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b. done in the lecture notes on derangements.

c. $D_0 = 1$ $D_1 = 0$ $D_2 = 1$ $D_3 = 2$
$D_4 = 9$ $D_5 = 44$ $D_6 = 265$
$D_7 = 1854$ $D_8 = 14853$
$D_9 = 133496$ $D_{10} = 1,334,961$.

d.

Let $D(x) = \sum_{n=0}^{\infty} D_n \frac{x^n}{n!}$

$$D'(x) = \sum_{n=1}^{\infty} D_n \frac{x^{n-1}}{(n-1)!}$$

$$= \sum_{n=1}^{\infty} (n-1)(D_{n-1} + D_{n-2}) \frac{x^{n-1}}{(n-1)!}$$

$$D'(x) = \sum_{n=1}^{\infty} (n-1) D_{n-1} \frac{x^{n-1}}{(n-1)!} + \sum_{n=2} (n-1) D_{n-2} \frac{x^{n-1}}{(n-1)!}$$

$$D'(x) = \sum_{n=1}^{\infty} D_{n-1} \frac{x^{n-1}}{(n-2)!} + \sum_{n=2}^{\infty} D_{n-2} \frac{x^{n-1}}{(n-2)!}$$

$$D'(x) = x \cdot \sum_{n=1}^{\infty} D_{n-1} \frac{x^{n-2}}{(n-2)!} + x \cdot \sum_{n=2}^{\infty} D_{n-2} \frac{x^{n-2}}{(n-2)!}$$

$$D'(x) = x D'(x) + x D(x)$$

$$D'(x)(1-x) = x D(x)$$

$$\frac{D'(x)}{D(x)} = \frac{x}{1-x} = -1 + \frac{1}{1-x}$$

$$1-x \overline{\big) \, x} \quad \frac{-1}{x-1}, \quad 1$$

$$\int \frac{D'(x)}{D(x)}\,dx = \int \left(-1 + \frac{1}{1-x}\right) dx$$

$$\ln|D(x)| = -x + -\ln|1-x|$$

$$\ln|D(x)| = -x - \ln|1-x| + C$$

$$D(x) = C_1 e^{-x} \cdot e^{-\ln(1-x)}$$

$$D(x) = C_1 e^{-x} \cdot \frac{1}{1-x} = C_1 e^{-x}(1-x)^{-1}$$

$$D(0) = D_0 = 1 \Rightarrow C_1 = 1$$

$$D(x) = e^{-x}(1-x)^{-1}.$$

37. INTELLIGENT

II TT LL EE NN G

Let U = the universal set of all arrangements of the word intelligent with no restrictions.

The number of arrangements with atleast 2 pairs of consecutive identical letters

$$= N - N_1 - N_2$$

N: all arrangements with no restrictions.

N_1: number of arrangements with no pairs of consecutive identical letters.

N_2: number of arrangements with exactly one pair consecutive identical letters.

$$N = \frac{11!}{(2!)^5}$$

For N_1,

Let A_1 = set of arrangements with II together.
A_2 = " " " " TT "
A_3 = " " " " LL "
A_4 = " " " " EE "
A_5 = " " " " NN "

$N(A_1)$ = put II together. $\frac{10!}{(2!)^4}$ arrangements with II together.

$N(A_2) = N(A_3) = N(A_4) = N(A_5) = N(A_1) = \frac{10!}{(2!)^4}$
same argument.

$N(A_iA_j) = \frac{9!}{(2!)^3}$ example [put II, TT together. $\frac{9!}{(2!)^3}$ arrangements with II TT together]
argument is the same with any other pairs together.

$N(A_iA_jA_k) = \frac{8!}{(2!)^2}$ example. [put II TT EE together. $\frac{8!}{(2!)^2}$ arrangements with these together.]
argument is the same with any other pairs together.

$N(A_iA_jA_kA_l) = \frac{7!}{2!}$

$N(A_1A_2A_3A_4A_5) = 6!$

we want

$$N(\bar{A}_1\bar{A}_2\bar{A}_3\bar{A}_4\bar{A}_5) = \frac{11!}{(2!)^5} - \binom{5}{1}\frac{10!}{(2!)^4} + \binom{5}{2}\frac{9!}{(2!)^3} - \binom{5}{3}\frac{8!}{(2!)^2} + \binom{5}{4}\frac{7!}{2!} - 6!$$

For N_2, exactly one pair

$$= N(A_1\bar{A}_2\bar{A}_3\bar{A}_4\bar{A}_5) + N(A_2\bar{A}_1\bar{A}_3\bar{A}_4\bar{A}_5) + N(A_3\bar{A}_1\bar{A}_2\bar{A}_4\bar{A}_5)$$

$$+ N(A_4\bar{A}_1\bar{A}_2\bar{A}_3\bar{A}_5) + N(A_5\bar{A}_1\bar{A}_2\bar{A}_3\bar{A}_4)$$

These will all be equal to each other, since there are two letters of each type.

$$N(A_1\bar{A}_2\bar{A}_3\bar{A}_4\bar{A}_5) = N(A_1 \cap \overline{(A_2 \cup A_3 \cup A_4 \cup A_5)})$$

$$= N(A_1) - N(A_1 \cap (A_2 \cup A_3 \cup A_4 \cup A_5))$$

$$= N(A_1) - N((A_1\cap A_2) \cup (A_1\cap A_3) \cup (A_1 \cap A_4) \cup (A_1\cap A_5))$$

$$= N(A_1) - \Big[N(A_1\cap A_2) + N(A_1\cap A_3) + N(A_1\cap A_4) + N(A_1\cap A_5)$$
$$- N(A_1\cap A_2\cap A_3) - N(A_1\cap A_2\cap A_4) - n(A_1\cap A_2\cap A_5)$$
$$- N(A_1\cap A_3\cap A_4) - N(A_1\cap A_3\cap A_5) - n(A_1\cap A_4\cap A_5)$$
$$+ N(A_1\cap A_2\cap A_3\cap A_4) + N(A_1\cap A_2\cap A_3\cap A_5) + N(A_1\cap A_2\cap A_4\cap A_5)$$
$$+ N(A_1\cap A_3\cap A_4\cap A_5) - N(A_1\cap A_2\cap A_3\cap A_4\cap A_5) \Big]$$

$$= \frac{10!}{(2!)^4} - \left[4\cdot\frac{9!}{(2!)^3} - 6\cdot\frac{8!}{(2!)^2} + 4\cdot\frac{7!}{2!} - 6! \right]$$

$$N_2 = 5\cdot\frac{10!}{(2!)^4} - \left[20\cdot\frac{9!}{(2!)^3} - 30\cdot\frac{8!}{(2!)^2} + 20\cdot\frac{7!}{2!} - 5\cdot 6! \right]$$

$$N - N_1 - N_2$$

$$= \cancel{\frac{11!}{(2!)^5}} - \cancel{\frac{11!}{(2!)^5}} + \cancel{\binom{5}{1}\frac{10!}{(2!)^4}} - \binom{5}{2}\frac{9!}{(2!)^3} + \binom{5}{3}\frac{8!}{(2!)^2}$$
$$- \binom{5}{4}\frac{7!}{2!} + 6! - \cancel{5\cdot\frac{10!}{(2!)^4}} + 20\cdot\frac{9!}{(2!)^3} - 30\cdot\frac{8!}{(2!)^2} + 20\cdot\frac{7!}{2!}$$
$$- 5\cdot 6!$$

$$= 10\cdot\frac{9!}{(2!)^3} - 20\cdot\frac{8!}{(2!)^2} + 15\cdot\frac{7!}{2!} - 4\cdot 6!$$

41. The question should read

Prove that $S_m = \sum_{k=m}^{n} \binom{k}{m} N_k \qquad (*)$

recall that $$N_m = \sum_{k=m}^{n} (-1)^{k-m} \binom{k}{m} S_k .$$

RHS of $(*)$

$$= N_m + \binom{m+1}{m} N_{m+1} + \binom{m+2}{m} N_{m+2} + \cdots + \binom{n}{m} N_n$$

Since $N_m, N_{m+1}, N_{m+2}, \ldots N_n$

contain $S_m, S_{m+1}, S_{m+2}, \ldots, S_n$, Our objective is to show that the coefficient of $S_{m+1}, S_{m+2}, \ldots, S_n$ is 0.

$$\sum_{k=m}^{n} (-1)^{k-m} \binom{k}{m} S_k + \sum_{k=m+1}^{n} (-1)^{k-m-1} \binom{k}{m+1} S_k + \cdots + \sum_{k=n}^{n} (-1)^{k-n} \binom{k}{n} S_k$$

$$= S_m + C_{m+1} S_{m+1} + \cdots + C_k S_k + \cdots + C_n S_n$$

$\{ C_{m+1}, C_{m+2}, \ldots C_k, \ldots C_n$ are the coefficien[ts]. We must show these are 0.

the term with S_k, $k \geq m+1$ is

$$= (-1)^{k-m} \binom{k}{m} S_k + (-1)^{k-m-1} \binom{m+1}{m}\binom{k}{m+1} + (-1)^{k-m-2} \binom{m+2}{m}\binom{k}{m+2}$$
$$+ \cdots \pm \binom{k}{m} S_k$$

$$= (-1)^{k-m} S_k \left[\binom{k}{m} - \binom{m+1}{m}\binom{k}{m+1} + \binom{m+2}{m}\binom{k}{m+2} - \binom{m+3}{m}\binom{k}{m+3} + \cdots \pm \binom{k}{m} \right]$$

$$\binom{m+1}{m}\binom{k}{m+1} = \frac{(m+1)!}{m!\,1!} \cdot \frac{k!}{(k-m-1)!\,(m+1)!} = \frac{k!}{m!\,(k-m-1)!} = \frac{k!}{m!\,(k-m)!}(k-m)$$

$$= (k-m)\binom{k}{m}$$

$$\binom{m+2}{m}\binom{k}{m+2} = \frac{(m+2)!}{m!\,2!} \cdot \frac{k!}{(k-m-2)!\,(m+2)!}$$

$$= \frac{k!}{2!\,(k-m-2)!\,m!} = \frac{k!}{m!\,(k-m)!}\,\frac{(k-m)(k-m-1)}{2!}$$

$$= \binom{k}{m}\binom{k-m}{2}$$

it's not hard to see that

$$\binom{m+e}{m}\binom{k}{m+e} = \binom{k}{m}\binom{k-m}{e}$$

Hence the coefficient of S_k can be expressed as

$$= (-1)^{k-m}\left[\binom{k}{m} - \binom{k-m}{1}\binom{k}{m} + \binom{k-m}{2}\binom{k}{m} - \binom{k-m}{3}\binom{k}{m} + \cdots \pm \binom{k}{m}\right]$$

$$= (-1)^{k-m}\left[\ 1 - \binom{k-m}{1} + \binom{k-m}{2} - \binom{k-m}{3} + \cdots \pm 1\ \right]$$

$$= (-1)^{k-m}\,(1-1)^{k-m}$$

$$= (-1)^{k-m}\cdot 0 = 0$$

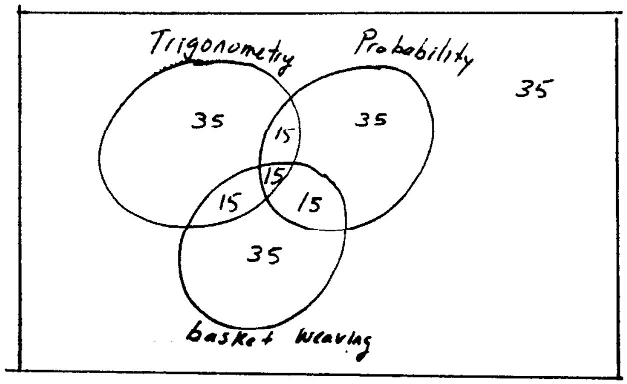
This shows that $c_k = 0 \quad k \geq m+1$.

$$\sum_{k=m}^{n}\binom{k}{m}N_k = S_m + 0\cdot S_{m+1} + 0\cdot S_{m+2} + \cdots + 0\cdot S_n$$

$$= S_m.$$

Additional Section 8.1 Exercises

13. Use a Venn Diagram

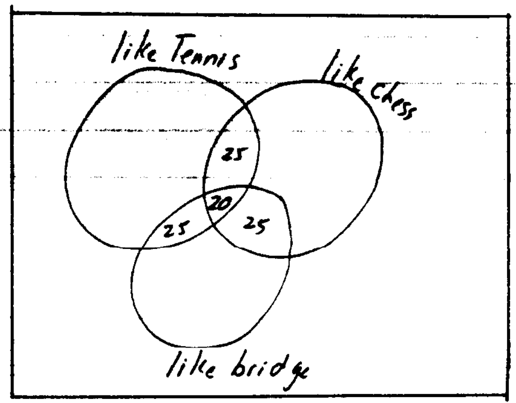


fill in the numbers

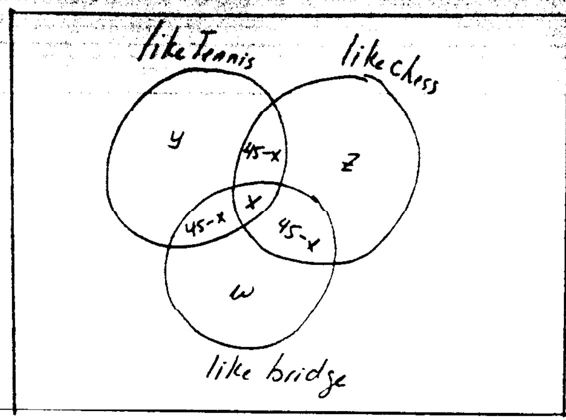
a. 35 are taking none of these subjects.

b. 35 are only taking probability.

14. Lets Say there are 100 college professors at the instutition where these results were obtained. Use a Venn Diagram



a. fill in 20 where all three circles intersect. Filling the other information, we are forced to conclude that at least 70 out of the 100 college professors like tennis. However, we were told that only 60 like tennis. We should be suspicious of these results.



2. fill in x where all three circles intersect.

Now

$$y + 45-x+x+45-x = 60 \quad , \quad z + 90-2x+x = 50$$

$$y + 90 - x = 60 \qquad z - x = -40$$

$$y - x = -30 \qquad x - z = 40 \quad (2)$$

i.e. $x - y = 30 \quad (1)$

$$w + 90-2x+x = 65$$

$$w - x = -35 \quad (3)$$

$$x - w = 35$$

$$x - y = 30 \quad (1)$$

$$x - z = 40 \quad (2)$$

$$x - w = 35 \quad (3)$$

Using (2) x must be at least 40.

Smallest percentage would be 40%.

21. Let A_1 = the set of arrangements of TAMELY with T before A.
Let A_2 = the set of arrangements of TAMELY with A before M.
Let A_3 = the set of arrangements of TAMELY with M before E.

We want $N(A_1 \cup A_2 \cup A_3)$.

$$N(A_1 \cup A_2 \cup A_3) = N(A_1) + N(A_2) + N(A_3) - N(A_1 \cap A_2) - N(A_1 \cap A_3) - N(A_2 \cap A_3) + N(A_1 \cap A_2 \cap A_3)$$

For $N(A_1)$,

Choose 2 positions out of the six available for the T and A.
There are $\binom{6}{2}$ ways to do this. Once these positions are selected, there is only one way (i.e. one order) in which we can place the T and the A. The other four letters can be arranged in 4! ways.

$$N(A_1) = \binom{6}{2} \times 4!$$

For $N(A_2)$, and $N(A_3)$, the argument is identical to the argument used for $N(A_1)$.

Hence $N(A_2) = N(A_3) = \binom{6}{2} \times 4!$

For $N(A_1 \cap A_2)$.

Choose three positions out of 6 for T, A, M.
There are $\binom{6}{3}$ ways to do this.

Once the positions are selected, there is only one order in which T, A, M can be placed in. The remaining 3 letters can be arranged in 3! ways.

$$N(A_1 \cap A_2) = \binom{6}{3} \times 3! .$$

The argument for $N(A_2 \cap A_3)$ is identical to the one used for $N(A_1 \cap A_2)$.

$$N(A_2 \cap A_3) = \binom{6}{3} \times 3! .$$

For $N(A_1 \cap A_3)$, choose 4 positions for T, A, M, E. This can be done in $\binom{6}{4}$ ways. Out of these 4 positions choose 2 for T, A, i.e. $\binom{4}{2}$ ways. Once the positions are selected there is only one order in which T, A can be placed in. The 2 remaining letters can be arranged 2! ways.

$$N(A_1 \cap A_3) = \binom{6}{4}\binom{4}{2} \times 2!$$

For $N(A_1 \cap A_2 \cap A_3)$, TAME must be together, so choose 4 positions out of six for these letters. There is only one order they can be placed in once the positions are selected. The remaining 2 letters can be arranged 2! ways.

$$N(A_1 \cap A_2 \cap A_3) = \binom{6}{4} \times 2!$$

$N(A_1 \cup A_2 \cup A_3)$

$$= 3 \times \binom{6}{2} \times 4! - \left[2 \times \binom{6}{3} \times 3! + \binom{6}{4}\binom{4}{2} \times 2! \right] + \binom{6}{4} \times 2!$$

22. Let A_1 = the set of arrangements of MATHEMATICS with both T's before both A's.

Let A_2 = the set of arrangements of MATHEMATICS with both A's before both M's.

Let A_3 = the set of arrangements of MATHEMATICS with both M's before E.

We want $N(A_1 \cup A_2 \cup A_3)$.

Use the same formula as in question 21.

$$N(A_1) = \binom{11}{4} \frac{7!}{2!}$$

11 positions choose four for T, T, A, A

arrange the rest of the letters.

$$N(A_2) = \binom{11}{4} \frac{7!}{2!}$$

11 positions choose four positions for M, M, A, A

arrange the rest of the letters.

in both cases once the positions are selected, there is only one way to place the letters.

$$N(A_3) = \binom{11}{3} \frac{8!}{2!2!}$$ ~~arrange rest of the letters~~

11 positions Choose four for M, M, E.

*- There is only one way to place the letters after the positions are chosen.

$$N(A_1 \cap A_2) = \binom{11}{6} \times 5!$$

arrange rest of letters.

11 positions chouse 6 for T, T A, A M, M.

See comment *

$$N(A_2 \cap A_3) = \binom{11}{5} \times \frac{6!}{2!}$$

arrange rest of the letters.

11 positions Chouse 5 for A, A, M, M, E

See comment *

arrange the four remaining le~~tters~~

$$N(A_1 \cap A_3) = \binom{11}{7} \binom{7}{4} \times 4!$$

11 positions chouse 7 for T, T, A, A, M, M, E,

out of these 7 positions pick four for T, T, A, A remaining positions filled with MME.

$$N(A_1 \cap A_2 \cap A_3) = \binom{11}{7} \times 4!$$

arrange the other letters.

11 positions choose 7 for TTAAMME. They must go in as TTAAMME.

$N(A_1 \cup A_2 \cup A_3)$

$$= 2 \times \binom{11}{4} \times \frac{7!}{2!} + \binom{11}{3} \times \frac{8!}{2!2!} - \binom{11}{6} \times 5!$$

$$- \binom{11}{5} \times \frac{6!}{2!} - \binom{11}{7}\binom{7}{4} \times 4!$$

$$+ \binom{11}{7} \times 4!$$