APM 346 (2012) Home Assignment 4

Deadline Wednesday, October 24; solution posting is allowed after 21:30 of this day

"Solve equation graphically" means that you plot a corresponding function and points $(z_n, 0)$ where it intersects with OX will give us all the frequencies $\omega_n = \omega(z_n)$.

"Simple solution" $u(x,t) = X(x)T(t)$.

You may assume that all eigenvalues are real (which is the case).

Problem 1 Justify examples 6–7 of Lecture 13: Consider eignevalue problem with Robin boundary conditions

$$\begin{aligned} &X'' + \lambda X = 0 && 0 < x < l, \\ &X'(0) = \alpha X(0), \quad X'(l) = -\beta X(l), \end{aligned}$$

$\alpha, \beta \in \mathbb{R}$.

(a) **Prove** that positive eigenvalues are $\lambda_n = \omega_n^2$ and the corresponding eigenfunctions are X_n where *omega*$_n$>0 are roots of

$$\tan(\omega l) = \frac{(\alpha+\beta)\omega}{\omega^2 - \alpha\beta};$$
$$X_n = \omega_n \cos(\omega_n x) + \alpha \sin(\omega_n x);$$

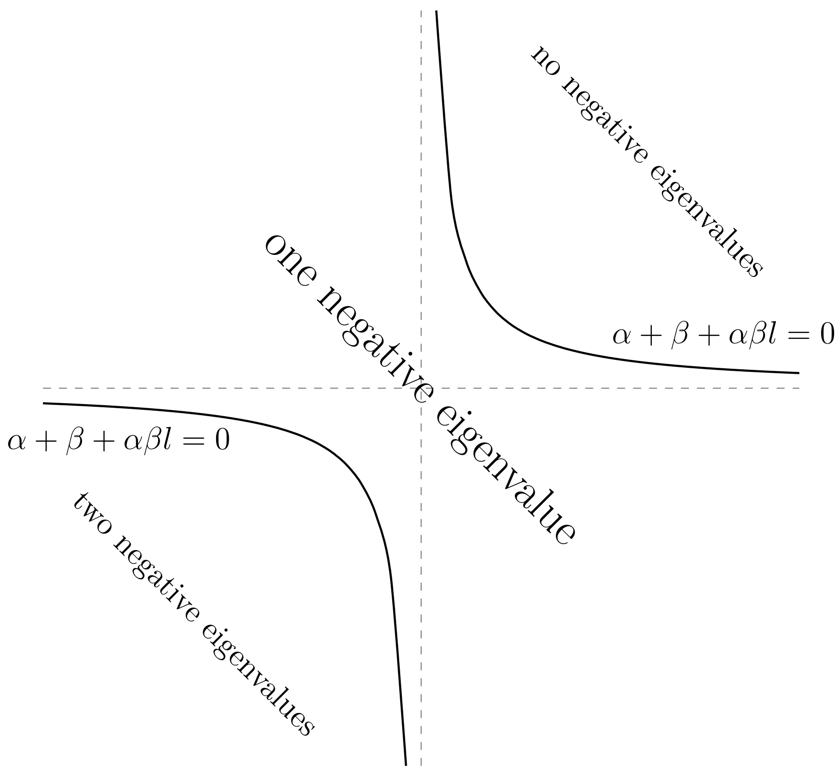
$(n = 1, 2, \ldots)$.

Solve this equation graphically.

(b) **Prove** that negative eigenvalues if there are any are $\lambda_n = -\gamma_n^2$ and the corresponding eigenfunctions are Y_n where $\gamma_n > 0$ are roots of

$$\tanh(\gamma l) = -\frac{(\alpha+\beta)\gamma}{\gamma^2 + \alpha\beta},$$
$$Y_n(x) = \gamma_n \cosh(\gamma_n x) + \alpha \sinh(\gamma_n x).$$

Solve this equation graphically.



(c) To investigate how many negative eigenvalues are, consider the threshold case of eigenvalue $\lambda = 0$: then $X = cx + d$ and plugging into b.c. we have $c = \alpha d$ and $c = -\beta(d + lc)$; this system has non-trivial solution $(c, d) \neq 0$ iff $\alpha + \beta + \alpha\beta l = 0$. This hyperbola divides (α, β)-plane into three zones: Check above arguments and **justify** that in the described zones there are really no, one, two negative eigenvalues respectively.

(d) **Prove** that eigenfunctions corresponding to different eigenvalues are orthogonal:

$$\int_0^l X_n(x)X_m(x), dx = 0 \text{as } \lambda_n \neq \lambda_m \tag{1}$$

where we consider now all eigenfunctions (no matter corresponding to positive or negative eigenvalues).

(e) **Bonus** Prove that eigenvalues are simple, i.e. all eigenfunctions corresponding to the same eigenvalue are proportional.

Problem 2 Oscillations of the beam which with both its ends having fixed positions and fixed directions (bricked into the walls: it is called "clamped") are described by an equation

$$u_{tt} + Ku_{xxxx} = 0, \qquad 0 < x < l,$$

with the boundary conditions

$$u(0,t) = u_x(0,t) = u(l,t) = u_x(l,t) = 0.$$

(a) **Find** equation describing frequencies and corresponding eigenfunctions (You may assume that all eigenvalues are real and positive).

(b) **Solve** this equation graphically.

(c) **Prove** that eigenfunctions corresponding to different eigenvalues are orthogonal (see (1)).

(d) **Bonus** Prove that eigenvalues are simple, i.e. all eigenfunctions corresponding to the same eigenvalue are proportional.

Problem 3 Consider wave equation with the Dirichlet boundary condition on the left and "weird" b.c. on the right:

$$\begin{aligned} &u_{tt} - c^2 u_{xx} = 0 && 0 < x < l,\\ &u(0,t) = 0, &&\\ &(u_x + i\alpha u_t)(l,t) = 0 && \end{aligned}$$

with $\alpha \in \mathbb{R}$.

(a) **Separate** variables;

(b) **Find** "weird" eigenvalue problem for ODE;

(c) **Solve** this problem;

(d) **Find** simple solution $u(x,t)$.