c) r distinct balls into n nondistinct boxes:

$$\sum_{k=1}^{n} S(r, k)$$

where S(r,k) = number of ways to partition an r-set into k (<u>non-empty</u>) parts. S(r,k) is called a Stirling number of the Second Kind.

Try to compute some S(r,k):

eg: $S(r,0) = 1 \quad \forall r \geq 0.$
$S(1,1) = 1 \quad S(r,2) = 2^{r-1} - 1$
$S(r,1) = 1 \quad S(r,3)$

(ii) 1
(iii) S(r,n)

d) r non-distinct balls into n nondistinct boxes:

(i) $\sum_{k=1}^{n} P_k^{(r)}$, where

$P_k^{(r)}$ = number of ways to partition the numbers r into k parts, <u>each part > 0</u>
Think of the number r as a sum of r ones:

$$r = \underbrace{1 + 1 + 1 + \ldots + 1}_{r \text{ times}}$$

We can group the ones as desired to form different parts. For example, for 3 we have:

3	1 + 2	- (2 parts, one each of size 1,2)
	1 + 1 + 1	- (3 parts, all of size 1)
	3	- (1 part of size 3)

For 4 we have:

4	1 + 3	
	2 + 2	- (2 parts, each of size 2)
	1 + 1 + 2	- (3 parts)

(ii) 1

(iii) $P_n^{(r)} = \sum_{k=1}^{r-n} P_k^{(r-n)}$

Ex. (i) How many non-negative integer solutions to $x_1 + x_2 + x_3 + x_4 = 12$?
(ii) $x_i > 0$
(iii) $x_1 \geq 2$, $x_2 \geq 2$, $x_3 \geq 4$, $x_4 \geq 0$

Solution (i) $\binom{15}{12} = \binom{15}{3}$

(ii) $\binom{11}{8} = \binom{11}{3}$

(iii) $\binom{7}{3} = \binom{7}{4}$

More Occupancy Problems (and Arrangements too!) Suppose we have r distinguishable balls, n distinct boxes.

How many ways are there to place r_i balls in

box i, i = 1, 2, … , n, where $\sum_{i=1}^{k} r_i = r$

$$\binom{r}{r_1} \bullet \binom{r - r_1}{r} \binom{r - r_1 - r_2}{r_3} - \binom{r - \sum_{i=1}^{n-1} r_i}{r_k}$$

(Note: there is a typo in the formula, change k to n in the last term)

$$= \frac{r!}{r_1!\ r_2! \_\ r_n!} \equiv \begin{pmatrix} r \\ r_1,\ r_2, \_ ,\ r_n \end{pmatrix}$$

"Multinomial Coeff"

Ex. Find the number of ways to arrange the letters of the word MISSISSIPPI

I M P S

$\frac{11!}{4!\ 4!\ 2!}$

4 distinct boxes

(corr. to letters)

11 distinct positions 11 balls (distinct)

4 distinct letters (corr. to positions)

$r_1 = 4 \quad r_2 = 1 \quad r_3 = 2 \quad r_4 = 4$

Alt-Solution: r objects, r_i type i (indistinct), n types.

$\sum_{i=1}^{n} r_i = r$. Find all possible arrangements

Let the number be x. Then there are $r_1!$, $r_2!$ … $r_k!$ ways of further arranging each such arrangement if we could distinguish among the identical objects. But by the product rule this yields $r_1!\ r_2!\ \dots\ r_n!$ x arrangements, which also corresponds to all possible arrangements of r objects. Since the total number of arrangements of r distinct items is r! we get $r_1!\ r_2!\ \dots\ r_n!\ x = r!$ which gives the desired value for x.

Binomial Expansion

$(a + b)^n$ = sum (j=1..n: n choose j) a^j b^(n-j)

(n is a positive integer)

Proof: (i) Induction

(ii) $(a + b)^n = (a + b)(a + b) \dots (a + b)$

n factors

to get a term $a^j b^{n-j}$, choose j of the a's from the factors, in $\binom{n}{j}$ ways (order counts, in the sense that the term with a from 1st bracket, b from 2nd, is different than term with b from 1st, a from

2nd).

$$(1 + x)^n = \sum_{j \geq 0} \binom{n}{j} x^{n - j} = \sum_{j \geq 0} \binom{n}{j} \text{xSUPj}$$

(use the fact that $\binom{n}{j} = \binom{n}{n - j}$)

Put x = 1

$$(1 + 1)^n = 2^n = \sum_{j \geq 0} \binom{n}{j}$$

Put x = -1

$$(1 - 1)^n = 0 = \sum_{j \geq 0} (-1)^j \binom{n}{j}$$

<u>Example</u> Find $\sum_{k \geq 0} k \binom{n}{k}$

Solution: (1) $k \binom{n}{k} = n \binom{n - 1}{k - 1}$

$$\therefore \sum_{k \geq 0} k \binom{n}{k} = \sum_{k \geq 1} n \binom{n - 1}{k - 1} \text{(k = 0 term is 0 on both sides)}$$

$$= n \sum_{k \geq 0} \binom{n - 1}{k}$$

$$= 2^{n - 1}$$

(2) $(1 + x)^n = \sum_{k \geq 0} \binom{n}{k} x^k$

$$n(1+x)^{n-1} = \sum_{k\geq 0} k\binom{n}{k}x^{k-1}$$

Set $x = 1 \Rightarrow n2^{n-1} = \sum_{k\geq 0} k\binom{n}{k}$

Exercise: $\sum_{k} k^2\binom{n}{k} = ?$