Recurrence Relations

Tower of Hanoi

Let T_n be the minimum number of moves required.

$T_0 = 0, T_1 = 1$ ← Initial Conditions

* $T_n = 2\,T_{n-1} + 1$ $n \geq 2$

T_n is a sequence (fn. on integers). Solve for T_n?

* is a recurrence, difference equation (linear, non-homogeneous, constant coefficient)

Set $U_0 = T_0 + 1$, $U_n = T_n + 1$ $n \geq 1$

Then $U_n = T_n + 1 = 2T_{n-1} + 1 + 1 = 2(T_{n-1} + 1)$

so $U_n = 2U_{n-1}$

$= 2^2U_{n-2} = \dots = 2^{n-1}U_1 = 2^n$

$\therefore$ $T_n = 2^n - 1$

Suppose $a_{n+1} = 2a_n + n$, $a_0 = 1$

Let $U_n = a_n + n$, $U_0 = 1$

Then $U_{n+1} = a_{n+1} + (n + 1) = 2a_n + n + (n + 1)$

$= 2(a_n + n) + 1$

$= 2\,U_n + 1$

Thus, U_n is like the T_n in the preceding example, except

$U_0 = 1$ while $T_0 = 0$. In fact, since $T_1 = 1$, the

$\{U_n\}$ is just $\{T_n\}$ “advanced one step”,

i.e. $U_n = T_{n+1} = 2^{n+1} - 1$

$\therefore$ $a_n = 2^{n+1} - 1 - n = 2^{n+1} - (n + 1)$

Notice how the solution of one recurrence often can be reduced to the solution of a simpler one.

Suppose the recursion were

$$L_n = L_{n-1} + n \quad , \qquad L_0 = 1$$

Then we can "<u>expand out</u>" as follows:

$$L_n = L_{n-2} + (n-1) + n$$

$$= L_{n-3} + (n-2) + (n-1) + n$$
$$=$$
$$= L_0 + 1 + 2 + \cdots + n$$

$$= 1 + \frac{n(n+1)}{2}$$

This describes the number of regions formed by n intersecting lines in the plane, no 2 parallel and no 3 intersect in a point (PIZZA CUTTING PROBLEM).

<u>General Problem</u>

(*) $F(Y_{n+k}, Y_{n+k-1}, \cdots, Y_n) = 0$

Difference equation of order k(DFE)
Assume F <u>linear</u>, <u>constant coefficients</u>

(**) $Y_{n+k} + a_1 Y_{n+k-1} + \cdots + a_k Y_n - \varphi(n) = 0$

If $\varphi(n) = 0$, Homogeneous; otherwise non-Homo.

Note the strong analogy with D.E.!
Suppose that $Y_n = S_1(n)$ is a "solution". Then

$S_1(n+k) + a_1 S_1(n+k-1) + \cdots + a_k S_1(n) - \varphi(n) = 0$

If $S_2(n)$ is any other solution, then
$[S_1(n+k) - S_2(n+k)] + a_1 [S_1(n+k-1) - S_2(n+k-1)] + \ldots + a_k [S_1(n) - S_2(n)] = 0$
It follows that $S_1(n) - S_2(n)$ is a solution of
the Homogeneous equation related to (**)
(obtained by (ignoring $\varphi(n)$).

What is the "General Solution" of a DFE?
It's a family of functions, usually characterized by a parameter(s) which can take on different values.

From the above, the general solution for (**) is just the general solution to the related HOMO equation plus any particular solution to the NON-HOMO equation (**), i.e.

$$S_{NH}(n) = S_H(n) + S_p(n)$$

where $S_p(n)$ is <u>any</u> solution of (**), $S_H(n)$ is <u>general</u> solution of related HOMO and $S_{NH}(n)$ is general solution of (**)

<u>Solving HOMO</u>

(i) First Order DFE

Suppose $Y_{n+1} + a_1Y_n = 0$ (k = 1)

Then $Y_{n+1} = -a_1Y_n = (-1)^2 a_1^2 Y_{n-1}$

$= \dots = (-1)^n a_1^n Y_1$

$= (-a_1)^{n+1} Y_0$

where Y_0 is an arbitrary number (initial value of sequence).

CHECK: $(-a_1)^{n+1}Y_0 + a_1(-a_1)^nY_0$

$= -a_1(-a_1)^nY_0 + a_1(-a_1)^nY_0 = 0$

(ii) Higher Order DFE

Notice that if U_1 (n) and $U_2(n)$ are both solutions of the <u>homo</u> equation

(H) $Y_{n+k} + a_1Y_{n-1+k} + \dots + a_kY_n = 0$ then so is

$C_1 U_1(n) + C_2 U_2(n)$ where $C_1, C_2 \in \mathbb{R}$. Two

solutions of (H) are <u>different</u> iff $\nexists$ C such that

$U_1(n) = CU_2(n)$

It can be shown that if $U_1(n), U_2(n), \ldots, U_k(n)$

are k <u>different</u> solutions of (H), then the general

solution is

$$S_H(n) = \sum_{i=1}^{k} C_i U_i\ (n) \qquad \text{where } C_i \in \mathbb{R}.$$

<u>Finding Different Solutions</u>

$$Y_{n+2} + a_1 Y_{n+1} + a_2 Y_n = 0$$

Characteristic polynomial $p(\lambda) = \lambda^2 + a_1\lambda + a_2$

Let $p(\lambda)$ have roots $\lambda_1 \neq \lambda_2$ (Real λ_1, λ_2).

Then $U_1(n) = \lambda_1^n \qquad U_2(n) = \lambda_2^n$ are different solutions, since

$\lambda_1^{n+2} + a_1\lambda_1^{n+1} + a_2\lambda_1^n = \lambda_1^n(\lambda_1^2 + a_1\lambda_1 + a_2) = 0$

and the same holds for λ_2 and clearly $U_1(n) \neq CU_2(n)$ for any $C \in \mathbb{R}$. Thus, the general solution is $C_1\lambda_1^n + C_2\lambda_2^n$

<u>Exercise</u>: $F_{n+2} = F_{n+1} + F_n \qquad F_0 = F_1 = 1$

$$F_{n+2} - F_{n+1} - F_n = 0.$$

$$P(\lambda) = \lambda^2 - \lambda - 1 \text{ , roots } \frac{1 \pm \sqrt{5}}{2}$$

General Solution:

$$S_H\ (n) = C_1\left(\frac{1+\sqrt{5}}{2}\right)^n + C_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$F_0 = S_H\ (0) = 1 \Rightarrow C_1 + C_2 = 1$

$F_1 = S_H\ (1) = 1 \Rightarrow C_1\left(\frac{1+\sqrt{5}}{2}\right) + C_2\left(\frac{1-\sqrt{5}}{2}\right) = 1$

$\therefore C_1 = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right) \quad C_2 = -\ \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)$

$$\therefore F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

Since $\frac{1+\sqrt{5}}{2} > 1$ and $\left|\frac{1-\sqrt{5}}{2}\right| < 1$, for n large

$F_n \sim \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$

In fact <u>you</u> can verify that for all n,

$$\left|\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right| < 0.5$$

so that F_n = "integer nearest" $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$

(Also notice that $1 - \sqrt{5} < 0$ so that

$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$ is alternately above or below F_n.)

Also, $\frac{F_{n+1}}{F_n} \to \frac{1+\sqrt{5}}{2} = \tau$ "<u>Golden Mean</u>"

$$\frac{AB}{AC} = \frac{AC}{CB}$$

Suppose the two roots were the same

$$p(\lambda) = (\lambda - \lambda_1)^2$$

Then $U_1(n) = \lambda_1^n$ is one solution, we need a second. Suppose the second looks like $\lambda_1^n V(n)$ for some V(n), then we have

$$V(n + 2)\, \lambda_1^{n+2} - 2\lambda_1 V(n + 1)\, \lambda_1^{n+1} + \lambda_1^2 V(n)\lambda_1^n = 0$$

Divide by λ_1^{n+2} to get

$$V(n + 2) - 2\, V(n + 1) + V(n) = 0$$

By inspection we notice that 2 possible solutions are
$V(n) = 1$ (!!) and $V(n) = n$. This latter solution for V(n) gives a second (different) solution to the original equation.

Thus, the general solution is

$$S_H(n) = C_1{}^n\lambda_1\ C_2 n\lambda_1{}^n$$
$$= \lambda_1{}^n(C_1 + C_2 n)$$

Exercise: $Y_{n+2} - 4Y_{n+1} + 4Y_n = 0$

$$P(\lambda) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$$

$$Y_n = (C_1 + C_2 n)2^n$$

The final possibility is that the 2 roots are distinct but not real. Then they must be complex conjugates (since the coefficients are real), say $\alpha \pm i\beta$.

The earlier analysis gives

$$S_H(n) = C_1(\alpha + i\beta)^n + C_2\,(\alpha - i\beta)^n$$

(since we never used the fact that the roots were real explicitly!). What's wrong? NOTHING, except the solution $S_H(n)$, may not be real. We want real solution for practical problems. What to do?

Recall: $\alpha + i\beta = r[\cos\theta + i\sin\theta]$

where $r = \sqrt{\alpha^2 + \beta^2}$ $\cos\theta = \dfrac{\alpha}{r}$ $\sin\theta = \dfrac{\beta}{r}$

De Moivre: $(\alpha + i\beta)^n = r^n[\cos n\theta + i \sin n\theta]$

$$(\alpha - i\beta)^n = r^n[\cos n\theta - i \sin n\theta]$$

Thus. $\dfrac{1}{2}\{(\alpha + i\beta)^n + (\alpha - i\beta)^n\} = r^n \cos n\theta$

$$\frac{1}{2i}\{(\alpha + i\beta)^n - (\alpha - i\beta)^n\} = r^n \sin n\theta$$

Thus, $r^n \cos n\theta$ and $r^n \sin n\theta$ are 2 diff. real solutions. General Solution

$$S_H(n) = c_1 r^n \cos n\theta + c_2 r^n \sin n\theta$$

Example: $Y_{n+2} = -(Y_{n+1} + Y_n)$

$$p(\lambda) = \lambda^2 + \lambda + 1 \text{ , roots } \frac{1}{2}\left(-1 \pm i\sqrt{3}\right)$$

$$r = \left(\frac{1}{4} + \frac{3}{4}\right)^{\frac{1}{2}} = 1 \quad , \quad \cos\theta = -\frac{1}{2} \quad , \quad \sin\theta = \frac{\sqrt{3}}{2}$$

$$\text{so } \theta = \frac{2\pi}{3} \qquad \therefore Y_n = c_1 \cos\frac{2n\pi}{3} + c_2 \sin\frac{2n\pi}{3}$$

Next Step: General for any k. This is relatively easy:

1) if all the k roots are real and distinct, say, $\lambda_1, \lambda_2, \ldots, \lambda_k$ then general solution is

$$S_H(n) = \sum_{i=1}^{k} c_i \lambda_i^n$$

2) if λ_i occurs with multiplicity m_i, the distinct solutions associated with it are

$$\lambda_i^n, n\lambda_i^n, n^2\lambda_i^n, \ldots, n^{m_i - 1}\lambda_i^n$$

Do this for all k roots

3) whenever a root is complex, its conjugate must also be a root (coeff of polynomial are real!) [Recall that any polynomial $p(\lambda)$ can be written as a product of linear and quadratic factors.] Use the approach described above for this pair of complex conjugate roots.

Exercise $Y_{n+3} - 2Y_{n+2} + Y_{n+1} - 2Y_n = 0$

$$p(\lambda) = \lambda^3 - 2\lambda^2 + \lambda - 2 = (\lambda - 2)(\lambda^2 + 1)$$

$$\text{roots are } 2, \pm i \quad , \quad r = 1 \quad , \quad \theta = \frac{\pi}{2}$$

$$Y_n = c_1 2^n + c_2 \cos\frac{n\pi}{2} + c_3 \sin\frac{n\pi}{2}$$

Exercise $H_{n+3} - H_{n+2} - H_n = 0 \text{ , } H_0 = H_1 = H_2 = 1$

Find a "neat" formula for H_n.

What about for the recursion:

$$G_{n+3} - G_{n+2} + G_n = 0 \text{ , } G_0 = G_1 = G_2 = 1$$

Try to generalize these results for :

$H_{n+k+1} = H_{n+k} + H_n$, $H_0 = H_1 = ... = H_k = 1$
$G_{n+k+1} = G_{n+k} - G_n$, $G_0 = G_1 = ... = G_k = 1$

Non-Homogeneous Equations

- no general solution, even in case of constant coefficients
- some special cases can be solved, when the function on the RHS is of a certain type, namely,

$\varphi(n)$ = polynomial in n, e.g. $n^2 + 1$
$\varphi(n)$ = exponomial in n, e.g. 3^n
$\varphi(n)$ = (poly.) (exp.), e.g. $n^2 \cdot 2^n - 1$

- many other special cases can also be solved, but we don't want to get into this (for those interested, see Kelley and Petersen, Difference Equations, Academic Press (1991))

Exercise 1 $Y_{n+1} - Y_n = 1$

Let's first find one particular solution.
Clearly Y_n is not a constant function (since then

$Y_{n+1} - Y_n = 0 \ \forall n$). Let's try $Y_n = b_1 n + b_0$, a linear polynomial

$Y_{n+1} - Y_n = [b_1(n + 1) + b_0] - [b_1 n + b_0] = b_1$

Thus, $b_1 = 1$, while b_0 is arbitrary.
But recall that the solution (general) to the homogeneous related equation is just a constant K. Thus, the general solution to (1) is

$$Y_n = K + n$$

CHECK: $Y_{n+1} - Y_n = K + (n + 1) - (K + n) = 1$

Exercise 2 $Y_{n+1} - 2Y_n = 3n + 2$
The solution to the related homogeneous equation
$Y_{n+1} - 2Y_n = 0$ is $K2^n$, where K is determined by the initial value of this sequence.

We need a particular solution for the non-homogeneous equation. Let's try

$$\tilde{Y}_n = b_0 + b_1 n$$

Then $\tilde{Y}_n - 2\tilde{Y}_n = b_0 + b_1(n + 1) - 2(b_0 + b_1{}^n)$

$= (b_0 + b_1) + n(-b_1)$

$\therefore 3n + 2 = -b_1 n + (b_1 - b_0)$

$\therefore b_1 = -3 \qquad b_0 = -5$

<u>CHECK</u>:

$[-3(n + 1) - 5] - 2[-3n - 5] = -3n - 8 + 6n + 10 = 3n + 2.$

<u>Exercise 3</u> $\quad Y_{n+1} + Y_n = 4^n$

Solution to the related homo equation is $K(-1)^n$.

A particular solution : try

$$\tilde{Y}_n = b.4^n$$

Then $\tilde{Y}_{n+1} + \tilde{Y}_n = b\cdot 4^{n+1} + b\cdot 4^n = b\cdot 4^n[4 + 1] = 5b4^n$

$\therefore \qquad 4^n = 5b\cdot 4^n \Rightarrow b = \frac{1}{5}$

CHECK: $\frac{1}{5}4^{n+1} + \frac{1}{5}4^n = \frac{1}{5}4^n[4 + 1] = 4^n$

General Solution: $Y_n = \frac{1}{5}4^n + K(-1)^n$

<u>Exercise 4</u> $\quad Y_{n+1} - 3Y_n = (n + 3)\cdot 7^n$

solution to related homo equation $K \cdot 3^n$

For a particular solution try:

$\tilde{Y}_n = (b_0 + b_1 n)7^n$

$\tilde{Y}_{n+1} - 3\tilde{Y}_n = [(b_0 + b_1(n + 1)]7^{n+1} - 3[b_0 + b_1 n]7^n$

$\therefore \quad (n + 3)7^n = (4b_0 + 7b_1)7^n + n\cdot 4b_1\cdot 7^n$

$\therefore \quad 4b_1 = 1 \qquad 4b_0 + 7b_1 = 3.$

$\therefore \quad b_1 = \frac{1}{4} \qquad b_0 = \frac{5}{16}$

CHECK: $\left[\frac{5}{16} + \frac{1}{4}(n + 1)\right]7^{n+1} - 3\left[\frac{5}{16} + \frac{1}{4}n\right]7^n$

$$= \left(\frac{35}{16} - \frac{15}{16} + \frac{7}{4}\right)7^n + \left[\frac{7}{4}n - \frac{3}{4}n\right]7^n$$

$$= 3\cdot 7^n + n\cdot 7^n = (n + 3)7^n$$

$$\therefore \quad Y_n = K\cdot 3^n + \left(\frac{5}{16} + \frac{1}{4}n\right)7^n$$

In general, for

$$Y_{n+1} + aY_n = P_m(n)\, s^n$$

where $P_m(n)$ is a pol. of degree m, a non-

homo solution is given by

$$\tilde{Y}_n = Q_m(n)s^n \qquad s \neq -a$$

$$\tilde{Y}_n = nQ_m(n)s^n \qquad s = -a$$

and the coefficients of the pol. $Q_m(n)$ (of deg m) can be determined by subst. $\tilde{Y}$ into above equation (method of undetermined coeff.)

For $Y_{n+2} + a_1Y_{n+1} + a_2Y_n = P_m(n)\, s^n$
we can show that a particular solution is

$$\tilde{Y}_n = Q_m(n)s^n \quad , \quad s \neq \lambda_1, \lambda_2$$

$$\tilde{Y}_n = nQ_m(n)s^n \quad , \quad s = \lambda_1, \neq \lambda_2$$

$$\tilde{Y}_n = n^2Q_m(n)s^n \quad , \quad s = \lambda_1 = \lambda_2$$

where λ_1, λ_2 are the roots of the characteristic pol. of the related HOMO equation.

Exercise 1 $Y_{n+2} + 2Y_{n+1} + Y_n = (n + 3)2^n$

Here $P_m(n) = n + 3$, s = 2 and the ch. pol. is $\lambda^2 + 2\lambda + 1$, with roots $\lambda_1 = \lambda_2 = -1$.
Thus, a particular solution has form

$$\tilde{Y}_n = (\alpha_0 + \alpha_1 n)2^n$$

Solve for α_0, $\alpha_1$by substitution.

$[\alpha_0 + \alpha_1 (n + 2)]2^{n + 2} + 2[\alpha_0 + \alpha_1(n + 1)]2^{n + 1} + (\alpha_0 + \alpha_1 n)2^n =(n + 3)2^n$

$\Rightarrow 4(\alpha_0 + \alpha_1 n + 2\alpha_1) + 4(\alpha_0 + \alpha_1 n + \alpha_1) + \alpha_0 + \alpha_1 n = n + 3$

$\Rightarrow 9\alpha_0 + 12\alpha_1 = 3$, $9\alpha_1 = 1$

$\therefore \alpha_1 = \frac{1}{9}$, $\alpha_0 = \frac{5}{27}$

Exercise 2 $Y_{n+2} + Y_{n+1} + Y_n = \alpha^n$

Here $P_m(n) = 1$, s = α , roots of characteristic polynomial

$\lambda^2 + \lambda + 1$ are $\frac{1}{2}\left(-1 \pm i\sqrt{3}\right)$. A particular

solution has form $c\alpha^n$. Substitution yields

$$c\alpha^{n+2} + c\alpha^{n+1} + c\alpha^n = \alpha^n$$

Assume $\alpha \neq 0$. Then $c\alpha^2 + c\alpha + c = 1$ or $c(\alpha^2 + \alpha + 1) = 1$. If α not a root of the ch. Pol.

then $c = \frac{1}{\alpha^2 + \alpha + 1}$. Since the roots of the ch. pol. are

complex, if $\alpha \in \mathbb{R}$ we know that α is not a root so we're done.

Exercise 3. $Y_{n+2} + 2Y_{n+1} + Y_n = (-1)^n$

Like Exercise 1 above. Try $\tilde{Y}_n = cn^2 (-1)^n$

$(-1)^{n+2}c (n + 2)^2 + 2(-1)^{n+1}c(n + 1)^2 + (-1)^n cn^2$

$= (-1)^n c[n^2 + 4n + 4 - 2n^2 - 4n - 2 + n^2]$

$= (-1)^n 2c$

$\therefore$ $2c = 1 \Rightarrow c = \frac{1}{2}$