Deadline Wednesday, November 28.

APM 346 (2012) Home Assignment 8

This assignment is based on Lecture 26.

1. Problem 1
2. Problem 2
3. Problem 3
4. Problem 4

Problem 1

(a) Find the solutions that depend only on r of the equation

$$\Delta u := u_{xx} + u_{yy} = 0.$$

(b) Find the solutions that depend only on ρ of the equation

$$\Delta u := u_{xx} + u_{yy} + u_{zz} = 0.$$

(c) (bonus) In n-dimensional case prove that if $u = u(r)$ with $r = (x_1^2 + x_2^2 + \ldots + x_n^2)^{\frac{1}{2}}$ then

$$\Delta u = u_{rr} + \frac{n-1}{r} u_r = 0. \tag{1}$$

(d) (bonus) In n-dimensional case prove ($n \neq 2$) that $u = u(r)$ satisfies Laplace equation as $x \neq 0$ iff $u = Ar^{2-n} + B$.

Problem 2 Using the proof of mean value theorem (see Section 4 of Lecture 26) prove that if $\Delta u \geq 0$ in $B(y, r)$ then

(a) $u(y)$ does not exceed the mean value of u over the sphere $S(y, r)$ bounding this ball:

$$u(y) \leq \frac{1}{\sigma_n r^{n-1}} \int_{S(y,r)} u \, dS. \tag{2}$$

(b) $u(y)$ does not exceed the mean value of u over this ball $B(y,r)$:

$$u(y) \leq \frac{1}{\omega\_n r^n} \int_{B(y,r)} u\, dV. \tag{3}$$

(c) Formulate similar statements for functions satisfying $\Delta u \leq 0$ (in the next problem we refer to them as (a)' and (b)').

Definition.

(a) Functions having property (a) (or (b) does not matter) of the previous problem are called *subharmonic.*

(b) Functions having property (a)' (or (b)' does not matter) are called *superharmonic.*

Problem 3

(a) Using the proof of maximum principle (see Section 5 of Lecture 26) prove the maximum principle for subharmonic functions and minimum principle for superharmonic functions.

(b) Show that minimum principle for subharmonic functions and maximum principle for superharmonic functions do not hold (*Hint*: construct counterexamples with $f = f(r)$).

(c) Prove that if u, v, w are respectively harmonic, subharmonic and superharmonic functions in the bounded domain Ω, coinciding on its boundary ($u|_\Sigma = v|_\Sigma = w|_\Sigma$) then in $w \geq u \geq v$ in Ω.

Problem 4 (bonus) Using Newton screening theorem (see Section 6 of Lecture 26) prove that if Earth was a homogeneous solid ball then the gravity pull inside of it would be proportional to the distance to the center.