APM 346 (2012) Home Assignment X

The purpose of this assignment is a survey of material in preparation to Term Test 1. Don't submit it–it will not be graded!

Problem 1

$$(t^2+1)xu_t + (x^2+1)tu_x = 0. \tag{1}$$

(a) Find the characteristic curves and sketch them in the (x,t) plane.

(b) Write the general solution.

(c) Where the solution is fully determined by the initial condition $u(x,0) = g(x)$.

(d) Solve equation (1) with the initial condition $u(x,0) = x^2$.

(e) Solve equation (1) with the initial condition $u(x,0) = x$.

Problem 2 Solve the wave equation with the following initial conditions

$$\begin{cases} u_{tt} - 25u_{xx} = e^{-x-t}, & -\infty < x < \infty \\ u(x,0) = xe^{-x}, \\ u_t(x,0) = e^{-x}. \end{cases} \tag{2}$$

Problem 3 Consider wave equation with boundary conditions:

$$\begin{cases} u_{tt} - c^2u_{xx} + \gamma u = 0, & 0 < x < L, \\ (u_x - \alpha u_t)(0,t) = 0, \\ (u_x - \beta u_t)(L,t) = 0, \end{cases} \tag{3}$$

with $\alpha, \beta \in \mathbb{C}$, $\gamma \in \mathbb{R}$ and $c > 0$.
Therefore u is a complex-valued function. Consider an energy $E(t)$ defined as

$$E(t) = \frac{1}{2}\int_0^L \left(|u_t|^2 + c^2|u_x|^2 + \gamma|u|^2\right)dx. \tag{4}$$

(a) Find conditions to α, β, γ such that $E(t)$ does not depend on t for any u satisfying (3);

(b) Find conditions to α, β, γ such that $E(t)$ is non-increasing function of t for any u satisfying (3);

Hint. Each end is independent and conditions are separate for α, and for β.

Problem 4 Find solutions u of IVP for a heat equation

$$\begin{cases} u_t - u_{xx} = 0 & -\infty < x < \infty, \\ u(0,x) = f(x) \end{cases} \tag{5}$$

where

(a) $f(x) = \theta(x) := \begin{cases} 0 & x < 0, \\ 1 & x > 0; \end{cases}$

(b) $f(x) = \begin{cases} 0 & |x| < 1, \\ 1 & |x| > 1. \end{cases}$

Hint. In (b) represent $f(x)$ as $\theta(x+1) - \theta(x-1)$.
Remark. Solution must be expressed through

$$\operatorname{erf}(z) = \sqrt{\frac{2}{\pi}} \int_0^z e^{-\frac{s^2}{2}}\, ds$$