Challenge set 6 solution sketches

1) n+1. If n people are chosen, they might all be women; but you can't choose n+1 people from among n couples without picking two from the same couple.

2) Consider how many partiers are friendless. If nobody is, then each of the 20 partiers has from 1 to 19 friends, so two must have the same number. If one person is friendless, then each of the remaining 19 has from 1 to 18 friends, and again two must have the same number. If two or more are friendless, then they all have the same number of friends.

3) If n players play, and each has between 1 and n-1 wins, then two must have the same number of wins, by the pigeonhole principle.

4) 52. The French and German books are red herrings. If we pick all 18 of them, then by the pigeonhole principle we can pick 33 but not 34 books of the remaining languages without picking 12 of the same language. 18+34=52.

5) If everyone knows at least five people, then there are at least 5*20/2 = 50 pairs, not 48.

6) 6. If she has five jokes, then there are only C(5,3)=10 different triples and she'll have to repeat twice in twelve years. If she has six, then she can choose from among C(6,3)=20 different triples without repetition.

7) We can assume all the numbers are positive, or add sufficiently large multiples of ten to those that aren't without changing the problem. Two of the seven numbers must then have last digits that belong to the same one of the six sets <0>, <1,9>, <2,8>, <3,7>, <4,6>, <5> and those two will have a sum or difference divisible by ten.

8) The sums of three consecutive numbers themselves sum to 165. By the pigeonhole principle, one of the ten sums must be at least 165/10=16.5.

9) At least 99/6 = 16.5 of the hours must fall into one of the six two-day periods <days 1 and 2>, <days 3 and 4>, <days 5 and 6>, ..., <days 11 and 12>.

10) The largest odd number that divides each number must be one of the seven numbers 1, 3, 5, 7, 9, 11, 13, so some two numbers must share the same largest odd divisor. The smaller divides the larger.

11) First note that no more than (n-2)/2 integers between 2 and 2n can share a common divisor, since there are that many even integers in that range. Now if each of C(n,2)=n*(n-1)/2 pairs had a common divisor in the set 2,3,4,...,n of n-1 possible divisors, then one divisor would have to be common to at least (n*(n-1)/2)/(n-1) = n/2 > (n-2)/2 integers, which cannot be.

12.

Such a set must contain two 1s, or else 15 numbers greater than one would sum to less than 30. Now starting with the subset $<1,1>$ and its sums $<1,2>$, add each remaining member to the subset in increasing order as long as the resulting sum set would still be a consecutive set $<1,2,...,m>$. If you stop before $m=30$ when you were about to add some r to a subset of n numbers, then $r-1$ is not in the sum set, or else you could sum $m+1$ by summing m, removing a subset of sum $r-1$ and adding the number r. Since the smallest number not yet in the subset is at least $m+2$, you must have $(16-n)(m+2)<=30-m$ or $n>=(17m+2)/(m+2)$. Combining this with $m>=n$ gives you $m**2-15m-2>=0$ or $m>15$. But then you are done, since if there's a subset that sums to any number up to 15, each number greater than 15 is the sum of the complement of such a subset.

13) Number the computers 1 to 15 and the printers 1 to 10. Connect printer n to computers n through $n+5$.

14) Without knowing which printers are connected to which computers, it's impossible to give an optimal answer to this question. In the worst case, if each computer was connected to only one printer, then each dummy printer must be connected to every real printer, in case its computer requests printing when any set of five real printers is already busy. In this case, 24 connections are required.

15) Each of the 20 digits on one disk can match 10 digits on the other disk when superposed in one of 20 different ways. One of the superpositions must therefore have at least 20*10/20 = 10 matching digits.

16) Let A(n) be the number of hours spend studying up to and including the nth day. The sets <A(i)|i=1..49> and <A(j)+20|j=1..49> both consist of forty-nine distinct integers in the range 1..97. By the pigeonhole principle there are i and j such that A(i)=A(j)+20, so from day i+1 to day j, the student studies exactly twenty hours.

17) Find the longest increasing subsequence containing each number. If all of these are less than n+1 numbers long, then by the pigeonhole principle there must be at least n+1 different subsequences. The subsequence formed by taking the last member of each of these increasing subsequences is then a decreasing subsequence of at least n+1 numbers.

18) Call one of the people A. By the pigeonhole principle, there are either three friends of A or three strangers to A at the party. If there are three friends, then either two of them form a trio of mutual friends with A, or they all form a trio of mutual strangers. If there are three strangers, then either two of them form a trio of mutual strangers with A, or they all form a trio of mutual friends.