# ***Pigeon hole principle***

1. *

At least n+1 people are needed since we could take the n wives and have no married couple. The pigeonhole principle will show that n+1 will do. Assign to each person the set {person, person's spouse}. For example, if Janet is married to Jim, both Janet and Jim are assigned the set {Janet, Jim}. Now there are n such sets (i.e. pigeon holes). If we have n+1 pigeons and n pigeon holes, then at least one pigeon hole contains two pigeons. So at least one married couple is contained in any subset of n+1 people from the set of n married couples.

If we tried to select n+1 people from a set of n married couples without selecting a married couple then after we have selected the nth person we would have nobody left to choose from since after we select one person, the person's spouse can't be selected next.

2.

If not then each person at the party has a different number of friends. No person can have more than 19 friends ( we will say that no person is a friend of him or herself). For each j = 0,1,2,...,19 there exists a person who has j number of friends. However the person who has 19 friends is a friend of everybody at the party and hence a friend of the person who has no friends. This is a contradiction and there must be two people who have the same number of friends.

3.

Let the pigeon holes represent the number of wins that could have occurred for any one of the n players involved in the tournament. Since no player loses all the matches, there can only be n-1 of these holes. Call the pigeons the n players that have participated in the tournament. If we distribute n pigeons into n-1 holes then there must be a hole that has two pigeons in it. Therefore there must be two players who won the same number of games.

4. *

First, any number smaller than 51 will not do since I can select 10 French books, 8 German books, 11 Russian books, 11 Spanish books and 11 Italian books and still not have 12 copies of a book in one language. This represents the worst case scenario. 52 books is the minimum required because if we select one more book then we will have 12 books of the same language.

5.

Let each of the people at the party be represented by a vertex in a graph. If two people know each other, then we draw an edge between them. Now suppose that each person has 5 or more acquaintances at the party. We have a graph with each vertex having a degree of at least five. The sum of all the degrees of the vertices of this graph will be at least 100. From theorem 1 in section 1.3, twice the number of edges in the graph will be at least 100 as well. This means that the number of edges must be at least 50. This contradicts the fact that there are only 48 different pairs of people who know each other. Hence there must be at least one person with four or fewer acquaintances.

6. *

The professor would need a set of at least 6 jokes since C(6,3)=20 and C(5,3)=10<12. If the professor's set of jokes is less than 6, then there are fewer than 10 triples of jokes available to tell to the class. If each triple of jokes represent the pigeon holes and the years represent the pigeons, then by the pigeonhole principle there must be a year in which the same triple of jokes was told.

7.

Look at the last digit of each number. If two numbers have the same last digit then their difference will contain 0 as the last digit. Any number with 0 as the last digit is divisible by 10. If the last digit of all seven numbers are different then one pair of them must add to 10. The reason for this is that if we select seven distinct digits from 0,1,2,3,4,5,6,7,8,9 then at least one pair of them will add to 10. To prove this, look at all the pairs of digits which add up to 10, i.e. {1,9},{2,8},{3,7}, {4,6}. If we select 0, and 5 as part of our

10.

The key to this problem is to realize that every even integer can be written in the form $2^i k_i$ where $k_i$ is an odd integer. All this means is that you can take any even number and divide it by a power of 2 and always get a odd quotient. For example divide 8 by $2^3$ and we get 1 as the quotient and 0 as the remainder, i.e. $8=2^3 x1+0$( quotient 1 and remainder is 0).
Now any set of 8 integers will have 8 odd numbers expressed either with a $2^j$ or on its own. Since there only seven odd numbers in the set {1,2,…,14}, an application of the pigeonhole principle tells us that two of these odd numbers must be the same and one of these two numbers will divide the other.

11.

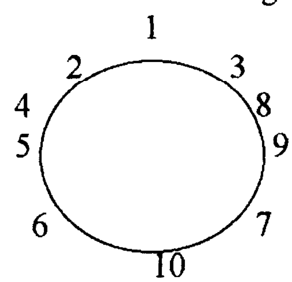
Suppose that in a subset of n integers between 2 and 2n (excluding 2 and 2n), every pair of integers has a common divisor between them. There are C(n,2)= n(n-1)/2 pairs of integers in a subset of n integers. The potential divisors for these pair of integers are 2,3,…,n. Call these n-1 potential divisors the pigeonholes and the C(n,2) pairs of integers, the pigeons. Since C(n,2)=n/2 x (n-1), it follows from the pigeonhole principle, that one of the holes has at least n/2 pairs of integers in it. This means that there are n numbers that have 2 or 3 or… n as a common divisor. The common divisor could not be larger than 2 otherwise one of the numbers would exceed 2n. This means that 2 must be the common divisor and our subset of n numbers must be 2,4,…,2n. However, we only have n-2 even integers available. We arrive at a contradiction and we are forced to conclude that there must be a pair of integers which have no common divisor.

12.

Let the numbers be $a_1,a_2,\ldots,a_{16}$ such that $1 \leq a_1 \leq a_2 \leq \ldots \leq a_{16}$ and $a_1+a_2+\ldots+a_{16}=30$. Let $a_{16}=15-k$ where $k=0,1,\ldots,13$. This implies that $a_1+a_2+\ldots+a_{15}=15+k$. There must be at least 15-k 1's in $a_1,a_2,\ldots a_{15}$. Hence $a_1=a_2=\ldots=a_{15-k}=1$ and $a_{15-k+1}+\ldots+a_{15}=2k$. We have a subset which sums to 1,2,…,15-k and 29,28,…,15+k. Any number between 15-k and 15+k can be seven digits, then we are selecting the remaining five numbers from the digits where we have 4 pairs which add up to 10. From the pigeonhole principle we must select one of these pairs. If 0 is selected and 5 is not, then we are selecting 6 digits from the remaining digits in which 4 pairs add up to 10. Another application of the pigeonhole principle tells us that we must select a pair that adds up to 10. Same argument if 5 is selected and 0 is not. If 5 and 0 are not selected then we are selecting 7 digits from the digits where we have 4 pairs adding up to 10. From above we found 5 to be sufficient for a selection of a pair that adds up to 10. So seven is more than enough. Getting back to our problem, we now know that there are two numbers that have a last digit of 0 when we add them. That number will be divisible by 10.

8.

Consider any arrangement of the digits 1,2,...,10 on a circle. These digits occupy 10 positions. Relabel each position with the sum of the three integers centered around it. For example, If we have the arrangement,




we replace each digit with the sum of the three digits centered around it. Replace 1 with 2+1+3=6. Replace 2 with 4+2+1=7, and so on. These sums will be our new labels. The sum of these labels is 3*(1+2+...+10)=165. By the pigeonhole principle, one of these labels must be an integer which exceeds 165/10 = 16.5.

9.

Let the pigeons represent the 99 hours and let the pigeon holes represent the consecutive days, day 1-2, day 3-4, day 5-6, day 7-8, day 9-10, and day 11-12. We have six pigeon holes because there are six pairs of consecutive days. Since 99=16x6 + 3, it follows from the pigeonhole principle that there must be pigeonhole with at least 17 pigeons in it. This pigeonhole will be the pair of consecutive days where the computer was used at least 17 hours.

obtained by summing up 15-j 1's and adding 2k to it from the subset that sums to 2k. For example 16-k is obtained by adding up 14-k 1's with the subset summing to 2k. Hence there is a subset summing up to any integer from 1 to 29.

13. *

Denote the printers $P_j$ , j=1,2,..,10 and the computers $C_i$ , i=1,2,…,15. In the following 15 x 10 matrix, an ijth entry of 1 means that computer $C_i$ is connected to printer $P_j$. To ensure that any printing job can be accommodated, connect the first printer to the first six computers, then connect the second printer to the next six computers, and onward in this manner.

| C\P | $P_1$ | $P_2$ | $P_3$ | $P_4$ | $P_5$ | $P_6$ | $P_7$ | $P_8$ | $P_9$ | $P_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $C_1$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $C_2$ | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $C_3$ | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $C_4$ | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $C_5$ | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| $C_6$ | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| $C_7$ | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |
| $C_8$ | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 |
| $C_9$ | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 |
| $C_{10}$ | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| $C_{11}$ | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| $C_{12}$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| $C_{13}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| $C_{14}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| $C_{15}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

14. *

There need to at least 24 connections. If not then one of the dummy printers, would be connected to at most 5 of the real printers. If each of the six computers calls upon this dummy printer, it couldn't accommodate all the printing jobs.

15. Suppose one disk is smaller than the other, but they have an equal number of sectors. Both disks have 10 0's and 10 1's on them. If the larger disk is fixed in place, then there are 20 possible positions for the small disk such that each sector of the small disk is contained in a sector of the large disk. We first count the total number of digit matches over all of the 20 possible positions of the disks. Since the large disk has 20 sectors with 10 0's and 10 1's, each sector of the small disk will match digits in exactly 10 out of the 20 possible positions. Thus the total number of digit matches over all the possible positions equals the number of sectors of the smaller disk multiplied by 10, and this equals 200. Therefore the average number of digit matches per position is 200/20=10. By the pigeon hole principle there must be some position with at least 10 digit matches.

16.

Let $a_1$ be the number of games played on the first day, $a_2$ be the number of games played on the first and second days, $a_3$ the number of games played on the first, second, and third days, and so on. The sequence of numbers $a_1, a_2, \ldots a_{49}$ is a strictly increasing sequence since at least one game is played on each day. Moreover, $a_1 \geq 1$ and the student never studies more than 11 hours in any one week, $a_{49} \leq 11x7=77$. Hence we have $1 \leq a_1 < a_2 < \ldots < a_{49} \leq 77$. The sequence $a_1+20, a_2+20, \ldots, a_{49}+20$ is also a strictly increasing sequence and $21 \leq a_1+20 < a_2+20 < a_3+20 < \ldots < a_{49}+20 \leq 97$. Thus each of the 98 numbers $a_1, a_2, \ldots, a_{49}, a_1+20, a_2+20, \ldots, a_{49}+20$ is an integer between 1 and 97. Since no two of the numbers $a_1, a_2, \ldots, a_{49}$ are equal and no two of the numbers $a_1+20, a_2+20, \ldots, a_{49}+20$ are equal, it follows from the pigeon hole principle that there must be and j and a k such that $a_j = a_k + 20$. Therefore on days k+1, k+2,…,j the student studied a total of 20 hours.

17.

Let $a_1, a_2, \ldots, a_{n^2+1}$ be the sequence of $n^2+1$ real numbers. We prove this as follows. We suppose there is no increasing subsequence of length n+1 and show that there must be a decreasing subsequence of length n+1. For each $k=1,2,\ldots,n^2+1$, let $m_k$ be the length of the longest increasing subsequence which begins at $a_k$. Suppose $m_k \leq n$ for each $k=1,2,\ldots,n^2+1$, so that there is no increasing subsequence of length n+1. Since $m_k \geq 1$ for each $k=1,2,\ldots,n^2+1$, the numbers $m_1, m_2, \ldots, m_{n^2+1}$ are $n^2+1$ integers between 1 and n. Since $n^2+1 = n(n)+1$, it follows from the pigeon hole principle that n+1 of the numbers $m_1, m_2, \ldots, m_{n^2+1}$ are equal. Let

$m_{k_1} = m_{k_2} = \ldots = m_{k_{n+1}}$ where

$1 \leq k_1 < k_2 < \ldots < k_{n+1} \leq n^2+1$. Suppose that for some $i = 1,2,\ldots,n$, $a_{k_i} < a_{k_{i+1}}$. Then since $k_i < k_{i+1}$ we could take a longest increasing subsequence beginning with $a_{k_{i+1}}$ and put $a_{k_i}$ in front to obtain an increasing subsequence starting with $a_{k_i}$. This implies that $m_{k_i} > m_{k_{i+1}}$, but from above we know that they are equal. Hence we are forced to conclude that $a_{k_i} \geq a_{k_{i+1}}$. Since this is true for each $i = 1,2,\ldots,n.$, we have that $a_{k_1} \geq a_{k_2} \geq \ldots \geq a_{k_n} \geq a_{k_{n+1}}$, thus $a_{k_1}, a_{k_2}, \ldots, a_{k_{n+1}}$ is a decreasing subsequence of length n

18.

We will prove that if there is no set of three mutual friends then there must be a set of three mutual strangers. It is sufficient to prove this for a party of six people. Let {A,B,C,D,E,F} be a group of 6 people. Suppose that the people known to A are seated in room Y and the people not known to A are seated in room Z. A is not seated in either room so there are at least 3 people in either room Y or room Z. Suppose B,C,D are in room Y. If they are all mutual strangers then we are done. If not, then at least two of them know each other. Lets say B and C know each other. Now we have A, B, and C form a group of 3 mutual acquaintances. To complete our solution, suppose B,C,D are in room Z. If they are all mutual acquaintances , then we are done. If not then at least two of them don't know each other. Lets say B and C don't know each other. Now we have A, B and C form a group of 3 mutual strangers.