Non-Linear DFE

Exercise 1 $Y_nY_{n+2} = Y^2_{n+1}$ $Y_0 = 1$, $Y_1 = 2$

Follows that $Y_n \neq 0\ \forall n$.

$$\therefore \quad \frac{Y_{n+2}}{Y_{n+1}} = \frac{Y_{n+1}}{Y_n}$$

Let $W_n = \dfrac{Y_{n+1}}{Y_n}$. Then $W_{n+1} = W_n$, $W_0 = 2$

$\therefore W_n = 2 \Rightarrow Y_{n+1} = 2Y_n \Rightarrow Y_n = 2^n$ $(Y_0 = 1)$

N.B. Could also linearize with logs.

Exercise 2 $Y^2_{n+2} + 3Y^2_{n+1} - 4Y^2_n = 0$

Set $W_n = Y_n^2$

$$W_{n+2} + 3W_{n+1} - 4W_n = 0$$

$W_n = c_1(-4)^n + c_2 \Rightarrow Y_n = \sqrt{c_1(-4)^n + c_2}$

Solving Recurrences Using Generating Fn

*Basic Idea: G.F. is formal power series with the coeff of interest to us related by the recursion. Using formal manipulations this leads to a formal expression (hopefully a recognizable closed form) from which the coefficients of the FPS can be determined.

Exercise: $a_{n+1} = 2a_n + 1$ $a_0 = 0$, $a_1 = 1$

Define: $A(x) = \sum_{n \geq 0} a_n x^n$ ("FPS")

$\therefore A(x) = a_0 + \sum_{n \geq 1} a_n x^n$ "Formal manipulation"

$$= a_0 + \sum_{n \geq 0} a_{n+1} x^{n+1}$$

$$= \sum_{n \geq 0} (2a_n + 1) x^{n+1}$$

$$= 2\sum_{n \geq 0} a_n x^{n+1} + \sum_{n \geq 0} x^{n+1}$$

$\therefore$ A(x) = 2x $\sum_{n\geq 0} a_n x^n + \frac{x}{1-x} = 2x\,A(x) + \frac{x}{1-x}$.

$\therefore$ A(x) = $\frac{x}{(1-x)(1-2x)}$ "Closed Form"

$$= x \sum_{n\geq 0} x^n \cdot \sum_{n\geq 0} 2^n x^n.$$

$$= x \sum_{n\geq 0} \left(\sum_{j=0}^{n} 2^j \right) x^n$$

$$\therefore \sum_{n\geq 0} a_n x^n = \sum_{n\geq 0} \left(2^{n+1} - 1\right) x^{n+1}$$

Comparing coefficients we see that $a_n = 2^n - 1$

N.B. If 2 FPS are equal, they are the same coefficient by coefficient.

Exercise: $F_{n+2} = F_{n+1} + F_n \; n \geq 0$, $F_0 = F_1 = 1$

Let F(x) = $\sum_{n\geq 0} F_n x^n$ "FPS"

$$= F_0 + F_1 x + \sum_{n\geq 2} F_n x^n$$

$$= 1 + x + \sum_{n\geq 0} F_{n+2} x^{n+2}$$

$$= 1 + x + x\sum_{n\geq 0} F_{n+1} x^{n+1} + x^2 \sum_{n\geq 0} F_n x^n$$

$$= 1 + x + \sum_{n\geq 0} \left(F_{n+1} + F_n\right) x^{n+2}$$

$$= 1 + x + x\,[F(x) - F_0] + x^2 F(x)$$

$\therefore$ F(x) = 1 + x F(x) + x^2F(x)

$\therefore$ F(x) = $\frac{1}{1 - x - x^2}$

How can we determine the coeff of the FPS for F(x) from this?

Use Partial Fraction Expansion of RHS!

$1 - x - x^2 = (1 - x R_+)(1 - x R_-)$ $\qquad R_\pm = \frac{1 \pm \sqrt{5}}{2}$

$$\therefore F(x) = \left(\frac{1}{R_+ - R_-}\right)\left\{\frac{R_+}{1 - xR_+} - \frac{R_-}{1 - xR_-}\right\}$$

$$= \frac{1}{\sqrt{5}}\left\{\sum_{n \ge 0}\left(R_+^{n+1} - R_-^{n+1}\right)x^n\right\}$$

$$\therefore F_n = \frac{\sqrt{1}}{5}\left(R_+^{n+1} - R_-^{n+1}\right)$$

Sometimes we use e.g.f. just as successfully.

$$F(x) = \sum_n \frac{F_n}{n!}x^n$$

$$= 1 + x + \sum_{n \ge 0}\frac{F_{n+2}}{(n+2)!}x^{n+2}$$

$$= 1 + x + \sum_{n \ge 0}\frac{F_{n+1} + F_n}{(n+2)!}x^{n+2}$$

$$\therefore F'(x) = 1 + \sum_{n \ge 0}\frac{F_{n+1} + F_n}{(n+1)!}x^{n+1}$$

$$= 1 + \sum_{n \ge 0}\frac{F_{n+1}}{(n+1)!}x^{n+1} + \sum_{n \ge 0}\frac{F_n}{(n+1)!}x^{n+1}$$

$$= 1 + (F(x) - 1) + \sum_{n \ge 0}\frac{F_n}{(n+1)!}x^{n+1}$$

$$\therefore F''(x) = F'(x) + \sum_{n \ge 0}\frac{F_n}{n!}x^n$$

$$= F'(x) + F(x)$$

This can be solved using ODE approach for FPS (same as if F(x) a real (or complex) function).

$F(x) = c_1 e^{R_+ x} + c_2 e^{R_- x}$

where $1 = F_0 = F(0) = c_1 + c_2$

$1 = F_1 = F'(0) = c_1 R_+ + c_2 R_-$

Solve for c_1 , c_2 :

$$c_1 = \frac{1}{\sqrt{5}} R_+ \qquad\qquad c_2 = - \frac{1}{\sqrt{5}} R_-$$

$$\therefore F(x) = \frac{1}{\sqrt{5}} (R_+ e^{R_+ x} - R_- e^{R_- x})$$

$$\therefore \sum_{n \geq 0} \frac{F_n}{n!} x^n = \frac{1}{\sqrt{5}} \sum_{n \geq 0} \frac{1}{n!} (R_+^{n+1} - R_-^{n+1}) x^n$$

DERANGEMENT RECURSION

$D_n \equiv$ number of perms in S_n with no fixed points.
$D_{n+1} = n (D_n + D_{n+1})$ $\qquad D_1 = 0, D_2 = 1$
↑

LINEAR BUT NOT CONSTANT COEFF.

Define $D_0 = 1$ to make recursion hold for
n = 1.

$$\text{Let } D(x) = \sum_{n \geq 0} \frac{D_n}{n!} x^n$$

$$= 1 + \sum_{n \geq 2} \frac{D_n}{n!} x^n$$

$$= 1 + \sum_{n \geq 0} \frac{D_{n+2}}{(n+2)!} x^{n+2}$$

$$\therefore D'(x) = \sum_{n \geq 0} \frac{D_{n+2}}{(n+1)!} x^{n+1}$$

$$= \sum_{n \geq 0} \frac{D_{n+1} + D_n}{n!} x^{n+1}$$

$$= \sum_{n \geq 0} \frac{D_{n+1}}{n!} x^{n+1} + x D(x)$$

Doesn't look too promising? What now?

$$= x\left(\sum_{n\geq 0} \frac{D_{n+1}}{n!} x^n\right) + xD(x)$$

$D'(x) = x\,(D(x) - D_0)' + xD(x)$

$= xD'(x) + xD(x)$

$\therefore D'(x)\,(1 - x) = xD(x)$

$$\therefore \frac{D'(x)}{D(x)} = \frac{x}{1-x} = \frac{1}{1-x} - 1$$

$\therefore \ell n\, D(x) = \ell n(1 - x) - x + c$

$$D(x) = \frac{1}{1-x} e^{-x} e^{c}$$

But $D(0) = D_0 = 1 \Rightarrow e^c = 1$ ($\Rightarrow c = 0$.)

$$\therefore D(x) = \frac{e^{-x}}{1-x}$$

If you don't know ODE, there is another way to get this result: (See Roberts, pp. 224 ff)

$$D(x) = 1 + \sum_{n\geq 0} \frac{D_{n+2}}{(n+2)!} x^{n+2}$$

$D_{n+2} = (n + 1)\,(D_{n+1} + D_n)$
↑
If only this were (n + 2) instead!? We can show (see Roberts, p.225)

$D_{n+2} = (n + 2)\,D_{n+1} + (-1)^{n+2}$

$$\therefore D(x) = 1 + \sum_{n\ge 0} \frac{D_{n+1}}{(n+1)!} x^{n+2} + \sum_{n\ge 0} \frac{(-1)^{n+2} x^{n+2}}{(n+2)!}$$

$$= 1 + x\,[D(x) - 1] + (e^{-x} - 1 + x)$$

$$\therefore D(x) = 1 + xD(x) - x + e^{-x} - 1 + x.$$

$$= x\,D(x) + e^{-x}$$

$$\therefore D(x) = e^{-x}/(1 - x)$$

$$= \left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \ldots\right)\left(1 + x + x^2 + \ldots\right)$$

$$\sum_{n\ge 0} \frac{D_k}{k!} x^k = \sum_{k=0}^{\infty} \left(1 - \frac{1}{1!} + \frac{1}{2!} \ldots + (-1)^k \frac{1}{k!}\right) x^k$$

$$\therefore D_k = k! \left(1 - \frac{1}{1!} + \frac{1}{2!} \ldots + (-1)^k \frac{1}{k!}\right)$$

<u>Counting Bracketings in Products</u>

In how many different ways can the "product" $x_1, x_2 \ldots x_n$ be parenthesized?

eg (x_1) 1

$(x_1 x_2)$ 1

$((x_1 x_2) x_3)$, $(x_1(x_2 x_3))$ } 2

Let the number be b_n. Then $b_1 = 1$, $b_2 = 1$, $b_3 = 2$. To bracket n letters, bracket first r, last n - r

$$b_n = \sum_{r=1}^{n-1} b_r\, b_{n-r} \quad , \quad \underline{\underline{n \ge 2}}$$

Let $b_0 = 0$. Then

$$b_n = \sum_{r=0}^{n} b_r\, b_{n-r} \quad , \quad n \ge 2$$

$$\text{Let } B(x) = \sum_{n\ge 0} b_n x^n$$

$$(B(x))^2 = \sum_{n\ge 0} c_n x^n \quad , \quad c_n = \sum_{r\ge 0}^{n} b_r\, b_{n-r}$$

$$= \sum_{n\ge 2} b_n x^n$$

$= B(x) - x$

$\therefore B(x)^2 - B(x) + x = 0.$

$$B(x) = \frac{1 \pm \sqrt{1-4x}}{2}$$

Two possible solutions - must check each

$$\sqrt{1-4x} = \sum_{n \geq 0} \binom{1/2}{n} (-4)^n x^n$$

Show $\binom{1/2}{n}(-4)^n = -\frac{2}{n}\binom{2n-2}{n-1}$, $n \geq 1$

$$\frac{1}{2}\sqrt{1-4x} = \frac{1}{2} - \sum_{n \geq 1} \frac{1}{n}\binom{2n-2}{n-1} x^n$$

$\therefore B(x) = \sum_{n \geq 1} \frac{1}{n}\binom{2n-2}{n-1} x^n$ (-ive root req'd!!)

$\therefore b_n = \frac{1}{n}\binom{2n-2}{n-1}$ Catalan (1814-94)

If we take the positive root we get

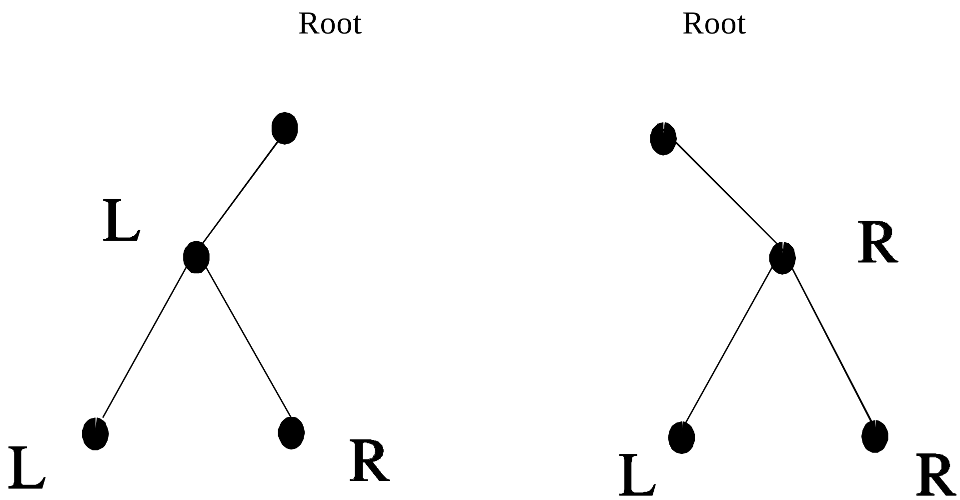
$$B(x) = 1 - \sum_{n \geq 1} \frac{1}{n}\binom{2n-2}{n-1} x^n$$

which give only negative values for the coefficients for
$n \geq 1$, which makes no sense.

A variety of problems lead to essentially the same recurrence as the above one:

(1) counting the number of simple, ordered rooted (SOR) trees

- unlabeled rooted trees, each vertex has 0, 1, or 2 descendents, “left” and “right” descendents distinguished



(2) Secondary structure in RNA [not precisely but **similar** - see Roberts]

(3) Triangulation of an n-gon by diagonals - division of the inside into triangles using only non-intersecting diagonals

(4) Let S_n be the number of distinct ordered sets of n integers $a_1, a_2, \ldots, a_n$ (allow some to be 0) such that

$$a_1 + \ldots + a_n = n, \; a_1 + a_2 + \ldots + a_k \geq k \text{ for each } k < n.$$

$$\text{Then } S_n = \frac{1}{n+1}\binom{2n}{n}$$

(5) Let S_n be the no. of sequences of length 2n,

$$a_1, a_2, \ldots, a_{2n}, \; a_i = +1 \text{ or } -1 \text{ and}$$

$$\sum_{j=1}^{2n} a_j = 0 \;, \quad \sum_{j=1}^{k} a_j \geq 0 \;, \quad k < 2n$$

$$\text{Then } S_n = \frac{1}{n+1}\binom{2n}{n}$$

<u>INCLUSION - EXCLUSION</u>

How many positive integers between 1 and 30 are not divisible by 2 or 3?

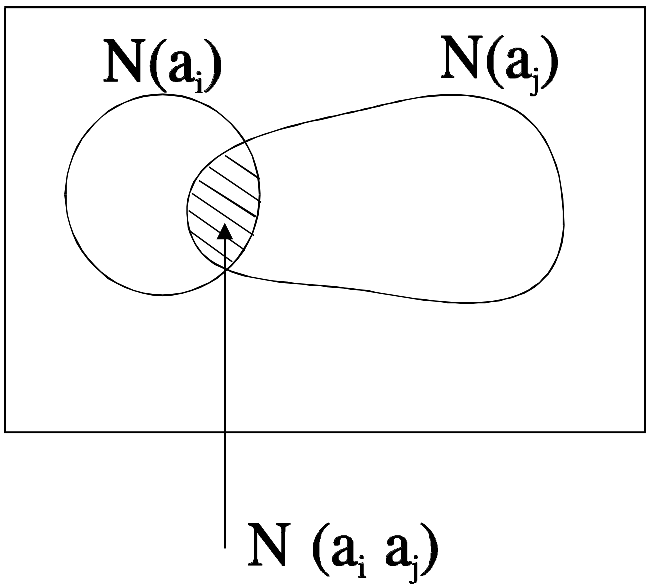
$6 = 2 \times 3$.

Divisible by 6 ⇔ divisible by 2 and 3.
Exactly 15 are divisible by 2
Exactly 10 are divisible by 3.
Exactly 5 are divisible by 6 (hence are divisible by <u>both</u> 2 and 3)

$\therefore 30 - (10 + 15) + 5 = 10$ are not divisible by 2 and 3

{1, 5, 7, 11, 13, 17, 19, 23, 25, 29}

Let A be a set of N objects.



Let $a_1, a_2, \ldots, a_r$ to be collection of r properties that each of the objects of A may have (but <u>need</u> not have)
Let $N(a_i)$ = # of objects of A <u>with</u> property a_i
Let $N(a'_i)$ = # of objects of A <u>without</u> property a_i
For all i, $N(a_i) + N(a'_i) = N$
Let $N(a_i a_j)$ = # of objects with both properties a_i, a_j

Obvious definition for $N(a_i a'_j)$, $N(a_i a_j a_k)$ and so on
$N(a_i) = N(a_i a_j) + N(a_i a'_j)$

$N(a'_i\, a'_j) = N - (N(a_i) + N(a_j)) + N(a_i a_j)$

Remove object with a_i or a_j — objects with a_i a_j included in this count — add back in objects with both a_i and a_j since these were removed twice

If we use above notation:

$$N(a'_1 a'_2 \cdots a'_r) = N - \sum_i N(a_i) + \sum_{i \neq j} N(a_i a_j) - \sum_{\substack{i,j,k \\ \text{different}}} N(a_i a_j a_k) + \ldots + (-1)^r N(a_1 a_2 \cdots a_r)$$

Proof: LHS counts each object without $a_1, a_2, \ldots, a_r$ exactly once. Show that the RHS does as well, and all others zero times.

→if an object β has none of the properties it is counted in N built never in any other term in RHS so we're OK.

→ if an object β has exactly p of the properties then it is counted once in N, $\binom{p}{1}$ = p times in $\sum_{i \neq j} N(a_i a_j)$, $\binom{p}{3}$ times in $\sum_{i \neq j \neq k} N(a_i a_j)$ and so on.

Thus it is counted $\sum_{j=0}^{p} \binom{p}{j} (-1)^j = 0$ times, as required

This is called the Principle of Inclusion-Exclusion (PIE)

<u>Corollary</u> The number of elements of A that have at least one of the properties is $N - N(a'_1 a'_2 \cdots a'_r)$.

<u>Exercise 1</u> Find the number of non negative integers satisfying solutions to $x_1 + x_2 + x_3 + x_4 = 18$ with <u>each</u> $x_i \leq 7$.

Solution: Without the restriction on all x_i the answer is

$$\binom{4+18-1}{18} = \binom{21}{18} = \binom{21}{3}$$

Define the property a_i for a <u>solution</u> to the equation if $x_i \geq 8$, i.e. a solution (x_1, x_2, x_3, x_4) satisfies a_i if $x_i \geq 8$, i = 1, 2, 3, 4. We want to count