Deadline Monday, October 1, 9 pm

APM 346 (2012) Home Assignment 2

Problem 1 Consider equation with the initial conditions

$$u_{tt} - 4u_{xx} = 0, \qquad t > 0, x > vt, \qquad (1)$$
$$u|_{t=0} = e^{-x}, \qquad x > 0, \qquad (2)$$
$$u_t|_{t=0} = e^{-x}, \qquad x > 0, \qquad (3)$$

(A) Let $v = 3$. Find which of these conditions (a)-(c) at $x = vt$, $t \geq 0$ could be added to (1)-(3) so that the resulting problem would have a unique solution:

(a) None,
(b) $u|_{x=vt} = 0$ $(t \geq 0)$,
(c) $u|_{x=vt} = u_x|_{x=vt} = 0$ $(t \geq 0)$.

Solve the problem you deemed as a good one.

(B) Let $v = 1$. Find which of these conditions (a)-(c) at $x = vt$, $t \geq 0$ could be added to (1)-(3) so that the resulting problem would have a unique solution:

(a) None
(b) $u|_{x=vt} = 0$ $(t \geq 0)$,
(c) $u|_{x=vt} = u_x|_{x=vt} = 0$ $(t \geq 0)$.

Solve the problem you deemed as a good one.

(C) Let $v = -3$. Find which of these conditions (a)-(c) at $x = vt$, $t \geq 0$ could be added to (1)-(3) so that the resulting problem would have a unique solution:

(a) None
(b) $u|_{x=vt} = 0$ $(t \geq 0)$,
(c) $u|_{x=vt} = u_x|_{x=vt} = 0$ $(t \geq 0)$.

Solve the problem you deemed as a good one.

Problem 2 A spherical wave is a solution of the three-dimensional wave equation of the form $u(r,t)$, where r is the distance to the origin (the spherical coordinate). The wave equation takes the form

$$u_{tt} = c^2\bigl(u_{rr} + \frac{2}{r}u_r\bigr) \qquad (\text{"spherical wave equation"}). \tag{4}$$

(a) Change variables $v = ru$ to get the equation for v: $v_{tt} = c^2 v_{rr}$.

(b) Solve for v using

$$v = f(x+ct) + g(x-ct) \tag{5}$$

and thereby solve the spherical wave equation.

(c) Use

$$v(r,t) = \frac{1}{2}\bigl[\phi(r+ct) + \phi(r-ct)\bigr] + \frac{1}{2c}\int_{r-ct}^{r+ct} \psi(s)\,ds \tag{6}$$

with $\phi(r) = v(r,0)$, $\psi(r) = v_t(r,0)$ to solve it with the initial conditions $u(r,0) = \Phi(r)$, $u_t(r,0) = \Psi(r)$.

(d) Find the general form of solution continuous as $r = 0$.

Problem 3

By method of continuation combined with D'Alembert formula solve each of the following four problems (a)–(d).

$$\begin{cases} u_{tt} - 4u_{xx} = 0, & x \geq 0, \\ u|_{t=0} = 0, & x \geq 0, \\ u_t|_{t=0} = 1, & x \geq 0, \\ u|_{x=0} = 0, & t \geq 0. \end{cases} \tag{7}$$

$$\begin{cases} u_{tt} - 4u_{xx} = 0, & x \geq 0, \\ u|_{t=0} = 0, & x \geq 0, \\ u_t|_{t=0} = 1, & x \geq 0, \\ u_x|_{x=0} = 0, & t \geq 0. \end{cases} \tag{8}$$

$$\begin{cases} u_{tt} - 4u_{xx} = 0, & x \geq 0, \\ u|_{t=0} = 0, & x \geq 0, \\ u_t|_{t=0} = x, & x \geq 0, \\ u|_{x=0} = 0, & t \geq 0. \end{cases} \tag{9}$$

$$\begin{cases} u_{tt} - 4u_{xx} = 0, & x \geq 0, \\ u|_{t=0} = 0, & x \geq 0, \\ u_t|_{t=0} = x, & x \geq 0, \\ u_x|_{x=0} = 0, & t \geq 0. \end{cases} \tag{10}$$

Problem 4

For a solution $u(x,t)$ of the wave equation with $\rho = T = 1 \implies c = 1$, the *energy density* is defined as $e = \frac{1}{2}\left(u_t^2 + u_x^2\right)$ and the *momentum density* as $p = u_t u_x$.

1. Show that
$$\frac{\partial e}{\partial t} = \frac{\partial p}{\partial x} \qquad \text{and} \qquad \frac{\partial p}{\partial t} = \frac{\partial e}{\partial x}. \tag{11}$$

2. Show that both $e(x,t)$ and $p(x,t)$ also satisfy the wave equation.