Deadline Wednesday, December 5.

APM 346 (2012) Home Assignment 9

Problem 1 Find function u harmonic in
$x^2+y^2+z^2\le 1$ and coinciding with $g=z^3$ as $x^2+y^2+z^2=1$.
Hint. According to Subsection 30.1 solution must be a harmonic polynomial of degree 3 and it should depend only on $x^2+y^2+z^2$ and z (Explain why). The only way to achive it (and still coincide with g on
$x^2+y^2+z^2=1$) is to find

$$u=z^3+az(1-x^2-y^2-z^2)$$

with unknown coefficient A.

Problem 2 Apply method of descent described in Subsection 28.4 but to Laplace equation in $\mathbb{R}^2$ and starting from Coulomb potential in $3D$

$$U_3(x,y,z)=-\frac{1}{4\pi}\log\bigl(x^2+y^2+z^2\bigr)^{-\frac{1}{2}}, \tag{1}$$

derive logarithmic potential in $2D$

$$U_2(x,y,z)=\frac{1}{2\pi}\log\bigl(x^2+y^2\bigr)^{-\frac{1}{2}}, \tag{2}$$

Hint. You will need to calculate diverging integral $\int_0^\infty U_3(x,y,z)$. Instead consider $\int_0^N U_3(x,y,z)$, subtract constant (f.e. $\int_0^N U_3(1,0,z)$) and then tend $N\to\infty$.

Problem 3 Using method of reflection (studied earlier for different equations) construct Green function for

(a) Dirichlet problem

(b) Neumann problem

for Laplace equation in

1. half-plane
2. half-space

as we know that in the whole plane and space they are just potentials

$$\frac{1}{2\pi}\log\bigl((x_1-y_1)^2+(x_2-y_2)^2\bigr)^{\frac{1}{2}}, \tag{3}$$

$$\tag{4}$$

$$-\frac{1}{4\pi}\bigl((x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2\bigr)^{-\frac{1}{2}} \tag{5}$$

respectively.

Problem 4 Apply method of descent but now looking for stationary solution of $-\Delta u = f(x_1, x_2, x_3)$ instead of non-stationary solution of

$$u_{tt} - \Delta u = f(x_1, x_2, x_3),$$

$$u|_{t=0} = g(x_1, x_2, x_3),$$

$$u_t|_{t=0} = h(x_1, x_2, x_3)$$

start from Kirchhoff formula (28.12) and derive for $n = 3$ (26.10) with $G(x, y)$ equal to (5) here.