COMBINATORICS IS ....

what combinatorialists do!

One definition:

arranging the elements of sets (usually "discrete", often finite ones) into definite patterns. Examples include:

[n] = {1, 2, …, n}.

N = {1, 2, 3, … }

N_0 = {0, 1, 2, 3, … }

Set of all binary sequences of length 10

What kind of issues arise?

- Counting (C)
- Existence (E)
- Optimization (O)

Applications

- Exam scheduling without conflicts (E)
- Transportation problem (O)
- License plate numbers (C)

Ancient Roots

- magic squares (China, 2200 BCE)
- permutations (China, 1100 BCE)
- n-permutations (Sefer Yetzirah, 200-500 CE)
  movement of heavenly bodies
- Pascal, Fermat (16-17th CE)
  gambling

Euler (Konigsberg bridge)

Bernoulli (1st book)

Hamilton (graph theory applications)

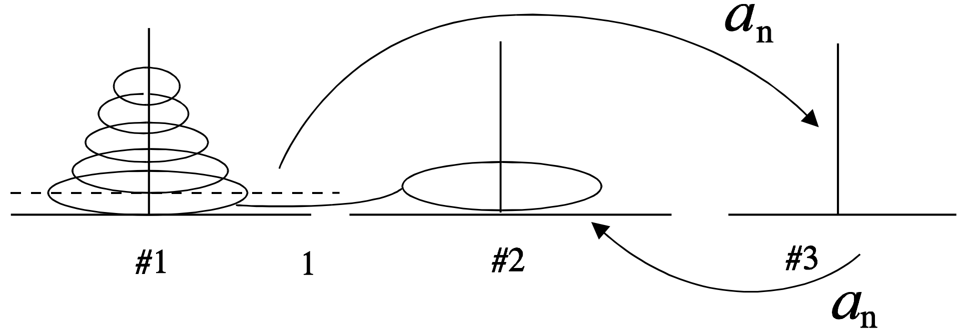
Let's examine some puzzles and play some games to start.

A. Puzzle

Find the fewest # of steps to get from (a) to (b) where a step is a movement of 1 ball.

Solution:
3 steps are required.

B. Tower of Hanoi

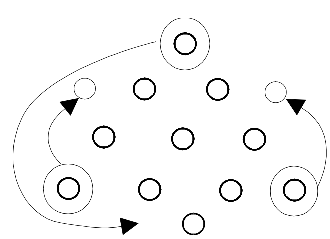


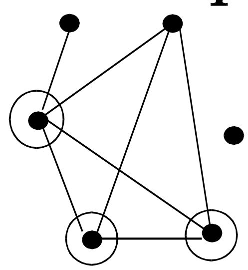
n = 1 ✓ $a_1 = 1$
n = 2 ✓ $a_2 \leq 3$ and $a_2 > 2 \Rightarrow a_2 = 3$.

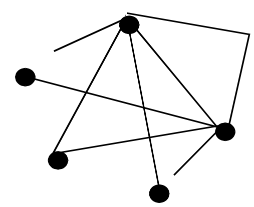
(a) (b)

Existence

Optimization $a_{n+1} \leq 2a_n + 1$ Recursion
$a_n = 2^n - 1$ $a_{n+1} = 2a_n + 1$, $a_1 = 1$.







$$\binom{6}{2} = 15$$

C. Game of SIM

Given 6 dots in the plane , and 2 players Red and Blue. Each move consists of joining a pair of dots by a Red or Blue line. Winner is the 1st person to form a triangle in a single colour.

What is the maximum number of moves?
Is there always a winner?
Is there a winning strategy? Does it matter whether you go first or second?
Winner? If not, end of game configuration has vertex #1 with 5 lines coming out of it, so at least 3 of them have the same colour, say blue. Look at the 3 end points of these lines. If all the lines joining them are red, we have a red _. Otherwise one line is blue, so together with the 2 blue lines from vertex #1, we have a blue _. Thus, there is always a winner.

Exercise: Is there a winning strategy for player #2?

Exercise: In SIM Misère, the object is to force the <u>other</u> player to complete the first triangle, who then is <u>loser</u>. Is there a winning strategy and if so for which player?

(G.J. Simmons, The game of SIM, J. Recreational Math 2 (1969), 66).

<u>THEOREM</u>: In SIM Misère, if both players play optimally, the <u>first</u> player will lose on the 15th move, at which time he/she will form 2 triangles of his/her colour.

These are examples of Achievement/Avoidance Games.

D. SPROUTS (Conway)

- n dots
- players move alternately by joining any pair of dots by a (possibly curved) line segment, put new dot in the "middle"
- restrictions:
  - no crossing except at dots
  - no dot has > 3 rays emanating from it.
- player to make last move wins.

Questions: Is there a winning strategy? If so, describe it.
Finite game? Max # of moves?

Answer: Sprouts must end in $\leq 3n-1$ moves if we start with n dots ($\Rightarrow$ Finite game)

Proof: Each dot has 3 "lives" so we begin with 3n lives. Each time we connect 2 dots we destroy two lives, but we add one since we put in new dot in the middle of the segment which already has 2 rays attached to it. So net loss of life associated with each move is 1.

Thus, game must end in $\leq 3n$ moves. In fact, we can sharpen this bound. Suppose we could have 3n moves. Final graph would have n (original) + 3n (new) = 4n vertices, and (3n)2 = 6n edges. Must have at least one vertex of degree 2 (the last one created) and all others have degree at most 3. But this is not possible - try to figure out why. (Hint: show that twice the number of edges must be equal to the sum of the degrees of all the vertices, so $12n \leq 3(4n - 1) + 2$, contradiction.)

Basic Counting Rules and Why They Work

Want to represent letters of alphabet with 2 digit binary strings. How many letters are possible?

□ □ 01 10

2×2 00 11

Product Rule: n_1 ways to do 1st thing, n_2 ways to do second (irrespective of outcome for first) then $n_1 \times n_2$ ways to do BOTH

Example: How many Licence Plates (Ont.) can be issued, if each consists of 3 letters followed by 3#'s?

□ □ □ □ □ □

$26 \times 26 \times 26 \times 10 \times 10 \times 10$

What is the solution if the three letters and three numbers can come in any order?

$$\binom{6}{3} 26^3 \cdot 10^3$$

Here's a more formal mathematical approach to the Product Rule: suppose S is the set of

solutions to some combinatorial problem. Suppose we can write S in the form $S = S_1 \times S_2$. Then card (S) = card (S_1) $\times$ card (S_2).
More generally, if $S = S_1 \times S_2 \times \ldots \times S_n$, then

$$\text{card}(S) = \prod_{i=1}^{n} \text{card}(S_i).$$

Exercise: Find the number of divisors of 21? of 168?
(Hint: $168 = 2^4 \cdot 3^3 \cdot 7^2$).

<u>Books</u>
5 Spanish, 4 French, 7 English
How many ways to choose 3, no two from same category?

$5 \times 4 \times 7$

Number of ways to choose a pair of books from the above collection not in the same language:
S - F 20 S - E 35 F - E 28 so total by the sum rule is 83

If you can do one thing in n_1 ways, second thing in n_2 ways, there are $n_1 + n_2$ ways that you can do either 1st or 2nd (<u>not</u> both). (SUM Rule)

Choose 3 letter sequences using a,b,c,d,e,f

a) allow repetition
b) no repetition
c) no repetition and must contain e
d) allow repetition and must contain e

a) 6^3 (product rule)
b) $6 \cdot 5 \cdot 4$ (product rule)
c) $60 = (5 \cdot 4) + (5 \cdot 4) + (5 \cdot 4)$
d) $6^3 - 5^3 = 91$

Exercise: Solve part d) by considering three separate cases, ie, 1e, 2e, or 3e.

Here is a move formal mathematical approach to the sum rule: let S be the set of solutions to some mathematical problem. Suppose we can write $S = S_1 \cup S_2$, where $S_1 \cap S_2 = \varnothing$. Then card (S) = card ($S_1$) + card ($S_2$).

More generally, if we have $S = \bigcup_{i=1}^{n} S_i$, and $S_i \cap S_j = \varnothing$ if

$i \neq j$, then $\text{card}(S) = \sum_{i=1}^{n} \text{card}(S_i)$.

Note: The collection $\{S_1, S_2, \dots, S_n\}$ is called an n-partition of the set S. The subsets S_i in the partition are pairwise disjoint (since $S_i \cap S_j = \emptyset$).

Exercise: Prove the general form of the product rule and the sum rule using induction.

Exercise: Suppose two baseball teams A and B play a “best of five” series, that is, the winner is the team to first win 3 games. Find the number of different possible outcomes in which team A wins.

<u>Permutations</u>
arrangements of a <u>set</u> of distinct objects (sometimes we also arrange "multisets", those are sets which allow multiple copies of the same element; another way of viewing this is we arrange with repetition of elements allowed).

n elt. set S, there are $n! = n\cdot (n-1)\cdot \dots 3 \cdot 2 \cdot 1$ arrangements of the elt of S.
There are: $n(n-1) \dots (n-r+1)$ ways to create r-elt. sequences from the elt. of S.

NOTATION $[n] = \{1,2,\dots, n\}$, $n^n = n!$
"Falling factorial" $n^{\underline{r}} = n(n-1) \dots (n-r+1)$
Rising " $n^{\overline{r}} = n(n+1) \dots (n+r-1)$

<u>Notice</u>: $n^{\underline{r}} = n!/(n-r)!$

$$n^{\overline{r}} = (n+r-1)!/(n-1)! = (n+r-1)^{\underline{r}}$$