$N(a'_1\, a'_2\, a'_3\, a'_4) = N - \sum N(a_i) + \sum N(a_i a_j) - \sum N(a_i a_j a_k)$

Here $N = \binom{21}{3}$

$N(a_i) = \binom{4+10-1}{10}$ (Just put 8 balls in box i and fill arbitrarily for the remaining 10)

$N(a_i\, a_j) = \binom{4+2-1}{2} = \binom{5}{2}$ (Put 8 balls in both i and j, fill arbitrarily from here on)

$N(a_i\, a_j\, a_k) = 0$ $(3 \times 8 = 24 > 18)$

$\therefore N(a'_1\, a'_2\, a'_3\, a'_4) = \binom{21}{3} - 4\binom{13}{10} + \binom{4}{2}\binom{5}{2} = 246$

Exercise 2 Let A,B be finite sets, $|A| = n$, $|B| = k$. Find the number of onto functions $f: A \to B$

Solution: If $n < k$, the answer is 0. Assume $n \geq k$. The number of functions without restriction is k^n.

For $1 \leq i \leq k$, let a_i denote the property that a function does not have the i^{th} elt of B in its range. Then $N(a_i) = (k - 1)^n$.
$N(a'_i)$ counts the functions which do have the i^{th} element of B in their range, and

$N(a'_1\, a'_2 \cdots a'_k)$ counts those functions which do have the 1st, 2nd, $\cdots$, k^{th} elt of B in their range, i.e. the onto functions.

Notice that $N(a_i\, a_j) = (k - 2)^n$ while $N(a_{i_1}\, a_{i_2}, \cdots, a_{i_r}) = (k - r)^n$

Also, the number of r-tuples is $\binom{k}{r}$.

Thus,
$N(a'_1\, a'_2 \cdots a'_k) = k^n - \binom{k}{1}(k - 1)^n + \binom{k}{2}(k - 2)^n - \cdots$

$$= \sum_{r=0}^{k} (-1)^r \binom{k}{r} (k-r)^n$$

$= k!(S(n,k)) \equiv k!$(Stirling # of 2nd Kind).

Exercise: <u>The number of derangements D_n</u>

Let a_i be the property that i is in its natural position in the permutation π, i.e. $\pi(i) = i$.
Then D_n is the no. of perms with none of the properties a_i.

Then $N = n!$, $N(a_i) = (n - 1)!$, $N(a_i\, a_j) = n - 2)!$

$N\left(a_{i_1}\, a_{i_2}, \cdots, a_{i_r}\right) = (n- r)!$ The number of r-tuples is $\binom{k}{r}$. Thus,

$D_n = N(a'_1\, a'_2 \cdots a'_n) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} (n-r)!$

Exercise: (<u>Euler Phi Function</u>) Let $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$

Denote by $\phi(n)$ the number of integers

from 1 to n (incl.) rel. prime to n. Find $\phi(n)$.

Let a_i be the property that an integer in [n] is divisible by p_i. Then $\phi(n) = N(a'_1\, a'_2 \cdots a'_k)$. Here

$N = n$, $N(a_i) = \dfrac{n}{p_i}$, $N(a_i a_j) = \dfrac{n}{p_i p_j}$, etc.

Thus, $\phi(n) = n - \sum_i \dfrac{n}{p_i} + \sum_{i_1 \neq j_2} \dfrac{n}{p_{i_1} p_{i_2}} - \sum \dfrac{n}{p_{i_1} p_{i_2} p_{i_3}} \cdots$

$$= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right)$$

<u>Generalization</u>
Suppose we want the number of objects which have <u>exactl</u>y m of the properties (any m of the r properties will do, $m \in [0,r]$. Let e_m be this number. Let s_m be the number of objects

$$s_m = \sum N\left(a_{i_1}\, a_{i_2}, \cdots, a_{i_m}\right)$$

where the sum is taken over all choices of m distinct properties $\left(a_{i_1}, \cdots, a_{i_m}\right)$.

<u>N.B.</u> s_m counts elements more than once. Every element counted by s_m has <u>at least</u> m properties but those with more than m properties get counted many times. For example, if an object has the properties $a_{1,}a_2, \cdots, a_{m+1}$, it gets counted in $N(a_1\, a_2 \cdots a_m)$,

$N(\hat{a}_1\, a_2\, a_3 \cdots a_{m+1})$, $N(a_1\, \hat{a}_2\, a_3 \cdots a_{m+1})$ etc.
↑ ↑

"a_1 missing" "a_2 missing"

<u>Theorem</u>:

$$e_m = s_m - \binom{m+1}{1} s_{m+1} + \binom{m+2}{2} s_{m+2} - \cdots$$

$$\cdots + (-1)^p \binom{m+p}{p} s_{m+p} + \cdots + \cdots + (-1)^{r-m} \binom{m+r-m}{r-m} s_r$$

If $s_0 = N$, this yields the inclusion - exclusion formula for m = 0.
<u>Proof</u>: Let's consider any object x.

1) If it has fewer than m of the properties it contributes 0 to LHS and 0 to each of $s_m, s_{m+1}, \cdots$. So 0 to RHS.

2) If x has <u>exactly</u> m properties it's counted once in e_m, once in s_m, 0 times in $s_{m+1}, \cdots s_r$ so we're OK.

3) If x has t properties, $m < t \le r$, then x contributes

   0 to LHS, but is counted $\binom{t}{m}$ times

   in s_m, $\binom{t}{m+1}$ times in $s_{m+1}, \ldots, \binom{t}{t}$ times

in s_t and 0 times in $s_{m+1}, \ldots, s_r$.

Thus, on RHS x is counted.

$$\binom{t}{m} - \binom{m+1}{1}\binom{t}{m+1} + \binom{m+2}{2}\binom{t}{m+2} - \cdots$$

$$\cdots + (-1)^{t-m} \binom{m+(t-m)}{t-m} \binom{t}{t} \text{ times.}$$

For $0 \le k \le t - m$, use

$$\binom{m+k}{k}\binom{t}{m+k} = \frac{(m+k)!}{k!\,m!}\,\frac{t!}{(m+k)!(t-m-k)!}$$

$$= \frac{t!}{m!}\,\frac{1}{k!(t-m-k)!}$$

$$= \binom{t}{m}\binom{t-m}{k}$$

to show that above sum is 0.

Corollary Let L_m be the number of elements. that satisfy at least m of the r properties. Then

$$L_m = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{n-1} S_{m+2} - \dots + (-1)^{r-m} \binom{r-1}{m-1} S_r$$

Exercise: In how many ways can one arrange the letters in CORRESPONDENTS so

a) no pair of identical letters is consecutive
b) exactly 2 pairs of identical letters are consecutive
c) at least 3 pairs of identical letters are consecutive

Solution Let a_i be the property that 2 identical letters i are consecutive. Then we can treat these as a single unit in any arrangement of letters.
Five pairs (O,R,E,S,N), four singles (C,P,D,T).

$N(a_i) = 13!/2^4$ $\quad N = \dfrac{14!}{2^5}$

$N(a_i\, a_j) = 12!/2^3$

$N(a_i\, a_j\, a_k = 11!/2^2$ $\quad s_2 = \binom{5}{2}\dfrac{12!}{2^3}$

$N(\underbrace{\quad}_{4}) = 10!/2$

$N(\underbrace{\quad}_{5}) = 9!$

a) $N(a'_0\, a'_R\, a'_E\, a'_S\, a'_N) = \dfrac{14!}{2^5} - \binom{5}{1}\dfrac{13!}{2^4} + \binom{5}{2}\dfrac{12!}{2^3} - \binom{5}{3}\dfrac{11!}{2^2} + \binom{5}{4}\dfrac{10!}{2} - \binom{5}{5} 9$

MORE 'DERANGEMENTS"

Suppose the perm on [n] has exactly k fixed points. Define properties a_i as before, $N(a_i) = (n - 1)!$ and so on. Then

$$e_k = S_k - \binom{k+1}{1} S_{k+1} + \binom{k+2}{2} S_{k+2} - \ldots (-1)^{n-k}\binom{n}{n-k} S_k$$

$$= \binom{n}{k}(n-k)! - \binom{k+1}{1}\binom{n}{k+1}(n-k-1)! + \binom{k+2}{2}\binom{n}{k+2}(n-k-2)! \ldots + (-1)^{n-k}\binom{n}{n-k} 0! \binom{n}{n}$$

$$= \binom{n}{k}\sum_{j \geq 0} (-1)^j \binom{n-k}{j}(n-k-j)!$$

$$= \binom{n}{k} D_{n-k}$$

(Of course! Just choose the k fixed points in $\binom{n}{k}$ ways, "derange" all the other points in D_{n-k} ways!)

Let $E(x) = \sum e_m x^m$.

$$E(x) = (S_0 - S_1 + S_2 - \ldots + (-1)^r S_r) + \left[S_1 - \binom{2}{1} S_2 + \binom{3}{2} S_3 - \ldots + (-1)^{r-1}\binom{r}{r-1} S_r\right]x$$

$$+ \left[S_2 - \binom{3}{1} S_3 + \binom{4}{2} S_4 - \ldots + (-1)^{r-2}\binom{r}{r-2} S_r\right]x^2 + \ldots$$

$$\left[S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} - \ldots + (-1)^{r-m}\binom{r}{r-m} S_r\right]x^r + \ldots + S_r x^r$$

$$= S_0 + S_1(x - 1) + S_2\left(x^2 - \binom{2}{1}x + 1\right) + S_3\left(x^3 - \binom{3}{1}x^2 + \binom{3}{2}x - 1\right) + \ldots$$

$$+ S\left(x^m - \binom{m}{1}x^{m-1} + \binom{m}{2}x^{m-2} \ldots + (-1)^{m-1}\binom{m}{m-1}x + (-1)^m\right) + \ldots$$

$$S_r\left(x^r - \binom{r}{1}x^{r-1} + \binom{r}{2}x^{r-2} \ldots + (-1)^{r-1}\binom{r}{r-1}x + (-1)^r\right)$$

$$\therefore E(x) = \sum_{m=0}^{r} S_m (x - 1)^m$$!Wow - looks simple!

$$\therefore \sum_j e_{2j} = \frac{1}{2}(E(1) - E(-1)), \sum_j e_{2j+1} - \frac{1}{2}(E(1) = E(-1))$$

Exercise: Is it true that

$$S_m = \sum_{j=m}^{r} \binom{j}{m} e_j \quad ?$$

How might this relate, if true, to the earlier formula for e_m ?

Exercise: Show that if (2k + 1) objects are place in k drawers, at least one drawer will contain 3 or more objects.

Generalize the above to (mk + 1) objects in k drawers.

Exercise: Suppose a circle is divided into 200 congruent sectors and 100 are coloured red, other 100 blue. A smaller circle is also so divided and coloured (i.e. 100 sectors red, 100 blue) and placed concentrically on the larger circle. Prove that no matter how the 100 red sectors are chosen for each circle, the smaller circle can be rotated so that at least 100 sectors of the two circles match in colour.

(Hint: How many matches do you get in total as the smaller circle is rotated through 360° while the larger circle remains fixed?)

DIRICHLET DRAWER (PIGEONHOLE) PRINCIPLE

"k + 1 pigeons in k pigeonholes

⇒ at least one pigeonhole has 2 or more pigeons"
Peter Gustuv Lejeune Dirichlet (1805 - 1859)

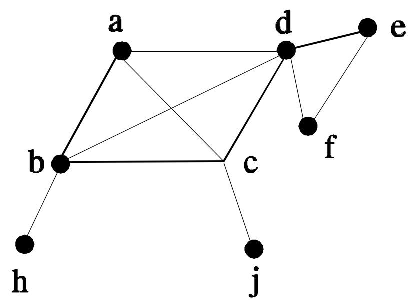
Exercise 1: Select n + 1 numbers from {1, 2, ... , 2n}
Then 2 are relatively prime.

Solution: 2 must be consecutive, hence this pair are relatively prime.

Exercise 2: Let G be a graph. A clique in G is a complete generated subgraph in G, i.e. a collection of vertices in G where each pair of vertices is joined by an edge.

Let $\omega(G)$ be the size (# of vertices) of the largest clique in G.
Then $\chi(G) \geq \omega(G)$, where $\chi(G)$ is the chromatic number of G.
(Vertices of clique $\equiv$ pigeons, colours $\equiv$ holes)

not {a,b,c,d,e,f}



{a, b, c, d}

{d, e, f}

{b, h}

Generalization 1 If n pigeons are placed into k pigeonholes, then at least one hole contains more than $\left\lfloor \frac{m-1}{k} \right\rfloor$ pigeons. $\lfloor x \rfloor \equiv$ greatest integer $\leq$ x.

Proof: If not, then there are at most $k \left\lfloor \frac{m-1}{k} \right\rfloor$ $\leq$ m - 1 < m pigeons, contradiction

Corollary: Given any set of numbers, there is always a number whose value is $\geq$ (also $\leq$) the average value of the numbers in the set.

Application: G a graph, W a set of vertices. W is an independent set of G if no two vertices in W are joined by an edge. Let $\alpha(G)$ be the size of the largest independent set in G (independence no.)
If G is coloured with $\chi(G)$ colours, then each subset containing all vertices of a fixed colour is an independent set, and V(G) is partitioned into $\chi(G)$ independent (disjoint) subsets.

Average size of each subset is $\frac{|V(G)|}{\chi(G)}$ and since
$\alpha(G)$ is the size of largest independent set, $\alpha(G) \geq \frac{|V(G)|}{\chi(G)}$ or $\chi(G)\alpha(G) \geq |V(G)|$

Theorem (Erdos, Szekeres) Given a sequence of $n^2 + 1$ distinct integers, there is an increasing subsequence of length n + 1 or a decreasing subsequence of length n + 1.

Example:
($n^2 = 16$: 4 3 2 1 8 7 6 5 12 11 10 9 16 15 14 13)

Proof: Denote the sequence by $x_1, x_2, \ldots, x_{n^2+1}$. Let t_i be the length of the largest increasing subsequence beginning with x_i. If any $t_i \geq n + 1$, we're done. Thus, assume $1 \leq t_i \leq n$. Then we have

($n^2 + 1$) values t_i all between 1 and n so that at least

$$\left\lfloor \frac{(n^2+1)-1}{n} \right\rfloor + 1 = n + 1$$

of the increasing subsequences have the <u>same</u> <u>length</u>. It follows that the x_i 's associated with these subsequences (that is, the initial term of each one) form a decreasing subsequence (of length n + 1).

To see this, note that if $t_i = t_j$ and $i < j$ then $x_i > x_j$. (For if not then $x_i \leq x_j$ and $i < j$ so the subsequence starting with x_i and then x_j and all the $t_j = t_i$ elements associated with x_j as initial point is an increasing subsequence of length $t_j + 1$, contradiction since t_i was supposed to be largest increasing subsequence starting at i.) Now, all the x_i form a decreasing subsequence as required.

<u>Theorem</u> Suppose $p_1, p_2, \ldots, p_k$ are positive integers. If $p_1 + \ldots + p_k - (k - 1)$ pigeons are put into k holes then either the the 1st hole contains $\geq p_1$, the 2nd $\geq p_2 \ldots$, the kth $\geq p_k$.

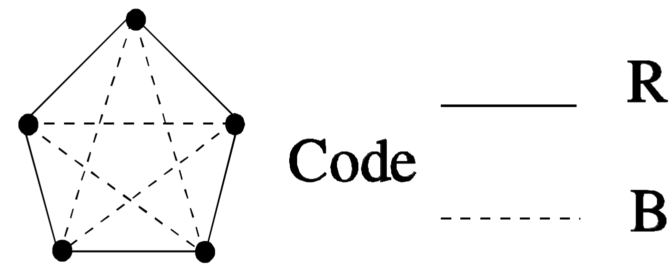
<u>Proof</u> If not, the # of pigeons is at most

$$\sum_{i=1}^{k} (p_i - 1) = \sum_{i=1}^{k} p_i - k \text{ which is 1 too small!}$$

Recall the game of SIM - remember that there was always a winner (form a red or blue $\triangle$)

<u>Theorem</u> In a group of 6 people there are either 3 mutual friends or 3 mutual strangers. [Equiv., colour the edges of the complete graph on 6 points with 2 colours R and B. Then there is a R or B $\triangle$.]

N.B. 6 is the fewest for which this is true

5: 

Suppose S is any set of 6 elements. Let T be the two

elements of R, $|T| = \binom{|S|}{2}$ Let $T = X \cup Y$, $X \cap Y = \emptyset$. Then

a) there is a 3 element subset of S all of whose 2 element subsets are in X

or: b) there is a 3 element subset of S all whose 2 elt. subsets are in Y.

MORE GENERAL

Suppose p, q $\geq$ 2, integers.

Divide S as above, assume $|S| = N$.
Then N has the (p,q) Ramsey property (Frank Plumpton Ramsey 1903 - 1930) if:

a) there is a p element subset of S all whose 2 element subsets are in X.

or:
b) there is a of element subset of S all of whose 2 element subsets are in Y.

Exercise: N = 6 has the (3,3) Ramsey property
N = 5 does not have the (3,3) Ramsey property.

N.B. If N is (p,q) - R and M > N then M is
(p,q) - R.

Proof Let $|S| = M$, $S_1 \subseteq S$, $|S_1| = N$

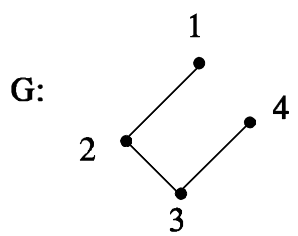
Divide the 2 elements subsets of S_1 into X,Y.
Then there is a p element subset of S_1 whose 2 element subsets are in X or a q element subset of S_1 whose 2 element subsets are in Y.
Put all the remaining 2 element subsets of S arbitrarily in X and Y.
Since any p element or q element subset of S_1 is also such a subset S of the desired property described above for S_1 follows for S. Thus, M is (p,q) - R.

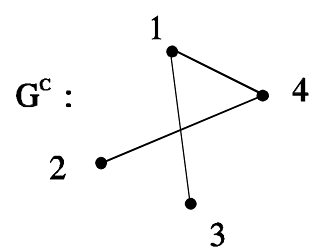
Theorem (Ramsey): If p ,q $\geq$ 2 then $\exists$ N $\ni$ N is (p,q) - R.

The smallest such number N is denoted by R(p,q), called a Ramsey number .

Finding Ramsey Numbers

1. R(3,3) = 6, since 6 is (3,3) - R while 5 (thus all smaller) is not.





If G = (V,E), then $G^c = (V,E^c)$ (complement) has same vertex set as G and all the edges in G^c are edges not in G (complement of E w.r.t. complete graph on vertex set V).

Theorem: N is (p,q) - R $\Leftrightarrow$ for every graph G with N vertices, either G has a complete subgraph of p vertices (p-gon) or G^c has a complete q-gon. (CLIQUE)
S $\equiv$ vertex set of graph G.
2 element subsets of S $\equiv$ edges in G.
(Clique G^c $\equiv$Independent set in G.)

Theorem: i) R (p,2) = p
ii) R (2,q) = q.

Proof: i) R(p,2) $\geq$ p (if not, clearly no p element subset at all, and could put all the two element subsets into one set X, haveY = $\varnothing$)
Every graph of p vertices, either it's complete or its complement contains at least one edge. Thus, R(p,2) $\leq$ p. $\therefore$ R(p,2) = p.

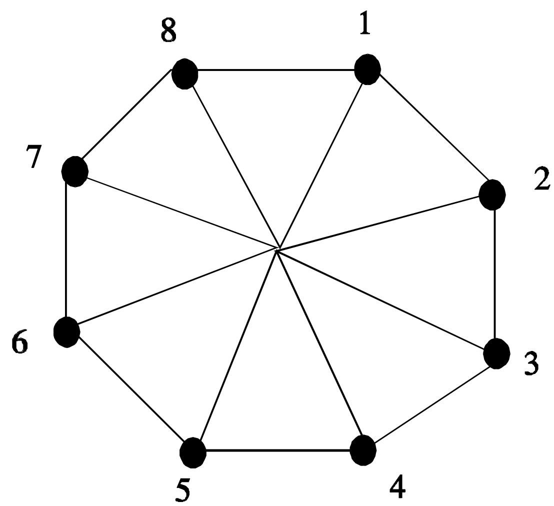
(ii) In general R(p,q) = R(q,p) by symmetry of the definition of R(p,q).

Theorem: N is (p,q) - R ⇔ for every graph G of N vertices, either G has a clique of p vertices or G has an independent set with q vertices.

Theorem: N is (p,q) - R ⇔ whenever we colour the edges of K_N either R or B then either K_N has a complete R p-gon or a complete B q-gon.

Proof: Given any edge colouring of K_N, let G be the graph whose vertices are those of K_N, and whose edges are R. Then by earlier result, G has a clique of p vertices (R p-gon) or an independent set of q vertices (B q-gon).

N.B. R(p,q) is HARD to determine.



This graph has 8 vertices

It has no △, no independent set of 4 vertices

∴ R(3,4) > 8 so R (3,4) ≥ 9
In fact, R (3,4) = 9
Take regular 13-gon to show R(3,5) ≥ 14
Take regular 27-gon to show R(4,4) ≥ 18
In both cases equality holds.

Theorem: R(p,q) ≤ R(p,q - 1) + R (p - 1, q)

If p ≥ 2, q ≥ 2 then

$R(p,q) \leq \binom{p+q-2}{p-1}$

From these it follows that

$$R(3,4) \leq R(3,3) + R(2,4) = 6 + 4 = 10$$

In fact, can show that when R(p,q - 1) and
R(p - 1, q) are both even, and p,q ≥ 3 then
R(p,q) ≤ R(p,q - 1) + R(p - 1, q) - 1
∴ R(3,4) ≤ 9
But we showed R(3,4) ≥ 9, hence R(3,4) = 9.
Similarly, R(3,5) ≤ R(3,4) + R(2,5) = 9 + 5 =14

<u>Generalizations</u>

1) Consider r element subsets of N element set (rather than just r = 2):

p ≥ r, q ≥ r , r ≥ 1

N is (p,q;r) - R ⇔ :

a) given any set S of N elements, if we divide the r element subsets of S into two collections X, Y then either :

- p element subset of S, all of whose r-element subsets are in X
- q element subset of S, all whose r-element subsets are in Y

N exists, least N is R (p,q;r)
e.g. R(3,3;2) = R(3,3) = 6
R(p,q;r) = R(p,q)

<u>Theorem</u> R(p,q;1) = p + q - 1

Proof: Let N = p + q - 1. Then N is (p,q;1) - R. To see this, suppose |S| = N. Divide the 1-element subsets of S into two classes X and Y. Then |X| ≥ p or |Y| ≥ q by the pigeonhole principle (actually a corollary), so one of the 2 conditions of (p,q;1) - R hold. Thus, R(p,q;1) ≤ p + q - 1. To see that R(p,q,1) - R. To see that R(p,q;1) ≥ p + q - 1, it is enough to show that
p + q - 2 is not (p,q;1) - R. To see this, just split the 1 element subsets of S into two classes X,Y with |X| = p - 1, |Y| = q - 1.

<u>N.B</u>. This is why Ramsey is thought of as generalized P.P.