Section 7.4

1

a. $a_n = a_{n-1} + 3(n-1) \qquad a_0 = 1$

$$a_n = a_0 + \sum_{k=1}^{n} 3(k-1) = 1 + \sum_{k=1}^{n} (3k-3)$$

$$= 1 + 3 \cdot \sum_{k=1}^{n} k - 3 \cdot \sum_{k=1}^{n} 1$$

$$= 1 + \frac{3n(n+1)}{2} - 3n$$

$$= 1 + \frac{3(n^2-n)}{2} = 1 + 3C(n,2)$$

b. $a_n = a_{n-1} + n(n-1) \qquad a_0 = 3$

$$a_n = a_0 + \sum_{k=1}^{n} k(k-1)$$

$$= 3 + \sum_{k=1}^{n} (k^2-k) = 3 + \sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k$$

$$= 3 + \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}$$

c. $a_n = a_{n-1} + 3n^2 \qquad a_0 = 10$

$$a_n = a_0 + \sum_{k=1}^{n} 3n^2$$

$$= a_0 + 3 \cdot \sum_{k=1}^{n} n^2$$

$$= 10 + \frac{3n(n+1)(2n+1)}{6} = 10 + \frac{n(n+1)(2n+1)}{2}$$

2

9.

a. $a_n = 3a_{n-1} - 2$ $\qquad a_0 = 0$

homogeneous equation,

$$a_n = 3a_{n-1}$$

$a_n = A(3)^n$ is a solution.

try $a_n^* = B_0$

$$B_0 = 3B_0 - 2$$

$$B_0 = 1$$

$$a_n = A(3)^n + 1$$

$$a_0 = A + 1 = 0$$

$$A = -1$$

$$a_n = -(3)^n + 1$$

b. $a_n = 2a_{n-1} + (-1)^n$ $\qquad a_0 = 2$

homogeneous equation

$$a_n = 2a_{n-1}$$

$a_n = A(2^n)$ is a solution.

$$a_n^* = B(-1)^n$$

$$B(-1)^n = 2B(-1)^{n-1} + (-1)^n$$

$$-B = 2B - 1$$

$$B = 1/3$$

$$a_n = A(2^n) + 1/3(-1)^n$$

$$a_0 = 2$$

$$\Rightarrow A + 1/3 = 2$$

$$A = 5/3$$

$$a_n = 5/3(2)^n + 1/3(-1)^n.$$

c. $a_n = 2a_{n-1} + n \qquad a_0 = 1$

homogeneous equation,

$a_n = 2a_{n-1}$

$a_n = A(2)^n$ solution of homogeneous equation.

$a_n^* = B_0 + B_1 n$

$B_0 + B_1 n = 2(B_0 + B_1(n-1)) + n$

$2B_0 - 2B_1 = B_0$

$2B_1 + 1 = B_1$

$B_0 = -2$

$B_1 = -1$

$a_n = A(2^n) - n - 2$

$a_0 = 1 = A - 0 - 2$

$\Rightarrow A = 3$

$a_n = 3 \cdot 2^n - n - 2$

d. $a_n = 2a_{n-1} + 2n^2 \qquad a_0 = 3$

$a_n = A(2)^n$ solution to the homogeneous equation.

$a_n^* = B_0 + B_1 n + B_2 n^2$

$B_0 + B_1 n + B_2 n^2 = 2(B_0 + B_1(n-1) + B_2(n-1)^2) + 2n^2$

$2B_2 + 2 = B_2$ ①

$2B_1 - 4B_2 = B_1$ ②

$2B_0 - 2B_1 + 2B_2 = B_0$ ③

$B_0 = -12 \qquad B_2 = -2 \qquad B_1 = -8$

$a_n = A(2)^n - 12 - 8n - 2n^2$

$a_0 = 3 = A - 12$

$A = 15$

$a_n = 15 \cdot 2^n - 2n^2 - 8n - 12$

11. $a_n = 3a_{n-1} - 2a_{n-2} + 3$ $\quad a_0 = a_1 = 1$

homogeneous equation

$$a_n = 3a_{n-1} - 2a_{n-2}$$

$$\alpha^n = 3\alpha^{n-1} - 2\alpha^{n-2}$$

$$\alpha^2 = 3\alpha - 2$$

$$\alpha^2 - 3\alpha + 2 = 0$$

$$(\alpha - 2)(\alpha - 1) = 0$$

$$\alpha = 2 \text{ or } \alpha = 1$$

$$a_n = A(2)^n + B(1)^n = A(2)^n + B$$

$$a_n^* = C$$

$C = 3C - 2C + 3$ doesn't work since C is a solution of the homogeneous equation.

try $a_n^* = nC$

$$nC = 3(n-1)C - 2(n-2)c + 3.$$

$$nC = nc + c + 3$$

$$\Rightarrow c = -3$$

$$a_n = 3 \cdot 2^n - 2 - 3n$$

17. $a_n = 5a_{n-1} - 6a_{n-2} + 3n + 2$

homogeneous equation

$a_n = 5a_{n-1} - 6a_{n-2}$

$a_n = \alpha^n$

$\alpha^2 = 5\alpha - 6$

$\alpha^2 - 5\alpha + 6 = 0$

$(\alpha - 3)(\alpha - 2) = 0$

general solution $a_n = A(3)^n + B(2)^n$

particular solution

$a_n^* = B_0 + B_1 n + B_2$

$B_0 + B_1 n + B_2 = 5(B_0 + B_1(n-1) + B_2) - 6(B_0 + B_1(n-2) + B_2) + 3n + 2$

$-B_1 + 3 = B_1$

$\Rightarrow B_1 = 3/2$

$2(B_0 + B_2) = 25/2$

$B_0 + B_2 = 25/4$

$a_n = A(3)^n + B(2)^n + \frac{25}{4} + \frac{3}{2}n$

19

a. $a_n^2 = 2a_{n-1}^2 + 1$ let $b_n = a_n^2$

$b_n = 2b_{n-1} + 1$

homogeneous : $b_n = 2b_{n-1}$

$b_n = A(2)^n$

$b_n^* = B$

$B = 2B + 1$

$B = -1$

$b_n = 2^{n+1} - 1$

$a_n^2 = 2^{n+1} - 1$

$\Rightarrow a_n = \sqrt{2^{n+1} - 1}$

$$a_n = -n\,a_{n-1} + n!$$

$a_0 = 0! = 1$
$a_1 = -1\cdot 1 + 1! = 0$
$a_2 = -1\cdot a_1 + 2! = 2!$

conjecture: $a_n = n!$ n-even
$a_n = 0$ n-odd

prove it by induction:

Suppose $a_n = 0$ n-odd.
$a_n = n!$ n-even

show $a_{n+1} = (n+1)!$ [n-odd]

$$a_{n+1} = -(n+1)\,a_n + (n+1)!$$

Since n is odd $a_n = 0$

$$a_{n+1} = -(n+1)\cdot 0 + (n+1)! = (n+1)!$$

Show $a_{n+1} = 0$ [n-even] $a_n = n!$

$$a_{n+1} = -(n+1)\,a_n + (n+1)!$$
$$= -(n+1)\,n! + (n+1)! = -(n+1)! + (n+1)! = 0$$

Exercises 7.5

1

a. $a_n = a_{n-1} + 2 \qquad a_0 = 1$

$$g(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$g(x) - a_0 = \sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} (a_{n-1} + 2) x^n$$

$$= \sum_{n=1}^{\infty} (a_{n-1} x^n + 2x^n)$$

$$= x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + 2 \cdot \sum_{n=1}^{\infty} x^n$$

$$g(x) - 1 = x \cdot \sum_{m=0}^{\infty} a_m x^m + 2 \cdot \left(\frac{1}{1-x} - 1 \right)$$

$$g(x) - 1 = x\, g(x) + \frac{2x}{1-x}$$

(answer in back of text is incorrect)

b. $a_n = 3a_{n-1} - 2a_{n-2} + 2 \qquad a_0 = a_1 = 1$

$$g(x) = a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n$$

$$= 1 + x + \sum_{n=2}^{\infty} (3a_{n-1} - 2a_{n-2} + 2) x^n$$

$$= 1 + x + 3 \sum_{n=2}^{\infty} a_{n-1} x^n - 2 \cdot \sum_{n=2}^{\infty} a_{n-2} x^n + 2 \cdot \sum_{n=2}^{\infty} x^n$$

$$= 1 + x + 3x \cdot \sum_{n=2}^{\infty} a_{n-1} x^{n-1} - 2x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} + 2\left(\frac{1}{1-x} - 1 - x \right)$$

$$= 1 + x + 3x \cdot \sum_{m=1}^{\infty} a_m x^m - 2x^2 \sum_{m=0}^{\infty} a_m x^m + \frac{2 \cdot x^2}{1-x}$$

⑧

$$g(x) = 1+x+3x(g(x)-1) - 2x^2 g(x) + \frac{2x^2}{1-x}$$

$$g(x) = 1 + 3xg(x) - 2x^2 g(x) - 2x + \frac{2x^2}{1-x}$$ (answer in back of text is incorrect)

I

c. $a_n = a_{n-1} + n(n-1) \qquad a_0 = 1$

$$g(x) - a_0 = \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} (a_{n-1} + n(n-1))x^n$$

$$g(x) - 1 = \sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=1}^{\infty} n(n-1)x^n$$

$$g(x) - 1 = x \cdot \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + x^2 \sum_{n=2}^{\infty} n(n-1) x^{n-2}$$

$$g(x) - 1 = x \cdot \sum_{m=0}^{\infty} a_m x^m + x^2 \cdot \frac{d^2}{dx^2}\left(\frac{1}{1-x}\right)$$

$$g(x) - 1 = x g(x) + \frac{2x^2}{(1-x)^3}$$

d. $a_n = 2a_{n-1} + 2^n \qquad a_0 = 1$

$$g(x) - a_0 = \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} (2a_{n-1} + 2^n)x^n$$

$$g(x) - 1 = 2\sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=1}^{\infty} (2x)^n$$

$$g(x) - 1 = 2x \cdot \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + \left(\frac{1}{1-2x} - 1\right)$$

$$g(x) - 1 = 2x\, g(x) + \left(\frac{1 - (1-2x)}{1-2x}\right)$$

$$g(x) - 1 = 2x\, g(x) + \frac{2x}{1-2x}$$ (answer in back of text is incorrect)

3.

(c) $a_n = \sum_{i=1}^{n-1} 2^i a_{n-i}$ valid for $n \geq 2$

$$g(x) - a_0 - a_1 x = \sum_{n=2}^{\infty} a_n x^n = a_2 x^2 + a_3 x^3 + \cdots$$

$$a_2 = \sum_{i=1}^{1} 2^i a_{n-i} \underset{n=2}{=} 2^1 a_1 \qquad a_3 = \sum_{i=1}^{3} 2^i a_{3-i} = 2^1 a_2 + 2^2 a_1$$

$$a_4 = \sum_{i=1}^{3} 2^i a_{4-i} = 2^1 a_3 + 2^2 a_2 + 2^3 a_1$$

$$a_5 = \sum_{i=1}^{4} 2^i a_{5-i} = 2^1 a_4 + 2^2 a_3 + 2^3 a_2 + 2^4 a_1$$

$$g(x) - 1 - x = (2^1 a_1) x^2 + (2^1 a_2 + 2^2 a_1) x^3 + (2^1 a_3 + 2^2 a_2 + 2^3 a_1) x^4 + \cdots$$

$$g(x) - 1 - x = x^2 (2^1 a_1 + (2^1 a_2 + 2^2 a_1) x + (2^1 a_3 + 2^2 a_2 + 2^3 a_1) x + \cdots$$

Inside is the Cauchy Product of

$$(2^1 + 2^2 x + 2^3 x^2 + \cdots) (a_1 + a_2 x + a_3 x^2 + \cdots)$$

$$= 2 (1 + 2x + 2^2 x^2 + \cdots) \frac{1}{x} (a_1 x + a_2 x^2 + a_3 x^3 + \cdots)$$

$$= 2 \cdot \frac{1}{1-2x} \cdot \frac{1}{x} [g(x) - a_0]$$

$$= \frac{2}{1-2x} \cdot \frac{g(x)-1}{x}$$

$$g(x) - 1 - x = x^2 \cdot \left[\frac{2}{1-2x} \cdot \frac{g(x)-1}{x} \right]$$

$$g(x) - 1 - x = \frac{2x}{1-2x} (g(x) - 1)$$

3

a. $$a_n = \sum_{i=0}^{n-1} a_i \, a_{n-1-i} \qquad a_0 = 1$$

$$a_n = a_0 a_{n-1} + a_1 a_{n-2} + \cdots + a_{n-1} a_0$$

$$g(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \left(\sum_{i=0}^{n-1} a_i \, a_{n-1-i} \right) x^n$$

$$g(x) - 1 = x \cdot \sum_{n=1}^{\infty} \left(\sum_{i=0}^{n-1} a_i \cdot a_{n-1-i} \right) x^{n-1}$$

$$g(x) - 1 = x \, [g(x)]^2$$

6. done in the lecture.
Check your lecture notes.

7. We need three vertices to create a triangle.

$$a_0 = a_1 = a_2 = 0$$

$a_3 = 1$ only one way to get a triangle out of 3 vertices.

Let $A_1, A_2, \ldots, A_n$ be the vertices of an n-gon.
Take out three vertices.

$$\underbrace{A_{i_1} A_{i_2} A_{i_3}}_{3\text{-gon}} \qquad \underbrace{A_{i_4} \cdots A_{i_n}}_{n-3 \text{ gon}}$$

$a_3 \cdot a_{n-3}$ triangles obtained by noncrossing diagonals.

generalize the argument to k-vertices.
Take out k-vertices

$A_{i_1} A_{i_2} \cdots A_{i_k}$ $\qquad$ $A_{i_{k+1}} A_{i_{k+2}} \cdots A_{i_n}$

k-gon $\qquad$ $n-k$ gon

a_k $\qquad$ a_{n-k}

$a_k \cdot a_{n-k}$ triangles obtained by noncrossing diagonals.

$$a_n = a_3 a_{n-3} + a_4 a_{n-4} + \dots + a_k a_{n-k} + \dots + a_{n-3} a_3$$

8. recurrence relation is

$$a_n = 3a_{n-1} - a_{n-3} \quad n \geq 3 \qquad a_1 = 0, \quad a_2 = 0$$

(see exercise 21 page 290)

$$a_3 = 1 = 3a_2 - a_0$$
$$\Rightarrow a_0 = -1$$

$$g(x) - a_0 - a_1 x - a_2 x^2 = \sum_{n=3}^{\infty} a_n x^n = \sum_{n=3}^{\infty} (3a_{n-1} - a_{n-3}) x^n$$

$$g(x) + 1 = 3\sum_{n=3}^{\infty} a_{n-1} x^n - \sum_{n=3}^{\infty} a_{n-3} x^n$$

$$g(x) + 1 = 3x \cdot \sum_{n=3}^{\infty} a_{n-1} x^{n-1} - x^3 \sum_{n=3}^{\infty} a_{n-3} x^{n-3}$$

$$g(x) + 1 = 3x\,(g(x)+1) - x^3 g(x)$$

$$g(x) + x^3 g(x) - 3x\,g(x) = 3x - 1$$

$$g(x) = \frac{3x-1}{1+x^3-3x}$$

15. n- distinct balls
a. and n- indistinguishable boxes.

a_n = number of ways to partition n distinct objects among n- indistinguishable boxes.

Take a box, put one ball in it.
$\binom{n}{1}$ ways to put the ball in it.
n-1 balls left, n-1 indistinguishable boxes
$\binom{n}{1} a_{n-1}$.

Take $\binom{n}{k}$ k-boxes and put a ball in them.
$\binom{n}{k}$ ways to do this.

n-k distinct balls
and n-k indistinguishable boxes.

$\binom{n}{k} a_{n-k}$

k-balls out of n. number of ways to partition n-k distinct objects into n-k indistinguishable boxes.

$$a_n = \binom{n}{1} a_{n-1} + \binom{n}{2} a_{n-2} + \cdots + \binom{n}{n-1} a_1$$

$$a_n = \sum_{k=1}^{n-1} \binom{n}{k} a_{n-k}$$

11. $a_{n,k}$ = probability that k successes occur in n-trials.

each trial has probability p of a success. $q = 1-p$

if the first trial is a failure then we need k successes in n-1 trials.

$q\, a_{n-1,k}$

if the first trial is a success then we need k-1 successes in n-1 trials.

$p\, a_{n-1,k-1}$

$$a_{n,k} = p\, a_{n-1,k-1} + q\, a_{n-1,k} \qquad a_{n,0} = 0$$

$$F_n = \sum_{k=0}^{n} a_{n,k} x^k$$

$$F_n - a_{n,0} = \sum_{k=1}^{n} a_{n,k} x^k = \sum_{k=1}^{n} p\, a_{n-1,k-1} x^k + \sum_{k=1}^{n} q\, a_{n-1,k} x^k$$

$$= px \cdot \sum_{k=1}^{n} a_{n-1,k-1} x^{k-1} + q \cdot \sum_{k=1}^{n} a_{n-1,k} x^k$$

$$= px \cdot F_{n-1} + q \cdot F_{n-1}$$

$F_n = (px+q) F_{n-1}$, solve it like $F_n = c F_{n-1}$

$F_n = (px+q)^n$.