from $\partial\Omega$ and partition it into cubic boxes such that

(a) The side of any box is at least ε;

(b) If the box is not inside of Ω completely its side is exactly ε.

The boxes which are completely inside of Ω are called *inner boxes* and the boxes which intersect $\partial\Omega$ are called *boundary boxes*.

We consider now *only* Dirichlet conditions. Let us us Definition 13.2.1 for $N(\lambda)$; for Dirichlet boundary conditions we can consider H as the space of all functions which vanish outside of Ω.

Let us tighten constrains to these functions: we assume that they are 0 in the boundary boxes as well and that they vanish on the walls of all boxes. Then $N(\lambda)$ can only decrease: $N(\lambda) \ge N_*((\lambda)$ where $N_*(\lambda)$ is an eigenvalue counting function under these new constrains. But under these constrains calculation in each box becomes independent and we arrive to

$$N_*(\lambda) = \Big[(2\pi)^{-d}\omega_d\lambda^{d/2} - c\lambda^{(d-1)/2}\varepsilon^{-1}\Big] \sum_\iota \operatorname{mes}_d(B_\iota)$$

where we sum over all inner boxes. As $\varepsilon \ge c_1\lambda^{-1/2}$ this expression is greater than $\Big[(2\pi)^{-d}\omega_d\lambda^{d/2} - c\lambda^{(d-1)/2}\varepsilon^{-1}\Big]\operatorname{mes}_d(\Omega -_\varepsilon)$ where Ω_ε^- is the set of all points of Ω on the distance $> \varepsilon$ from $\partial\Omega$. Thus

$$N_D(\lambda) \ge \Big[(2\pi)^{-d}\omega_d\lambda^{d/2} - c\lambda^{(d-1)/2}\varepsilon^{-1}\Big]\big(\operatorname{mes}_d(\Omega) - \operatorname{mes}_d(\Omega\setminus\Omega_\varepsilon^-)\big). \quad (13.2.9)$$

Let us loosen constrains to these functions: we assume that they are arbitrary in the boundary boxes as well and that they can be discontinuous on the walls of all boxes (then $Q(u)$ is calculated as a sum of $Q_\iota(u)$ all over boxes). Then $N(\lambda)$ can only increase: $N(\lambda) \le N^*((\lambda)$ where $N^*(\lambda)$ is an

eigenvalue counting function under these new constrains. But under these constrains calculation in each box becomes independent and we arrive to

$$N_*(\lambda) = \Big[(2\pi)^{-d}\omega_d\lambda^{d/2} + c\lambda^{(d-1)/2}\varepsilon^{-1}\Big] \sum_{\iota} \mathrm{mes}_d(B_\iota)$$

where we sum over all inner and boundary boxes. Thus

$$N_D(\lambda) \le \Big[(2\pi)^{-d}\omega_d\lambda^{d/2}+c\lambda^{(d-1)/2}\varepsilon^{-1}\Big]\big(\mathrm{mes}_d(\Omega)+\mathrm{mes}_d(\Omega^+_\varepsilon\setminus\Omega)\big). \quad (13.2.10)$$

But what is the gap between the right-hand expressions of (13.2.9) and (13.2.10)? It does not exceed

$$C\lambda^{(d-1)/2}\mathrm{mes}_d\varepsilon^{-1}(\Omega^+_\varepsilon) + C\lambda^{d/2}\mathrm{mes}_d(\Sigma_\varepsilon)$$

where Σ_ε is the set of all points on the distance $\le C_0\varepsilon$ from $\Sigma = \partial\Omega$.

We definitely want $\mathrm{mes}_d(\Sigma_\varepsilon) \to 0$ as $\varepsilon \to +0$. According to *Real Analysis* we say that it means exactly that Σ *is a set of measure* 0.

Thus we arrive to

Theorem 13.2.5. *If Ω is a bounded domain and $\partial\Omega$ is a set of measure* 0 *then*

$$N_D(\lambda) = (2\pi)^{-d}\omega_d\lambda^{d/2} + o\big(\lambda^{d/2}\big) \quad (13.2.11)$$

as $\lambda \to +\infty$.

In particular, it holds if Ω is a bounded domain and $\partial\Omega$ is smooth.

13.2.4 Remarks

Remark 13.2.4. (a) This Theorem 13.2.5 (in a bit less general form) was proven by H.Weyl in 1911;

(b) In fact we need neither of assumptions: even if $\partial\Omega$ is not a set of measure 0 asymptotics (13.2.11) holds but in this case $\mathrm{mes}_d(\Omega)$ means a Lebesgue measure of Ω (see *Real Analysis*).

(c) Further, as $d \ge 3$ we can assume only that $\mathrm{mes}_d(\Omega) < \infty$.

Remark 13.2.5. (a) We can always make a more subtle partition into cubic boxes: we can assume additionally that the size of the inner box

B_ι is of the same magnitude as a distance from B_ι to $\partial\Omega$; then we can get a remainder estimate

$$C\lambda^{(d-1)/2}\iiint_{\{x:\,\gamma(x)\ge c\lambda^{-1/2}\}}\gamma(x)^{-1}\,dx+C\lambda^{d/2}\mathrm{mes}_d(\Sigma_{c\lambda^{-1/2}}) \tag{13.2.12}$$

where $\gamma(x)$ is a distance from x to $\Sigma=\partial\Omega$.

(b) If Ω is a bounded set and $\partial\Omega$ is smooth then for Dirichlet and Robin boundary condition

$$N_D(\lambda)=(2\pi)^{-d}\omega_d\lambda^{d/2}+O\big(\lambda^{(d-1)/2}\ln\lambda\big). \tag{13.2.13}$$

(c) Even if the boundary is highly irregular one can define Neumann Laplacian. However in this case even if domain is bounded it *can* have non-empty continuous spectrum an then $N_N(\lambda)=\infty$ for $\lambda\ge C$.

Remark 13.2.6. Let Ω be bounded domain and $\partial\Omega$ be smooth.

(a) Much more delicate proof based on completely different ideas shows that for Dirichlet and Robin boundary condition

$$N(\lambda)=(2\pi)^{-d}\omega_d\lambda^{d/2}+O\big(\lambda^{(d-1)/2}\big). \tag{13.2.14}$$

(b) Even more delicate proof shows that under some *billiard condition* asymptotics (13.2.8) holds with sign "+" for Robin boundary condition and "−" for Dirichlet boundary conditions.

(c) This billialrd condition is believed to be fulfilled for any domain in $\mathbb{R}^d$. However it is not necessarily true on manifolds; for Laplacian on the sphere only (13.2.14) holds but (13.2.14) fails.

(d) Actually, one can allow Dirichlet boundary condition on some part of the boundary (Σ^-) and Robin boundary condition on the remaining part of the boundary ($\Sigma^+=\Sigma\setminus\Sigma^-$) provided transition between those parts is regular enough. Then the formula becomes

$$\begin{aligned}&N(\lambda)=(2\pi)^{-d}\mathrm{mes}_d(\Omega)\lambda^{d/2}+\\&\frac{1}{4}(2\pi)^{1-d}\big(\mathrm{mes}_{d-1}(\Sigma^+)-\mathrm{mes}_{d-1}(\Sigma^-)\big)\lambda^{(d-1)/2}+o\big(\lambda^{(d-1)/2}\big)\end{aligned} \tag{13.2.15}$$

(e) There are generalizations allowing some kinds or "regular singularities" of $\partial\Omega$.

Remark 13.2.7. There are many generalizations of these asymptotics; let us mention only few. We consider Schrödinger operator

$$H = -\hbar^2\Delta + V(x). \tag{13.2.16}$$

Then Weyl formula for this operator is

$$N(\lambda,\hbar) \approx (2\pi\hbar)^{-d}\omega_d \iiint (\lambda - V(x))_+^{d/2}\, dx \tag{13.2.17}$$

where $z_+ = \max(z,0)$. This formula is justified

(i) As $\hbar \to +0$ and $V(x) \ge \lambda + \epsilon$ as $|x| \ge C$;

(ii) As $\hbar$ is either fixed or tends to $+0$ and $\lambda \to +\infty$ provided $V(x) \to +\infty$ as $|x| \to \infty$;

(iii) As $\hbar$ is either fixed or tends to $+0$ and $\lambda \to -0$ provided $V(x) \to 0$ and $|x|^2(-V(x))_+ \to \infty$ as $|x| \to \infty$.

13.3 Properties of eigenfunctions

13.3.1 Base state

We assume that Ω is connected domain. Consider the lowest eigenvalue λ_1 and a corresponding eigenfunction u_1.

Theorem 13.3.1. *Let Ω be a connected domain. Then*

(a) Eigenfunction u_1 does not change its sign;

(b) λ_1 is a simple eigenvalue.

Proof. (a) Let $v = |u_1|$. Observe that $\|v\| = \|u_1\|$ and $Q(v) = Q(u_1)$. Then $|u_1| = v$ is also an eigenfunction corresponding to eigenvalue λ_1 and then $u_{1,\pm} = \frac{1}{2}(|u_1| \pm u_1) = \max(\pm u_1, 0)$ are also eigenfunctions corresponding to λ_1. At least one of these two eigenfunctions, say $u_{1,+}$ is positive on some open set $\Omega' \subset \Omega$ and then $u_{1,-} = 0$ on Ω'. However $(\Delta + \lambda_1)u_{1,-} = 0$ and

there are many different theorems implying that since $u_{1,-} = 0$ on Ω' it must be 0 on Ω. (For example: solutions of $(\Delta + \lambda_1)v = 0$ are analytic functions and there for analytic functions there is a unique continuation theorem.) Then $u_1 \geq 0$ in Ω.

(b) If u_1 and v_1 are two linearly independent eigenfunctions corresponding to the same lowest eigenvalue λ_1, then one can find $w_1 = u_1 + \alpha u_1$ which also is an eigenfunction and $(u_1, w_1) = 0$ which is impossible since both of them have constant signs in Ω and therefore $u_1 w_1$ has a constant sign in Ω and does not vanish identically in virtue of argument of (a).

□

Remark 13.3.1. (a) Let $\lambda_1 \geq 0$. Assuming that $u_1 \geq 0$ we see that u_1 is superharmonic function ($\Delta u_1 \leq 0$) but such functions cannot reach minimum inside domain unless constant. So $u_1 > 0$ in Ω.

(b) Let $\lambda_1 < 0$. Assuming that $u_1 \geq 0$ we see that u_1 is subharmonic function ($\Delta u_1 \geq 0$) and such functions cannot reach maximum inside domain unless constant.

(c) While λ_1 is always simple λ_n with $n \geq 2$ may be multiple.

Corollary 13.3.1. *u_n with $n \geq 2$ changes sign.*

Indeed, it is orthogonal to u_1 which does not change a sign.

13.3.2 Nodal sets

Definition 13.3.1. Let u_n be an eigenfunction. Then $\{x : u_n(x) = 0\}$ is called a *nodal set* (*nodal line* as $d = 2$) and connected components of $\{x \in \Omega, u_n(x) \neq 0\}$ are called *nodal domains*.

We know that for $n = 1$ is just one nodal domain and for $n \geq 2$ there are at least 2 nodal domains. We need the following theorem from *Ordinary Differential Equations*:

Theorem 13.3.2. *For $d = 1$ there are exactly $n - 1$ nodal points and n nodal intervals for u_n.*

Theorem 13.3.3. *For $d \geq 2$ if u is an eigenfunction with an eigenvalue λ_n then u_n has no more than n nodal domains.*

Proof. (i) Let u have $m \ge n$ nodal domains. Consider w_k coinciding with u in k-th nodal domain Ω_k of u. Then $w_1, \dots, w_m$ are linearly independent and

$$\|c_1 w_1 + \dots + c_m w_m\|^2 = \sum_{1\le j\le m} c_j^2 \|w_j\|^2,$$
$$Q(c_1 w_1 + \dots + c_m w_m) = \sum_{1\le j\le m} c_j^2 Q(w_j).$$

Consider the space L of linear combinations of $w_1, \dots, w_m$ which are orthogonal to $u_1, \dots, u_{n-1}$; then $L \ne \{0\}$. By definition $Q(v) \ge \lambda_n \|v\|^2$ on L. Since $u \in L$ we conclude that $Q(u) \ge \lambda_n \|u\|^2$; however $Q(u) = \lambda \|u\|^2$ and then $\lambda \ge \lambda_n$. It proves theorem if $\lambda_{n+1} > \lambda_n$.

Observe that since $(\Delta + \lambda_n) w_k = 0$ in Ω_k and $w_k = 0$ in $\Omega \setminus \Omega_k$ we integrating by parts see that $Q(w_k) = \lambda_n \|w_k\|^2$ for all k. Then $Q(v) = \lambda_n \|v\|^2$ for all $v \in L$.

Assume now that $m > n$. Then there exists $0 \ne v \in L$ which is a linear combination of $w_1, \dots, w_n$. Then v is an eigenfunction but it is 0 in Ω_{n+1} and therefore it must be 0. Contradiction.

Therefore if $\lambda_{n-1} < \lambda_n = \dots = \lambda_m$ then each eigenfunction corresponding to multiple eigenvalue λ_n has no more than n nodal domains.

Then we can use it in the case when variables are separated (we consider only $d = 2$ and only Dirichlet boundary condition: WORK

Example 13.3.1. Let $\Omega = \{0 < x < a, 0 < y < b\}$ be a rectangular box. Let us separate variables; then

$$u_{pq} = \sin\Big(\frac{p\pi x}{a}\Big)\sin\Big(\frac{q\pi y}{b}\Big), \qquad \mu_{pq} = \pi^2\Big(\frac{p^2}{a^2} + \frac{q^2}{b^2}\Big). \tag{13.3.1}$$

Then nodal lines form a rectangular grid (see below). Let $a = b = \pi$.

(a) Then $\lambda_1 = 2$ and $\lambda_2 = \lambda_3 = 10$ (where λ_n are μ_{pq} ordered). First figure shows nodal lines for u_{21} (and nodal lines for u_{12} are exactly like this but flipped over $x = y$). Consider now linear combinations of u_{21} and u_{12};

(b) Next $\lambda_4 = 8$;

(c) Further $\lambda_5 = \lambda_6 = 10$. First figure shows nodal lines for u_{31} (and nodal lines for u_{13} are exactly like this but flipped over $x = y$). Consider

now linear combinations of u_{31} and u_{13}; Comparing two last pictures we see that crossing opens under small perturbations.

(d) Further $\lambda_7 = \lambda_8 = 13$, First figure shows nodal lines for u_{32} (and nodal lines for u_{23} are exactly like this but flipped over $x = y$). Consider now linear combinations of u_{32} and u_{23};

(e) Further $\lambda_9 = \lambda_{10} = 17$, First figure shows nodal lines for u_{41} (and nodal lines for u_{14} are exactly like this but flipped over $x = y$). Consider now linear combinations of u_{412} and u_{14};

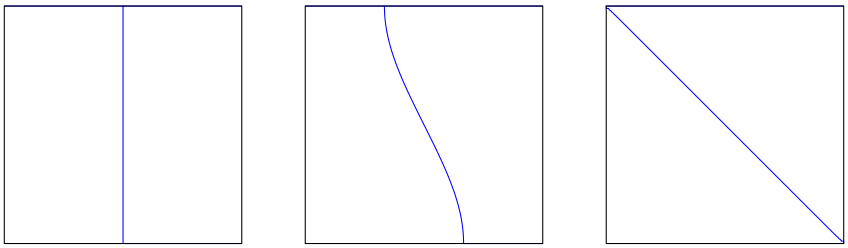
(f) Further $\lambda_{11} = 18$ is simple $p = q = 3$ and thus trivial; furthermore $\lambda_{12} = \lambda_{13} = 20$ with $p = 4, q = 2$ is also trivial: we need just to take any picture for $p = 2, q = 1$ and make its double mirror reflection (we do not draw u_{pq} anymore).

(g) Further $\lambda_{14} = \lambda_{15} = 25$ does not produce anything much different from (d) but simply more nodal domains:

(h) Further $\lambda_{16} = \lambda_{17} = 26$;

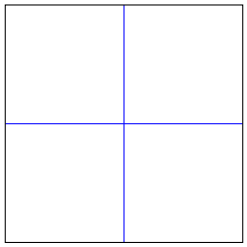
(i) Skipping $\lambda_{18} = \lambda_{19} = 29$, $\lambda_{20} = 32$, $\lambda_{21} = \lambda_{22} = 34$, consider $\lambda_{23} = \lambda_{24} = 37$.

(j) Starting from $\lambda = 50 = 7^2 + 1^2 = 5^2 + 5^2$ multiplicities could be larger than 2 and the following gallery is just a tiny sample



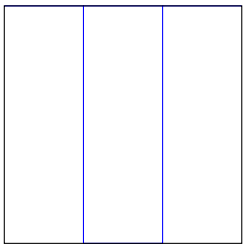
(a) u_{21} (b) $u_{21} + \frac{1}{2}u_{12}$ (c) $u_{21} + u_{12}$

Figure 13.1: $\lambda = 5$

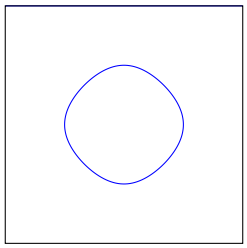


(a) u_{22}

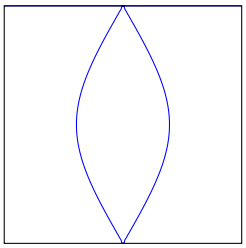
Figure 13.2: $\lambda = 8$



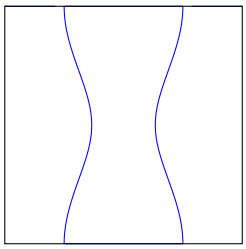
(a) u_{31}



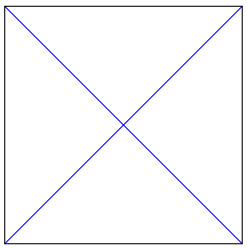
(b) $u_{31} + u_{13}$



(c) $u_{31} + \frac{1}{3}u_{13}$

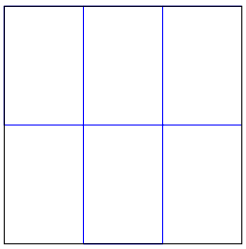


(d) $u_{31} - \frac{1}{3}u_{13}$

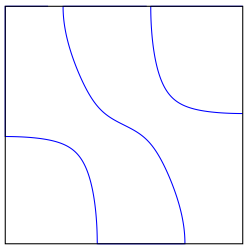


(e) $u_{31} - u_{13}$

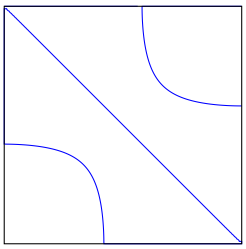
Figure 13.3: $\lambda = 10$



(a) u_{32}

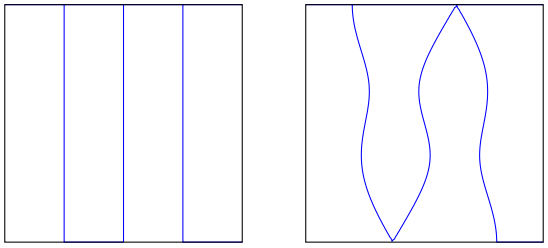


(b) $u_{32} + \frac{1}{2}u_{23}$



(c) $u_{32} + u_{23}$

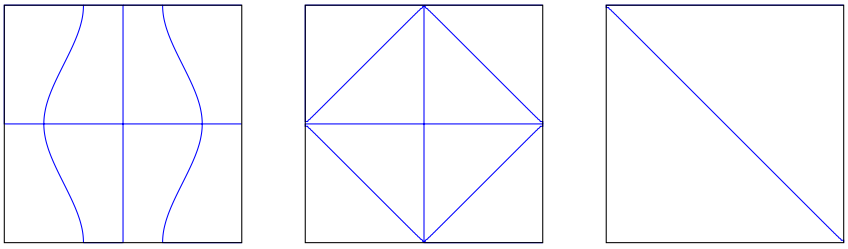
Figure 13.4: $\lambda = 13$



(a) u_{41} (b) $u_{41} + \sqrt{\frac{2}{27}}u_{14}$

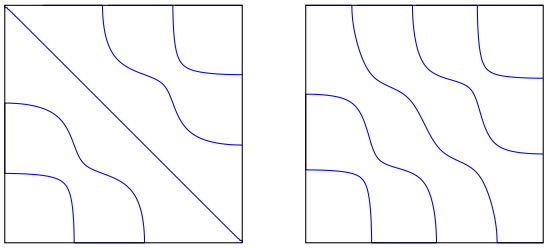
(c) $u_{41} + \frac{1}{2}u_{14}$ (d) $u_{41} + u_{14}$

Figure 13.5: $\lambda = 17$



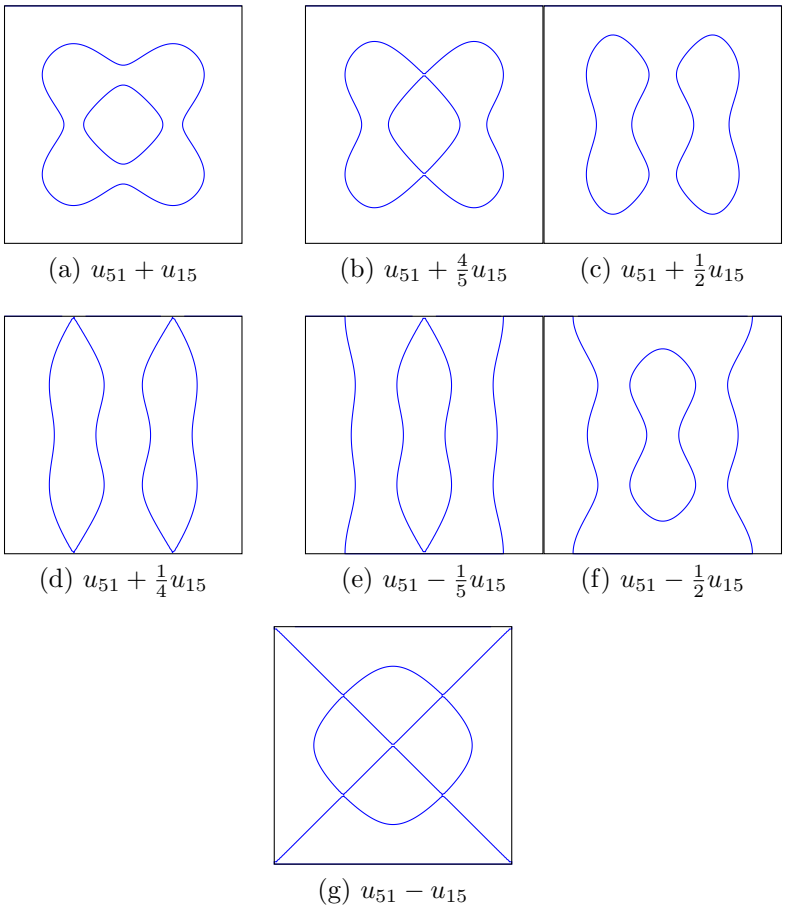
(a) $u_{42} + u_{24}$ (b) $u_{42} + \frac{1}{2}u_{24}$ (c) $u_{42} - u_{24}$

Figure 13.6: $\lambda = 20$



(a) $u_{43} + u_{34}$ (b) $u_{43} + \frac{1}{2}u_{34}$

Figure 13.7: $\lambda = 25$

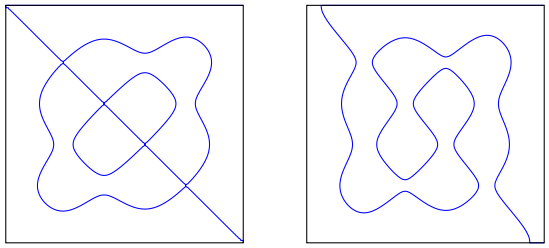


(a) $u_{51} + u_{15}$ (b) $u_{51} + \frac{4}{5}u_{15}$ (c) $u_{51} + \frac{1}{2}u_{15}$

(d) $u_{51} + \frac{1}{4}u_{15}$ (e) $u_{51} - \frac{1}{5}u_{15}$ (f) $u_{51} - \frac{1}{2}u_{15}$

(g) $u_{51} - u_{15}$

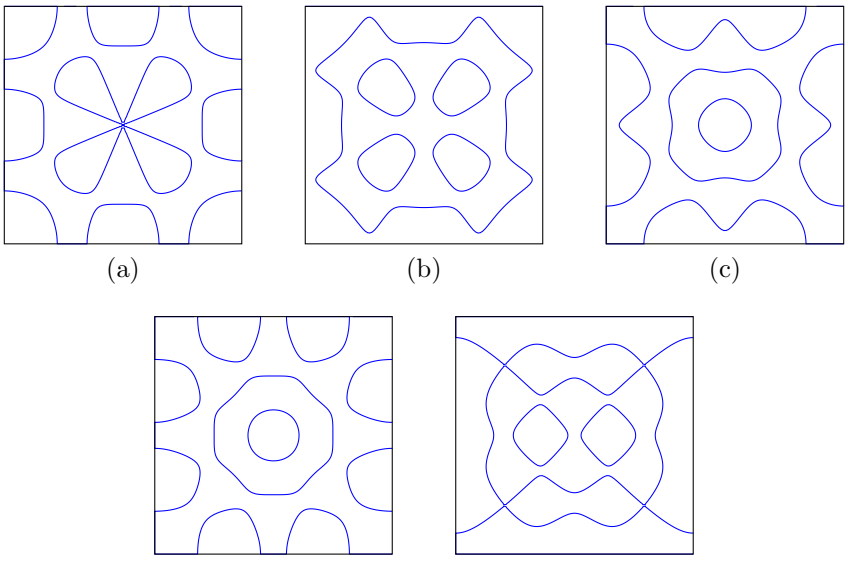
Figure 13.8: $\lambda = 26$



(a) $u_{61} + u_{16}$ (b) $u_{61} + \frac{4}{5}u_{16}$

(c) $u_{61} + \frac{1}{5}u_{16}$ (d) $u_{61} + \frac{1}{10}u_{16}$

Figure 13.9: $\lambda = 37$



(a) (b) (c)

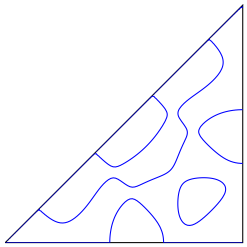
(d) (e)

Figure 13.10: $\lambda = 50$

Example 13.3.2. Consider $\Omega = \{0 < y < x < \pi\}$ which is a triangle.

(a) If on the diagonal $x = y$ a Dirichlet condition is required, then eigenfunctions are $u_{pq}(x,y) - u_{pq}(y,x)$ with $p \neq q$ (or their linear combination like $u_{83}(x,y) - u_{83}(y,x)$ and $u_{74}(x,y) - u_{74}(y,x)$).

(b) If on the diagonal $x = y$ a Neumann condition is required, then eigenfunctions are $u_{pq}(x,y) + u_{pq}(y,x)$ (or their linear combination like $u_{71}(x,y) + u_{71}(y,x)$ and $u_{55}(x,y)$).



(a)

Example 13.3.3. (a) Let $\Omega = \{r \leq b\}$ be a disk. Then nodal lines form a circular grid: $u_{pq}(r,\theta) = \cos(p\theta)J_p(k_{pq}r)$ where $J_p(z)$ are Bessel functions and $k_{pq}b$ is q-th root of $J_p(z)$. The similar statement is true for circular sectors, rings etc.

(b) Let Ω be an ellipse. Then (see 6.3.3) in the elliptical coordinates Laplacian is

$$\Delta = \frac{1}{c^2\big(\sinh^2(\sigma) + \sin^2(\tau)\big)}(\partial_\sigma^2 + \partial_\tau^2) \tag{13.3.2}$$

and separating variables $u = S(\sigma)T(\tau)$ we get

$$S'' + \big(\lambda c^2 \sinh^2(\sigma) - k\big)S = 0, \tag{13.3.3}$$

$$T'' + \big(\lambda c^2 \sin^2(\tau) + k\big)T = 0. \tag{13.3.4}$$

For T we have either π-periodic or π-antiperiodic boundary condition: $T(\tau + \pi) = \pm T(\tau)$ and for S we have Dirichlet boundary condition as $\cosh(\sigma) = a/c$ and respectively Neumann and Dirichlet boundary condition as $\sigma = 0$ arising from $S(\pm\sigma) = \pm S(\sigma)$. So we have a grid consisting from confocal ellipses and hyperbolas. The similar statement is true for elliptical "sectors", rings etc.

(c) Let Ω be an parabolic lense. Then (see Subsection 6.3.3) in the parabolic coordinates

$$\Delta = \frac{1}{\sigma^2+\tau^2}(\partial_\sigma^2+\partial_\tau^2) \tag{13.3.5}$$

and separating variables $u = S(\sigma)T(\tau)$ we get

$$S'' + \big(\lambda\sigma^2 - k\big)S = 0, \tag{13.3.6}$$
$$T'' + \big(\lambda c^2\tau^2 + k\big)T = 0. \tag{13.3.7}$$

So we have a grid consisting from confocal parabolas.

Remark 13.3.2. For generic domais (any sufficiently small perturbation of generic domain is a generic domain again but an arbitrarily small perturbation of non-generic domain may result in generic domain)

(a) All eigenvalues are simple;

(b) Nodal lines do not have self-intersections.

Remark 13.3.3. Historically interest to nodal lines appeared from Chladni plates but those are nodal lines for biharmonic operator

$$(\Delta^2 - \lambda)u = 0 \tag{13.3.8}$$

with free boundary conditions (appearing from variational problem

$$\delta\Big(\iint \big(u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2 - \lambda u^2\big)\,dxdy\Big) = 0 \tag{13.3.9}$$

(without boundary conditions). This is much more complicated question which was (partially) solved by Marie-Sophie Germain.

13.3.3 Hearing the shape of the drum

In 1965(?) Marc Kac asked the question"Can one hear the shape of the drum" which meant: if we know all the eigenvalues of the Dirichlet Laplacian Δ: $0 < -\lambda_1 < -\lambda_2 \le \lambda_3 \le \dots$ in the connected domain Ω, can we restore Ω (up to isometric movements—shift and rotations). The extensive study was using the method of *spectral invariants*, which are numbers

which have some geometric meaning and which can be calculated from $0 < -\lambda_1 < -\lambda_2 \le \lambda_3 \le \dots$.

The main source of such invariants was the *method of heat equation*: Namely let $G(x,y,t)$ with $x,y \in \Omega$ and $t>0$ be a Green function:

$$G_t - \Delta_x G = 0, \tag{13.3.10}$$
$$G|_{t=0} = \delta(x-y), \tag{13.3.11}$$
$$G|_{x\in\partial\Omega} = 0 \tag{13.3.12}$$

and let

$$\sigma(t) = \iint u(x,x,t)\,dx = \sum_{n\ge 1} e^{-\lambda_n t} \tag{13.3.13}$$

be a *heat trace*; then

$$\sigma(t) \sim \sum_{k\ge -d} c_k t^k \qquad \text{as } \ t\to +0 \tag{13.3.14}$$

Check! where c_k are heat invariants. It was proven that (as $d=2$, for $d\ge 3$ similarly) area, perimeter, number of holes and many other geometric characteristic are spectral invariants but the final answer was negative: there are *isospectral* domains Ω and Ω' (so that eigenvalues of Laplacians in those are equal) which are not *isometric* (have different shapes).

13.4 About spectrum

13.4.1 Definitions and classification

13.4.1.1 Definitions

Let H be a Hilbert space (see Definition 4.3.3)).

Definition 13.4.1. Linear operator $L : \mathsf{H} \to \mathsf{H}$ is *bounded* if

$$\|Lu\| \le M\|u\| \qquad \forall u \in \mathsf{H}; \tag{13.4.1}$$

the smallest constant M for which it holds is called *operator norm* of L and denoted $\|L\|$.

Definition 13.4.2. Let $L : \mathsf{H} \to \mathsf{H}$ be a bounded linear operator.

(a) Adjoint operator L^* is defined as

$$(Lu, v) = (u, L^*v) \qquad \forall u, v \in \mathsf{H}; \tag{13.4.2}$$

(b) Operator L is *self-adjoint* if $L^* = L$.

However one needs to consider also unbounded operators. Such operators not only fail (13.4.1) but they are not defined everywhere.

Definition 13.4.3. Consider a linear operator $L : D(L) \to \mathsf{H}$ where $D(L)$ is a linear subset in H (i.e. it is a linear subspace but we do not call it this way because it is not closed) which is *dense* in H (i.e. for each $u \in \mathsf{H}$ there exists a sequence $u_n \in D(L)$ converging to u in H). Then

(a) Operator L is *closed* if $u_n \to u$, $Lu_n \to f$ imply that $u \in D(L)$ and $Lu = f$;

(b) Operator L is *symmetric* if

$$(Lu, v) = (u, Lv) \qquad \forall u, v \in D(L); \tag{13.4.3}$$

(c) Symmetric operator L is *self-adjoint* if $(L \pm i)^{-1} : \mathsf{H} \to D(L)$ exist: $(L \pm i)(L \pm i)^{-1} = I$, $(L \pm i)^{-1}(L \pm i) = I$ (identical operator)

Remark 13.4.1. (a) For bounded operators "symmetric" equals "self-adjoint";

(b) Not so for unbounded operators. F.e. $Lu = -u''$ on $(0, l)$ with $D(L) = \{u(0) = u'(0) = u(l) = u'(l) = 0\}$ is symmetric but not self-adjoint;

(c) Self-adjoint operators have many properties which symmetric but not self-adjoint operators do not have;

(d) In Quantum Mechanics *observables* are self-adjoin operators.

Theorem 13.4.1. *The following statements are equivalent:*

(a) L is self-adjoint;

(b) L generates unitary group e^{itL} ($t \in \mathbb{R}$: $\|e^{itL}u\| = \|u\|$, $e^{i(t_1+t_2)L} = e^{it_1L}e^{it_2L}$, $u \in D(L) \implies e^{itL}u \in D(L)$, $\frac{d}{dt}e^{itL}u = Le^{itL}u$ for all $u \in D(L)$ (and conversely, if $e^{itL}u$ is differentiable by t then $u \in D(L)$;

(c) *Exist spectral projectors – operators* $\theta(\tau - L)$ *(*$\theta(\tau) = 0$ *as* $\tau \le 0$ *and* $\theta(\tau) = 1$ *as* $\tau > 0$*) such that* $\theta(\tau - L)$ *are orthogonal projectors,* $\theta(\tau_1 - L)\theta(\tau_2 - L) = \theta(\tau - L)$ *with* $\tau = \min(\tau_1, \tau_2)$*,* $\theta(\tau - L)u \to 0$ *as* $\tau \to -\infty$*;* $\theta(\tau - L)u \to u$ *as* $\tau \to +\infty$*;* $\theta(\tau - L)u \to \theta(\tau^* - L)$ *as* $\tau \to \tau^* - 0$ *and*

$$f(L) = \int f(\tau) d_\tau \theta(\tau - L) \tag{13.4.4}$$

Definition 13.4.4. Let us consider operator L (bounded or unbounded). Then

(a) $z \in \mathbb{C}$ belongs to the *resolvent set* of L if $(L-z)^{-1} : \mathsf{H} \to D(L)$ exists and is a bounded operator: $(L-z)^{-1}(L-z) = I$, $(L-z)(L-z)^{-1} = I$.

(b) $z \in \mathbb{C}$ belongs to the *spectrum* of L if it does not belong to its resolvent set. We denote spectrum of L as $\sigma(L)$.

Remark 13.4.2. (a) Resolvent set is always open (and spectrum is always closed) subset of $\mathbb{C}$;

(b) If L is self-adjoint then $\sigma(L) \subset \mathbb{R}$;

(c) If L is bounded then $\sigma(L)$ is a bounded set.

13.4.1.2 Classification

Not all points of the spectrum are born equal! From now on we consider only self-adjoint operators.

Definition 13.4.5. (a) z is an *eigenvalue* if there exists $u \neq 0$ s.t. $(A - z)u = 0$. Then u is called *eigenvector* (or *eigenfunction* depending on context) and $\{u : (A - z)u = 0\}$ is an *eigenspace* (corresponding to eigenvalue z). The dimension of the eigenspace is called a *multiplicity* of z. Eigenvalues of multiplicity 1 are *simple*, eigenvalues of multiplicity 2 are *double*, ... but there could be eignvalues of infinite multiplicity!

(b) The set of all eigenvalues is called *pure point spectrum*;

(c) Eigenvalues of the finite multiplicity which are isolated from the rest of the spectrum form a *discrete spectrum*; the rest of the spectrum is called *essential spectrum*.

Definition 13.4.6. $z \in \mathbb{C}$ belongs to *continuous spectrum* if z is not an eigenvalue and inverse operator $(L-z)^{-1}$ exists but is not a bounded operator (so its domain $D((L-z)^{-1}$ is dense).

Remark 13.4.3. Continuous spectrum could be classified as well. The difference between *absolutely continuous* and *singular continuous* spectra will be illustrated but one can define also multiplicity of continuous spectrum as well. However one needs a *Spectral Theorem* to deal with these issues properly.

13.4.2 Spectrum: examples

Example 13.4.1. Schrödinger operator

$$L = -\frac{1}{2}\Delta + V(x) \tag{13.4.5}$$

with potential $V(x) \to +\infty$ as $|x| \to \infty$ has a discrete spectrum: its eignevalues $E_n \to +\infty$ have finite multiplicities. In dimension $d = 1$ all these eigenvalues are simple, not necessarily so as $d \geq 2$.

Example 13.4.2. Consider Laplacian on 2-dimensional sphere which appears after separation of variables for Laplacian in $\mathbb{R}^3$ in spherical coordinates in Subsection 6.3.2. Then $-\Delta$ has a spectrum $\{E_n = n(n+1) : n = 0, 1, \dots\}$; E_n is an eigenvalue of multiplicity $(2n+1)$. Corresponding eigenfunctions are *spherical harmonics*. See Definition 8.1.1.

Example 13.4.3. Schrödinger operator in 2D with a constant magnetic and no electric field

$$L = \frac{1}{2}(-i\partial_x - \frac{1}{2}By)^2 + \frac{1}{2}(-i\partial_y + \frac{1}{2}By)^2 \tag{13.4.6}$$

with $B > 0$ (or $B < 0$) has a pure point spectrum. Eigenvalues $E_n = |B|(n + \frac{1}{2})$, $n = 0, 1, 2, \dots$ have infinite multiplicity and are called *Landau levels.*

Example 13.4.4. "Free" particle Schrödinger operator $L = -\frac{1}{2}\Delta$ in $\mathbb{R}^d$ has a continuous spectrum $[0, +\infty)$.

Example 13.4.5. Schödinger operator (13.4.5) with potential $V(x) \to 0$ as $|x| \to \infty$ has a continuous spectrum $[0, +\infty)$ but it can have a finite or infinite number of negative eignvalues $E_n < 0$.

(a) If $|V(x)| \le M(|x|+1)^{-m}$, $m > 2$ the number of eigenvalues is finite.

(b) For *Coulomb potential* $V(x) = -Z|x|^{-1}$ $(Z > 0)$ $E_n = -\frac{Z^2}{4n^2}$ of multiplicity n^2, $n = 1, 2, \ldots$.

Example 13.4.6. Free particle *Dirac operator*

$$L = \sum_{j=1}^{3} \gamma^j(-i\partial_{x_j}) + \gamma^0 m, \qquad m > 0 \tag{13.4.7}$$

(where γ^j are *Dirac matrices* has a continuous spectrum $(-\infty, -m]\cup[m, \infty)$.

Perturbing it by a potential $V(x)$, $V(x) \to 0$ as $|x| \to \infty$

$$L = \sum_{j=1}^{3} \gamma^j(-i\partial_{x_j}) + m\gamma^0 + V(x)I, \qquad m > 0 \tag{13.4.8}$$

can add a finite or infinite number of eigenvalues in *spectral gap* $(-m, m)$. They can accumulate only to the borders of the spectral gap.

Example 13.4.7. Perturbing Example 13.4.3 by a potential $V(x)$, $V(x) \to 0$ as $|x| \to \infty$

$$L = \frac{1}{2}(-i\partial_x - \frac{1}{2}By)^2 + \frac{1}{2}(-i\partial_y + \frac{1}{2}By)^2 + V(x) \tag{13.4.9}$$

breaks Landau levels into sequences of eigenvalues $E_{n,k}$, $n = 0, 1, \ldots$, $k = 1, 2, \ldots$, $E_{n,k} \to E_n = |B|(n + \frac{1}{2})$ as $k \to \infty$.

Example 13.4.8. Consider Schrödinger operator (13.4.5) with periodic potential in $\mathbb{R}^d$: $V(x + a) = V(x)$ for all $a \in \Gamma$ where Γ is a *lattice of periods*, see Definition 4.B.1. Then L has a *band spectrum*.

Namely on the *elementary cell* (Definition 4.B.3) Ω consider operator $L(k)$ where $k \in \Omega^*$ is a *quasimomentum*; $L(k)$ is given by the same formula as L but s defined on functions which are *quasiperiodic with quasimomentum* k. Its spectrum is discrete: $\sigma(L(k)) = \{E_n(k) : n = 1, 2, \ldots\}$.

Then spectrum $\sigma(L)$ consists of *spectral bands*

$$\sigma_n := [\min_{k\in\Omega^*} E_n(k), \max_{k\in\Omega^*} E_n(k)], \tag{13.4.10}$$

$$\sigma(L) = \bigcup_{n=1}^{\infty} \sigma_n; \tag{13.4.11}$$

these spectral bands can overlap. The spectrum $\sigma(L)$ is continuos.

1. As dimension $d = 1$ we can do better than this: $E_n(k)$ are increasing (decreasing) functions of k on $(0, \pi/a)$ (where a is the period) as n is odd (respectively even) and
$$E_n^* := \max_{k\in[0,\pi/a]} E_n(k) \le E_{(n+1)*} := \min_{k\in[0,\pi/a]} E_{n+1}(k) \tag{13.4.12}$$
and for *generic potential* $V(x)$ all inequalities are strict and all *all spectral gaps* $(E_n^*, E_{(n+1)}^*$ are open.
2. As dimension $d \ge 2$ only finite number of spectral gaps could be open.
3. Perturbation of such operator L by another potential $W(x)$, $W(x) \to 0$ as $|x| \to \infty$ could can add a finite or infinite number of eigenvalues in *spectral gaps*. They can accumulate only to the borders of the spectral gaps.

Example 13.4.9. In the space $\ell^2(\mathbb{Z})$ (which is the space of sequences u_n, $n = \dots, -2, -1, 0, 1, 2, \dots$ such that $\|u\|^2 := \sum_{n=-\infty}^{\infty} |u_n|^2 < \infty$) consider *almost Mathieu operator* (which appears in the study of *quantum Hall effect*).
$$(Lu)_n = u_{n+1} + u_{n-1} + 2\lambda\cos(2\pi(\theta + n\alpha)) \tag{13.4.13}$$
with $|\lambda| \le 1$. Assume that α is a *Diophantine number* (which means it is an irrational number which cannot be approximated well by rational numbers; almost all irrational numbers (including all algebraic like $\sqrt{2}$) are Diophantine).

Then the spectrum $\sigma(L)$ is continuous (no eigenvalues!) but it is *singular continuous*: for any $\varepsilon > 0$ it can be covered by the infinite sequence of segments of the total length $< \varepsilon$. As an example of such set see *Cantor set*.

Remark 13.4.4. (a) Example 13.4.8 was completely investigated only in the end of the 20-th century.

(b) Example 13.4.9 was completely investigated only in the 21-st century.

13.4.3 Spectrum: explanations

13.4.3.1 Landau levels

Consider Example 3: Schrdinger operator in 2D with a constant magnetic and no electric field
$$L = \frac{1}{2}(-i\partial_x - \frac{1}{2}By)^2 + \frac{1}{2}(-i\partial_y + \frac{1}{2}By)^2 \tag{13.4.14}$$

with $B > 0$ (or $B < 0$) has a pure point spectrum. For simplicity assume that $B > 0$. We apply a *gauge transformation* which for Schrdinger operator means multiplying it from the left and right by $e^{i\hbar^{-1}\phi(x)}$ and $e^{-i\hbar^{-1}\phi(x)}$ respectively with a real-valued ϕ (which is an unitary transformation) and replaces $-i\hbar\nabla$ by $-i\hbar\nabla - (\nabla\phi)$ (which is equivalent to changing vector potential $\mathbf{A}(x)$ by $\nabla\phi$ which in turn does not change $\nabla \times \mathbf{A}$. Taking $\hbar = 1$ and $\phi = \frac{1}{2}Bxy$ we arrive to

$$L' = \frac{1}{2}(-i\partial_x - By)^2 + \frac{1}{2}(-i\partial_y)^2;$$

then making Fourier transform by $x \mapsto \xi$ we get

$$L'' = \frac{1}{2}(-\xi - By)^2 + \frac{1}{2}(-i\partial_y)^2;$$

and plugging $y = B^{-\frac{1}{2}}(y_{\mathsf{new}} - B^{-1}\xi)$ we get

$$\frac{1}{2}B(-\partial_y^2 + y^2)$$

which is a harmonic oscillator multiplied by B and in virtue of Section 4.C its spectrum consists of eigenvalues $E_n = |B|(n + \frac{1}{2})$, $n = 0, 1, 2, \dots$ which are called *Landau levels.*

However there is a "hidden variable" ξ, so eigenfunctions Hermite functions of y but multiplied by arbitrary functions $C(\xi)$ rather than by constants which implies that these eigenvalues have constant multiplicities.

13.4.3.2 Band spectrum

Consider Example 13.4.8: Schrödinger operator with periodic potential in $\mathbb{R}^d$: $V(x + a) = V(x)$ for all $a \in \Gamma$ where Γ is a lattice of periods.

Let us decompose function $u(\mathbf{x})$ into n-dimensional Fourier integral

$$u(x) = \iiint_{\mathbb{R}^n} e^{i\mathbf{k}\cdot\mathbf{x}}\hat{u}(\mathbf{k})\, d^n\mathbf{k},$$

then replace this integral by a sum of integrals over dual elementary cell Ω^* shifted by $\mathbf{n} \in \Gamma^*$

$$\sum_{\mathbf{n}\in\Gamma^*} \iiint_{\Omega^*+\mathbf{n}} e^{i\mathbf{k}\cdot\mathbf{x}}\hat{u}(\mathbf{k})\, d^n\mathbf{k},$$

then change variable $\mathbf{k} = \mathbf{k}_{\mathsf{new}} + \mathbf{n}$

$$\iiint_{\Omega^*} e^{i(\mathbf{k}+\mathbf{n})\cdot\mathbf{x}} \underbrace{\Big(\sum_{\mathbf{n}\in\Gamma^*} e^{i\mathbf{n}\cdot\mathbf{x}}\hat{u}(\mathbf{k}+\mathbf{n})\Big)}_{=U(\mathbf{k},\mathbf{x})} d^n\mathbf{k},$$

we observe that $U(\mathbf{k},\mathbf{x})$ is quasiperiodic with quasimomentum $\mathbf{k}$.

In advanced Real Analysis it would be a decomposition of our Hilbert space $\mathsf{H} = L^2(\mathbb{R}^n)$ into *direct integral of Hilbert spaces* $\mathsf{H}(\mathbf{k})$ of such functions, and our operator is acting in each of those spaces separately, with a spectrum $\sigma(L(\mathbf{k})) = \{E_n(\mathbf{k}) : n = 1, 2, \dots\}$. This implies that L has a *band spectrum*: it consists of *spectral bands* $\sigma_n := [\min_{k\in\Omega^*} E_n(k), \max_{k\in\Omega^*} E_n(k)]$:

$$\sigma(L) = \bigcup_{n=1}^{\infty} \sigma_n;$$

these spectral bands can overlap. On can prove that $E_n(\mathbf{k})$ really depend on $\mathbf{k}$ and are not taking the same value on some set of *non–zero measure* (another notion from Real Analysis) which implies that the spectrum $\sigma(L)$ is continuos.

13.5 Continuous spectrum and scattering

13.5.1 Introduction

Here we discuss idea of scattering. Basically there are two variants of the *Scattering Theory*–non-stationary and stationary. We start from the former but then fall to the latter. We assume that there is *unperturbed operator* L_0 and *perturbed operator* $L = L_0 + V$ where V is a perturbation. It is always assumed that L_0 has only continuous spectrum (more precisely–absolutely continuous) and the same is true for L (otherwise our space H is decomposed into sum $\mathsf{H} = \mathsf{H}_{\mathsf{ac}} \oplus \mathsf{H}_{\mathsf{pp}}$ where L acts on each of them and on H_{ac}, H_{pp} it has absolutely continuous and pure point spectra respectively. Scattering happens only on the former.

Now consider $u = e^{itL}u_0$ be a solution of the perturbed non-stationary equation. In the reasonable assumptions it behaves as $t \to \pm\infty$ as solutions of the perturbed non-stationary equation:

$$\|e^{itL}u_0 e^{itL_0}u_\pm\| \to 0 \qquad \text{as} \quad t \to \pm\infty \tag{13.5.1}$$

or, in other words the following limits exist

$$u_{\pm} = \lim_{t\to\pm\infty} e^{-itL_0}e^{itL}u_0. \tag{13.5.2}$$

Then operators $W_{\pm} : u_0 \to u_p m$ are called *wave operators* and under some restrictions they are proven to be unitary operators from H onto H_{ac}. Finally $S = W_+W_-^{-1}$ is called a *scattering operator*.

Despite theoretical transparency this construction is not very convenient and instead are considered some test solutions which however do not belong to space H_{ac}.

13.5.2 One dimensional scattering

Let us consider on $\mathsf{H} = L^2(\mathbb{R})$ operators $L_0u := -u_{xx}$ and $L = L_0 + V(x)$. Potential V is supposed to be fast decaying as $|x| \to \infty$ (or even compactly supported). Then consider a solution of

$$u_t = iLu = -iu_{xx} + V(x)u \tag{13.5.3}$$

of the form $e^{ik^2t}v(x,k)$; then $v(x,k)$ solves

$$v_{xx} - V(x)v + k^2v = 0 \tag{13.5.4}$$

and it behaves as $a_{\pm}e^{ikx} + b_{\pm}e^{-ikx}$ as $x \to \pm\infty$.

Consider solution which behaves exactly as e^{ikx} as $x \to -\infty$:

$$v(k,x) = e^{ikx} + o(1) \qquad \text{as } x \to -\infty; \tag{13.5.5}$$

then

$$v(k,x) = A(k)e^{ikx} + B(k)e^{-ikx} + o(1) \qquad \text{as } x \to +\infty. \tag{13.5.6}$$

Complex conjugate solution then

$$\bar{v}(k,x) = e^{-ikx} + o(1) \qquad \text{as } x \to -\infty, \tag{13.5.7}$$

$$\bar{v}(k,x) = \bar{A}(k)e^{-ikx} + \bar{B}(k)e^{ikx} + o(1) \qquad \text{as } x \to +\infty. \tag{13.5.8}$$

Their Wronskian $W(v,\bar{v})$ must be constant (which follows from equation to Wronskian from ODE) and since $W(v,\bar{v}) = W(e^{ikx}, e^{-ikx}) + o(1) = -2ik + o(1)$ as $x \to -\infty$ and

$$W(v,\bar{v}) = W(A(k)e^{ikx} + B(k)e^{-ikx}, \tag{13.5.9}$$

$$\bar{A}(k)e^{-ikx} + \bar{B}(k)e^{ikx}) + o(1) = -2ik\big(|b(k)|^2 - |a(k)|^2\big) + o(1) \tag{13.5.10}$$

as $x \to +\infty$ we conclude that

$$|A(k)|^2 - |B(k)|^2 = 1. \tag{13.5.11}$$

We interpret it as the wave $A(k)e^{ikx}$ at $+\infty$ meets a potential and part of it e^{ikx} passes to $-\infty$ and another part $B(k)e^{-ikx}$ reflects back to $+\infty$.

We observe that (13.5.11) means that the energy of the passed (refracted) and reflected waves together are equal to the energy of the original wave. We can observe that

$$A(-k) = \bar{A}(k), \qquad B(-k) = \bar{B}(k). \tag{13.5.12}$$

Functions $A(k)$ and $B(k)$ are *scattering coefficients* and together with eigenvalues $-k_j^2$

$$\phi_{j,xx} - V_j(x)\phi_j - k_j^2\phi_j = 0, \qquad \phi_j \neq 0 \tag{13.5.13}$$

they completely define potential V.

13.5.3 Three dimensional scattering

Consider $-\Delta$ as unperturbed operator and $-\Delta + V(x)$ as perturbed where $V(x)$ is smooth fast decaying at infinity potential. We ignore possible point spectrum (which in this case will be finite and discrete). Let us consider perturbed wave equation

$$u_{tt} - \Delta u + V(x)u = 0; \tag{13.5.14}$$

it is simler than Schrdinger equation. Let us consider its solution which behaves as $t \to -\infty$ as a plane wave

$$u \sim u_{-\infty} = e^{ik(\boldsymbol{\omega}\cdot\mathbf{x}-t)} \qquad \text{as } t \to -\infty. \tag{13.5.15}$$

with $\boldsymbol{\omega} \in \mathbb{S}^2$ (that means $\boldsymbol{\omega} \in \mathbb{R}^3$ and $|\boldsymbol{\omega}| = 1$), $k \geq 0$.

Theorem 13.5.1. *If* (13.5.15) *holds then*

$$u \sim u_-+\infty = e^{ik(\boldsymbol{\omega}\cdot\mathbf{x}-t)} + v(x)e^{-ikt} \qquad \textit{as } t \to +\infty. \tag{13.5.16}$$

where the second term in the right-hand expression is an outgoing spherical wave i.e. $v(x)$ *satisfies Helmholtz equation* (9.1.19) *and Sommerfeld radiation conditions* (9.1.20)–(9.1.21) *and moreover*

$$v(x) \sim a(\boldsymbol{\theta}, \boldsymbol{\omega}; k)|x|^{-1}e^{ik|x|} \qquad \textit{as } x = r\boldsymbol{\theta}, r \to \infty, \boldsymbol{\theta} \in \mathbb{S}^2. \tag{13.5.17}$$

Sketch of Proof. Observe that $(u-u_{-\infty})_{tt}-\Delta(u-u_{-\infty})=f:=-Vu$ and $(u-u_{-\infty})\sim 0$ as $t\to-\infty$ and then applying Kirchhoff formula (9.1.12) with 0 initial data at $t=-\infty$ we arrive to

$$u-u_{-}-\infty=\frac{1}{4\pi}\iiint |x-y|^{-1}f(y,t-|x-y|)\,dy \tag{13.5.18}$$

and one can prove easily (13.5.17) from this. □

Definition 13.5.1. $a(\boldsymbol{\theta},\boldsymbol{\omega};k)$ is *Scattering amplitude* and operator $S(k):L^2(\mathbb{S}^2)\to L^2(\mathbb{S}^2)$,

$$(S(k)w)(\boldsymbol{\theta})=w(\boldsymbol{\theta})+\iint_{\mathbb{S}^2} a(\boldsymbol{\theta},\boldsymbol{\omega};k)w(\boldsymbol{\omega})\,d\sigma(\boldsymbol{\omega}) \tag{13.5.19}$$

is a *scattering matrix*.

It is known that

Theorem 13.5.2. *Scattering matrix is a unitary operator for each k:*

$$S^*(k)S(k)=S(k)S^*(k)=I. \tag{13.5.20}$$

Remark 13.5.1. (a) The similar results are proven when the *scatterer* is an obstacle rather than potential, or both.

(b) Determine scatterer from scattering amplitude is an important *Inverse scattering problem*.

(c) In fact *fast decaying at infinity* potential means decaying faster than Coulomb potential; for the latter theory needs to be heavily modified.

Chapter 14

Miscellaneous

14.1 Field theory

14.1.1 Green, Gauss, Stokes formulae

Let D be a bounded domain in $\mathbb{R}^2$ and $L = \partial D$ be its boundary. Then

$$-\int_L \mathbf{A}\cdot\mathbf{n}\,ds = \iint_D (\nabla\cdot\mathbf{A})\,dS \tag{14.1.1}$$

where the left-hand side expression is a linear integral, the right-hand side expression is an area integral and $\mathbf{n}$ is a unit inner normal to L. This is *Green formula*.

Let V be a bounded domain in $\mathbb{R}^3$ and $\Sigma = \partial V$ be its boundary. Then

$$-\iint_\Sigma \mathbf{A}\cdot\mathbf{n}\,dS = \iiint_D (\nabla\cdot\mathbf{A})\,dV \tag{14.1.2}$$

where the left-hand side expression is a surface integral, the right-hand side expression is a volume integral and $\mathbf{n}$ is a unit inner normal to Σ. This is *Gauss formula*.

Remark 14.1.1. (a) Gauss formula holds in any dimension.

(b) Here sign "$-$"appears because $\mathbf{n}$ is a unit *inner* normal.

Let D be a bounded domain in $\mathbb{R}^2$ and $L = \partial D$ be its boundary, counter–clockwise oriented (if L has several components then inner components

should be clockwise oriented). Then

$$\oint_L \mathbf{A}\cdot d\,\mathbf{r} = \iint_D (\nabla\times\mathbf{A})\cdot\mathbf{n}\,dS \tag{14.1.3}$$

where the left-hand side expression is a line integral, the right-hand side expression is an area integral and $\mathbf{n}=\mathbf{k}$. This is *Green formula* again.

Let Σ be a bounded piece of the surface in $\mathbb{R}^3$ and $L=\partial\Sigma$ be its boundary. Then

$$\oint_L \mathbf{A}\cdot d\,\mathbf{l} = \iint_\Sigma (\nabla\times\mathbf{A})\cdot\mathbf{n}\,dS \tag{14.1.4}$$

where the left-hand side expression is a line integral, the right-hand side expression is a surface integral and $\mathbf{n}$ is a unit normal to Σ; orientation of L should match to direction of $\mathbf{n}$. This is *Stokes formula*.

Remark 14.1.2. (a) We can describe orientation in the Green formula as "the pair $\{d\mathbf{r},\mathbf{n}\}$ has a right-hand orientation"

(b) We can describe orientation in the Stokes formula as "the triple $\{d\mathbf{r},\boldsymbol{\nu},\mathbf{n}\}$ has a right-hand orientation" where $\boldsymbol{\nu}$ is a normal to L which is tangent to Σ and directed inside of Σ.

(c) Stokes formula holds in any dimension of the surface Σ but then it should be formulated in terms of differential forms

$$\int_\Sigma d\omega = \int_{\partial\Sigma}\omega \qquad \text{(Stokes formula)}$$

which is the material of Analysis II class (aka Calculus II Pro).

14.1.2 Properties of nabla

14.1.2.1 Definitions

Definition 14.1.1. (a) Operator ∇ is defined as

$$\nabla = \mathbf{i}\partial_x + \mathbf{j}\partial_y + \mathbf{k}\partial_z. \tag{14.1.5}$$

(b) It could be applied to a scalar function resulting in its gradient ($\operatorname{grad}\phi$)

$$\nabla\phi = \mathbf{i}\partial_x\phi + \mathbf{j}\partial_y\phi + \mathbf{k}\partial_z\phi$$

(c) and to vector function $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$ resulting in its divergence ($\operatorname{div}\mathbf{A}$)

$$\nabla \cdot \mathbf{A} = \partial_x A_x + \partial_y A_y + \partial_z A_z$$

(d) and also in its curl ($\operatorname{curl}\mathbf{A}$) or rotor ($\operatorname{rot}\mathbf{A}$), depending on the mathematical tradition:

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix}$$

which is equal to

$$(\partial_y A_z - \partial_z A_y)\mathbf{i} + (\partial_z A_x - \partial_x A_z)\mathbf{j} + (\partial_x A_y - \partial_y A_x)\mathbf{k}.$$

14.1.2.2 Double application

Definition 14.1.2.

$$\Delta = \nabla^2 = \nabla \cdot \nabla = \partial_x^2 + \partial_y^2 + \partial_z^2. \tag{14.1.6}$$

is *Laplace operator* or simply *Laplacian*.

Four formulae to remember:

$$\nabla(\nabla\phi) = \Delta\phi, \tag{14.1.7}$$

$$\nabla \times (\nabla\phi) = 0, \tag{14.1.8}$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0, \tag{14.1.9}$$

$$\nabla \times (\nabla \times \mathbf{A}) = -\Delta\mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) \tag{14.1.10}$$

where all but the last one are obvious and the last one follows from

$$\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) = -\mathbf{a}^2\mathbf{b} + (\mathbf{a} \cdot \mathbf{b})\mathbf{a} \tag{14.1.11}$$

which is the special case of

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}). \tag{14.1.12}$$

14.1.2.3 Application to the product

Recall *Leibniz rule* how to apply the first derivative to the product which can be symbolically written as

$$\partial(uv) = (\partial_u + \partial_v)(uv) = \partial_u(uv) + \partial_v(uv) = v\partial_u(u) + u\partial_v(v) = v\partial u + u\partial v$$

where subscripts "u" or "v" mean that it should be applied to u or v only.

Since ∇ is a linear combination of the first derivatives, it inherits the same rule. Three formulae are easy

$$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi, \tag{14.1.13}$$

$$\nabla \cdot (\phi\mathbf{A}) = \phi\nabla \cdot \mathbf{A} + \nabla\phi \cdot \mathbf{A}, \tag{14.1.14}$$

$$\nabla \times (\phi\mathbf{A}) = \phi\nabla \times \mathbf{A} + \nabla\phi \times \mathbf{A}, \tag{14.1.15}$$

and the fourth follows from the Leibniz rule and (14.1.12)

$$\begin{aligned}\nabla \times (\mathbf{A} \times \mathbf{B}) =& (\mathbf{B} \cdot \nabla)A - \mathbf{B}(\nabla \cdot \mathbf{A}) - \\ & (\mathbf{A} \cdot \nabla)B - \mathbf{A}(\nabla \cdot \mathbf{B}).\end{aligned} \tag{14.1.16}$$

14.2 Conservation laws

14.2.1 Theory

We start from example. Consider the liquid with density $\rho = \rho(\mathbf{x}; t)$ and velocity $\mathbf{v} = \mathbf{v}(\mathbf{x}; t)$, $\mathbf{x} = (x_1, x_2, x_3)$. Consider a small surface dS with a normal $\mathbf{n}$ and let us calculate the quantity of the liquid passing through dS in the direction $\mathbf{n}$ for time dt. It will be a volume swept by this surfaceshifted by $\mathbf{v}dt$ and it will be equal to $\mathbf{v} \cdot \mathbf{n}dSdt$. It is negative as $\mathbf{v} \cdot \mathbf{n} < 0$ which means that the liquid passed in the opposite direction is counted with the sign "$-$". Then the mass will be $\rho\mathbf{v} \cdot \mathbf{n}dSdt$ and if we consider now surface Σ which is not small we get the flux through Σ in the direction of $\mathbf{n}$

$$\iint_\Sigma \rho\mathbf{v} \cdot \mathbf{n}\, dS \times dt. \tag{14.2.1}$$

Consider now volume V bounded by surface Σ. Then it contains $\iiint_V \rho dV$ of the liquid; for time dt it is increased by

$$\iiint \rho_t dV \times dt;$$

on the other hand, the quantity of the liquid which arrived to V through Σ for time dt is given by (14.2.1) where $\mathbf{n}$ is an inner normal to Σ and by Gauss formula (14.1.2) it is equal

$$-\iiint_V \nabla\cdot(\rho\mathbf{v})\,dV\times dt.$$

Equalizing these two expressions and we get

$$\iiint_V \big(\rho_t+\nabla\cdot(\rho\mathbf{v}\big)\,dV=0.$$

Since it holds for any volume V we conclude that

$$\rho_t+\nabla\cdot(\rho\mathbf{v})=0. \tag{14.2.2}$$

It is called *continuity equation* and it means the *conservation law*.

Remark 14.2.1. (a) It may happen that there is a *source* with the density $f=f(\mathbf{x},t)$ which means that for time dt in the volume V the quantity of the liquid $\iiint_V f dV\times dt$ emerges (if $f<0$ we call it *sink*). Then (14.2.2) should be replaced by

$$\rho_t+\nabla\cdot(\rho\mathbf{v}=f. \tag{14.2.3}$$

(b) Since ρ is a mass per unit volume, $\mathbf{p}=\rho\mathbf{v}$ is a momentum per unit volume i.e. *density of momentum*.

(c) One can rewrite (14.2.2) as

$$\rho_t+\mathbf{v}\cdot\nabla\rho+\rho\nabla\cdot\mathbf{v}=\frac{d\rho}{dt}+\rho\nabla\cdot\mathbf{v}=0 \tag{14.2.4}$$

where

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla \tag{14.2.5}$$

is a *differentiation along the trajectory* $\frac{d\mathbf{x}}{dt}=\mathbf{v}$.

(d) Instead of ρ usual density (of the mass) it could be an (electric) charge; then $\mathbf{j}=\rho\mathbf{v}$ a *density of the current*. In fact ρ could be density of anything which is preserved and flows(rather than disappears in one place and emerges in the other).

(e) In fact, we can consider a mixture of particles; then $\mathbf{v}$ is not defined but $\mathbf{p}$ still is and equation

$$\rho_t + \nabla \cdot \mathbf{p} = 0 \tag{14.2.6}$$

makes sense.

Definition 14.2.1. Equation (14.2.6) which could be written as

$$\rho_t + \sum_j p_{j,x_j} = 0 \tag{14.2.7}$$

is called a *conservation law*.

Remark 14.2.2. Since not only scalar but also vector quantities could be conserved such conservation laws could be written in the form

$$p_{i,t} + \sum_j F_{ij,x_j} = 0, \qquad i = 1,2,3. \tag{14.2.8}$$

Here F_{ij} is a *tensor* (more precise meaning of this word is not important here but those who pecialize in mathematics or theoretical physics will learn it eventually). Using Einstein summation rule (which also indicates the nature of vector and tensors) one can rewrite (14.2.7) and (14.2.8) as

$$\rho_t + p^j_{x^j} = 0, \tag{14.2.9}$$

$$p_{i,t} + F^j_{i,x_j} = 0 \tag{14.2.10}$$

respectively.

14.2.2 Examples

Example 14.2.1. For wave equation

$$u_{tt} - c^2 \Delta u = 0 \tag{14.2.11}$$

the following conservation laws hold:

$$\partial_t\big(\frac{1}{2}(u_t^2 + c^2|\nabla u|^2)\big) + \nabla \cdot \big(-c^2 u_t \nabla u\big) = 0; \tag{14.2.12}$$

and

$$\partial_t\big(u_t u_{x_i}\big) + \sum_j \partial_{x_j}\big(\frac{1}{2}(c^2|\nabla u|^2 - u_t^2)\delta_{ij} - c^2 u_{x_i} u_{x_j}\big) = 0; \tag{14.2.13}$$

Example 14.2.2. For elasticity equation

$$\mathbf{u}_{tt} - \lambda\Delta\mathbf{u} - \mu\nabla(\nabla\cdot\mathbf{u}) = 0 \tag{14.2.14}$$

one can write conservation laws similar to (14.2.12) and (14.2.13) but they are too complicated; we mention only that in (14.2.12) $\frac{1}{2}(u_t^2 + c^2|\nabla u|^2)$ is replaced by

$$\frac{1}{2}(|\mathbf{u}_t|^2 + \lambda|\nabla\otimes\mathbf{u}|^2 + \mu|\nabla\cdot\mathbf{u}|^2) \tag{14.2.15}$$

with $|\nabla\otimes\mathbf{u}|^2\sum_{i,j} u_{j,x_i}^2$ as $\mathbf{u} = (u_1, u_2, u_3)$.

Example 14.2.3. For membrane equation

$$u_{tt} + \Delta^2 u = 0 \tag{14.2.16}$$

the following conservation law holds:

$$\partial_t\big(\frac{1}{2}(u_t^2 + \sum_{i,j}|u_{x_ix_j}^2|)\big) + \sum_k \partial_{x_k}\big(\sum_j u_{x_jx_jx_k}u_t - u_{x_jx_k}u_{x_jt}\big) = 0. \tag{14.2.17}$$

Example 14.2.4. For Maxwell equations

$$\begin{cases} \mathbf{E}_t = \nabla\times\mathbf{H}, \\ \mathbf{H}_t = -\nabla\times\mathbf{E}, \\ \nabla\cdot\mathbf{E} = \nabla\cdot\mathbf{H} = 0 \end{cases} \tag{14.2.18}$$

the following conservation laws hold:

$$\partial_t\big(\frac{1}{2}(\mathbf{E}^2 + \mathbf{H}^2)\big) + \nabla\cdot\big(\mathbf{E}\times\mathbf{H}\big) = 0; \tag{14.2.19}$$

(where $\mathbf{P} = \mathbf{E}\times\mathbf{H}$ is a Pointing vector) and if $\mathbf{P} = (P_1, P_2, P_3)$ then

$$\partial_t P_k + \sum_j \partial_{x_j}\big(\frac{1}{2}(\mathbf{E}^2 + \mathbf{H}^2)\delta_{jk} - \mathbf{E}_j\mathbf{E}_k - \mathbf{H}_j\mathbf{H}_k\big) = 0. \tag{14.2.20}$$

Example 14.2.5. For Schrödinger equations

$$-i\hbar\psi_t = \frac{\hbar^2}{2m}\Delta\psi - V(\mathbf{x})\psi \tag{14.2.21}$$

the following conservation law holds:

$$(\bar\psi s)_t + \nabla\cdot\mathrm{Re}(-\frac{i\hbar}{m}\bar\psi\nabla\psi) = 0 \tag{14.2.22}$$

14.3 Maxwell equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{14.3.1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{14.3.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{14.3.3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{14.3.4}$$

where ρ and $\mathbf{J}$ are density of charge and current, respectively. See in Wikipedia

Remark 14.3.1. (a) Equation (14.3.1) is a *Gauss' law*, (14.3.2) is a *Gauss' law for magnetism,* (14.3.3) is a *Faraday's law of induction.*

(b) Equations (14.3.1) and (14.3.4) imply *continuity equation*

$$\rho_t + \nabla \cdot \mathbf{J} = 0. \tag{14.3.5}$$

14.3.1 No charge and no current

In absence of the charge and current we get

$$\mathbf{E}_t = \mu^{-1}\varepsilon^{-1}\nabla \times \mathbf{B}, \qquad \mathbf{B}_t = -\nabla \times \mathbf{E}$$

and then

$$\mathbf{E}_{tt} = \mu^{-1}\varepsilon^{-1}\nabla \times \mathbf{B}_t = -\mu^{-1}\varepsilon^{-1}\nabla \times (\nabla \times \mathbf{E}) = \mu^{-1}\varepsilon^{-1}\big(\Delta \mathbf{E} - \nabla(\nabla \cdot \mathbf{E})\big) = \mu^{-1}\varepsilon^{-1}\Delta \mathbf{E};$$

so we get a wave equation

$$\mathbf{E}_{tt} = c^2 \Delta \mathbf{E} \tag{14.3.6}$$

with $c = 1/\sqrt{\mu\varepsilon}$.

14.3.2 Field in the conductor

On the other hand, if we have a *relatively slowly changing field in the conductor*, then $\rho = 0$, $\mathbf{J} = \sigma \mathbf{J}$ where σ is a conductivity and we can neglect

the last term in (14.3.4) and $\nabla \times \mathbf{B} = \mu\sigma\mathbf{E}$ and

$$\mathbf{B}_t = -\nabla \times \mathbf{E} = \mu^{-1}\sigma^{-1}\nabla \times (\nabla \times \mathbf{B}) = \\ \mu^{-1}\sigma^{-1}(\Delta\mathbf{B} - \nabla(\nabla \cdot \mathbf{B})) = \mu^{-1}\sigma^{-1}(\Delta\mathbf{B})$$

so $\mathbf{B}$ satisfies heat equation

$$\mathbf{B}_t = \mu_0^{-1}\sigma^{-1}\Delta\mathbf{B}. \tag{14.3.7}$$