1. (20 pts) Consider the eigenvalue problem

$$x^2u'' + xu' + \lambda u = 0, \qquad x \in (1, 2),$$
$$u(1) = 0; \;\; u(2) = 0.$$

Assume $\lambda > 0$. Find all the eigenvalues and the corresponding eigenfunctions. (*Hint: look for solutions in the form* u^m).

2. (20 pts) Solve the eigenvalue problem

$$X'' + \lambda X = 0, \qquad x \in (0, l),$$
$$X(0) = 0; \quad X'(l) + \alpha X(l) = 0.$$

(a) For which values of α is $\lambda = 0$ an eigenvalue?

(b) For which values of α are there negative eigenvalues? How many?

(c) Find the positive eigenvalues.

(d) How do positive eigenvalues depend on α?

Note If the equations for eigenvalues cannot be solved explicitly, indicate in which interval they can be found, and write the corresponding eigenfunctions.

3. (20 pts) Consider the diffusion equation

$$u_t - ku_{xx} = 0 \qquad \text{for} \quad x \in (0, \pi)$$

with the boundary conditions $u_x(0, t) = 0$ and $u(\pi, t) = 0$ and the initial condition $u(x, 0) = x$.

(a) Write the associated eigenvalue problem.

(b) Find all eigenvalues and corresponding eigenfunctions.

(c) Show that the eigenfunctions associated to 2 different eigenvalues are orthogonal.

(d) Write the solution in the form of a series expansion.

(e) Write a formula for the coefficients of the series expansion. (You do *not* have to compute them.)

(Continued)

4. (20 pts) Let the function f be defined for $0 \leq x \leq l$ by

$$f(x) = \frac{l}{2} - x.$$

(a) Plot the function f.

(b) Write the Cosine Fourier series of the function $f(x)$, and compute the coefficients.

(c) Write the definition of the even extension $\tilde{f}$ of the function f in the interval $[-l, l]$ and represent it graphically.

(d) Let $F(x)$ be the periodic extension with period $2l$ of $\tilde{f}(x)$. Represent it graphically.

(e) Write the Full Fourier series of $F(x)$, and compare it with the series in Part b).

5. (20 pts) Consider the eigenvalue problem with periodic boundary conditions:

$$X'' + \lambda X = 0, \qquad x \in (-\pi, \pi);$$
$$X(-\pi) = X(\pi); \qquad X'(-\pi) = X'(\pi).$$

(a) Without computing the eigenvalues and eigenfunctions, prove that ll eigenvalues are strictly positive.

(b) Without computing the eigenvalues and eigenfunctions, prove that two eigenfunctions corresponding to two different eigenvalues are orthogonal. Explain the concept of *orthogonality of two functions*.