We could repeat the same argument for b_n. If the first digit is a 0 or 1, then we are left with an $n-1$ digit quaternary sequence requiring an odd total number of 0's and 1's. In each case we get b_{n-1} for a total of $2b_{n-1}$.

If the first digit is 0 or 1, then we are left an $n-1$ digit quaternory sequence requiring an even total number of 0's and 1's. In each case we get a_{n-1} for a total of $2a_{n-1}$. Hence $b_n = 2b_{n-1} + 2a_{n-1}$.

c. Let $a_n =$ the number of quaternary sequences with an even number of 0's and an even number of 1

If the first digit is a 2 or 3, then we are left with an $n-1$ digit quaternary sequence requiring an even number of 0's and an even number of 1's. In each case we get a_{n-1} for a total of $2a_{n-1}$.

Let $b_n =$ the number of quaternory sequences with an odd number of 0's and an even number of 1's.
Let $c_n =$ the number of quaternary sequences with an even number of 0's and an odd number of 1's.

If the first digit is 0, then we are left with an $n-1$ digit quaternary sequence requiring an odd number of 0's and an even number of 1's. These are b_{n-1} of these.
If the first digit is 1, then we are left with an $n-1$ digit quaternary sequence requiring an odd number of 1's and an even number of 0's. There are c_{n-1} of the

Hence $a_n = 2a_{n-1} + b_{n-1} + c_{n-1}$

We need to develop a system of recurrence relation for b_n.

If the first digit is a 2 or 3, then we are left with an $n-1$ digit quaternary sequence requiring an odd number of 0's and an even number of 1's. In each case we get b_{n-1} for a total of $2b_{n-1}$.
If the first digit is a 0, then we are left with an $n-1$ digit quaternary sequence requiring an even number of 0's and an even number of 1's. There are a_{n-1} of these.
If the first digit is a 1, then we are left with an $n-1$ digit quaternary sequence with an odd number of 1's and an odd number of 0's. We don't have anything for this type so we will consider subcases via the second digit.

If the second digit is 0, then we are left with an $n-2$ digit sequence requiring an even number of 0's and an odd number of 1's. There are c_{n-2} of these.
If the second digit is a 1, then we are left with an $n-2$ digit sequence with an even number of 1's and and an odd number of 0's. There are b_{n-2} of these.

Hence $b_n = 2b_{n-1} + a_{n-1} + c_{n-2} + b_{n-2}$

We need to develop a system of recurrence relations for c_n.

If the first digit is a 2 or 3, then we are left with an $n-1$ digit quaternary sequence requiring an even number of 0's and an odd number of 1's. In each case we get c_{n-1} for a total of $2c_{n-1}$.

If the first digit is a 1, then we are left with an $n-1$ digit quaternary sequence requiring an even number of 1's and an even number of 0's. There are a_{n-1} of these.
If the first digit is a 0, then we are left with an $n-1$ digit quaternary sequence requiring an odd number of 0's and an odd number of 1's. We don't have anything for this type so we will consider subcases via the second digit.
If the second digit is a 0, then we are left with an $n-2$ digit quaternary sequence requiring an even number of 0's and an odd number of 1's. There are c_{n-2} of these.
If the second digit is a 1, then we are left with an $n-2$ digit quaternary sequence requiring an odd number of 0's and an even number of 1's. There are b_{n-2} of these.

Hence $c_n = 2c_{n-1} + a_{n-1} + c_{n-2} + b_{n-2}$

41 b) Show that $\sum_{k=0}^{n/2} f(n,k) = F_{n+1}$

where F_{n+1} is the $n+1$ st Fibonocci number.
Since n is even, let $n = 2\ell$.

We will show that $\sum_{k=0}^{\ell} f(2\ell, k) = F_{2\ell+1}$

We need a result concerning $F_{2\ell+1}$.

$$F_{2\ell+1} = 2F_{2\ell-1} + F_{2\ell-3} + F_{2\ell-5} + \cdots + F_1 + F_0$$

Proof by Induction (Strong)

base case: $\ell = 1$

$$LHS = \sum_{k=0}^{1} f(2,k) = f(2,0) + f(2,1) = 1 + 2 = 3$$

$$RHS = F_3 = F_2 + F_1 = 3$$

Suppose that $\ell = 0, 1, 2, \ldots, n$

$$\sum_{k=0}^{n} f(2n,k) = F_{2n+1}$$

We want to prove that $\sum_{k=0}^{n+1} f(2n+2,k) = F_{2n+3}$.

$$\sum_{k=0}^{n+1} f(2n+2,k) = \sum_{k=0}^{n+1} f(2n+1,k) + \sum_{k=0}^{n+1} f(2n,k-1)$$

$$= \sum_{k=0}^{n+1} f(2n+1,k) + \sum_{k=1}^{n+1} f(2n,k-1)$$

Since $f(2n+1, n+1) = 0$

$$= \sum_{k=0}^{n} f(2n+1,k) + \sum_{j=0}^{n} f(2n,j)$$

$$= \sum_{k=0}^{n} f(2n+1,k) + F_{2n+1}$$

$$= \sum_{k=0}^{n} f(2n,k) + \sum_{k=0}^{n} f(2n-1,k-1) + F_{2n+1}$$

using I.H.

$$= 2F_{2n+1} + \sum_{k=1}^{n} f(2n-1,k-1)$$

$$= 2F_{2n+1} + \sum_{k=1}^{n} f(2n-2,k-1) + \sum_{k=2}^{n} f(2n-3,k-2)$$

using I.H.

$$= 2F_{2n+1} + F_{2n-1} + \sum_{k=2}^{n} f(2n-3,k-2)$$

$$= 2F_{2n+1} + F_{2n-1} + \sum_{k=2}^{n} f(2n-4,k-2) + \sum_{k=3}^{n} f(2n-5,k-3)$$

$$= 2F_{2n+1} + F_{2n-1} + F_{2n-3} + \sum_{k=3}^{n} f(2n-5,k-3)$$

$$= 2F_{2n+1} + F_{2n-1} + F_{2n-3} + \sum_{k=3}^{n} f(2n-6,k-3) + \sum_{k=4}^{n} f(2n-7,k-4)$$

using I.H.

$$= 2F_{2n+1} + F_{2n-1} + F_{2n-3} + F_{2n-5} + \sum_{k=4}^{n} f(2n-7,k-4)$$

Continuing this process, we will get to

$$2F_{2n+1} + F_{2n-1} + F_{2n-3} + F_{2n-5} + \cdots + F_1 + F_0 .$$

Reason:

$$\sum_{k=j}^{n} f(2n-(2j-1),k-j)$$

$$= \sum_{k=j}^{n} f(2n-2j,k-j) + \sum_{k=j}^{n} f(2n-2j-1,k-j-1)$$

$$= F_{2n-2j+1} + \sum_{k=j}^{n} f(2n-2j-1,k-j-1) .$$

Section 7.3 Exercises

3

a. $a_n = 3a_{n-1} + 4a_{n-2}$ $\qquad a_0 = a_1 = 1$

$a_n = \alpha^n$

$\alpha^n = 3\alpha^{n-1} + 4\alpha^{n-2}$

$\alpha^2 = 3\alpha + 4$

$\alpha^2 - 3\alpha - 4 = 0$

$(\alpha - 4)(\alpha + 1) = 0$

$\alpha = 4, \quad \alpha = -1$

general solution $\quad a_n = A_1(4)^n + A_2(-1)^n$

$a_0 = A_1 + A_2 = 1$

$a_1 = 4A_1 - A_2 = 1$

$A_1 = 2/5 \qquad A_2 = 3/5$

$a_n = 2/5\,(4)^n + 3/5\,(-1)^n$

b. $a_n = a_{n-2}$ $\qquad a_0 = a_1 = 1$

$a_n = \alpha^n$

$\alpha^n = \alpha^{n-2}$

$\alpha^2 = 1$

$\alpha = \pm 1$

general solution $\quad a_n = A_1(1)^n + A_2(-1)^n = A_1 + A_2(-1)^n$

$a_0 = A_1 + A_2 = 1$

$a_1 = A_1 - A_2 = 1$

$A_1 = 1 \qquad A_2 = 0$

$a_n = 1(1)^n = \underline{1}$

c. $a_n = 2a_{n-1} - a_{n-2}$ $\qquad a_0 = a_1 = 2$

$a_n = \alpha^n$

$\alpha^n = 2\alpha^{n-1} - \alpha^{n-2}$

$\alpha^2 = 2\alpha - 1$

$\alpha^2 - 2\alpha + 1 = 0$

$(\alpha - 1)^2 = 0$

$\alpha = 1, 1$ double root

general solution $\quad a_n = A_1(1)^n + A_2 n(1)^n$

$= A_1 + nA_2$

$a_0 = A_1 + 0 = 2$

$a_1 = 2 + A_2 = 2 \qquad \therefore \quad a_n = 2$

$A_2 = 0$

(20)

d $\quad a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3} \quad a_0 = a_1 = 1$

$a_2 = 2$

$a_n = \alpha^n$

$$\alpha^n = 3\alpha^{n-1} - 3\alpha^{n-2} + \alpha^{n-3}$$

$$\alpha^3 = 3\alpha^2 - 3\alpha + 1$$

$$\alpha^3 - 3\alpha^2 + 3\alpha - 1 = 0$$

$$(\alpha-1)^3 = 0$$

$\alpha = 1, 1, 1$ repeated roots.

general solution

$$a_n = A_1(1)^n + A_2 n(1)^n + A_3 n^2(1)^n$$

$$= A_1 + nA_2 + n^2 A_3$$

$$a_0 = A_1 = 1$$

$$a_1 = 1 + A_2 + A_3 = 1$$

$$A_2 + A_3 = 0$$

$$a_2 = 1 + 2A_2 + 4A_3 = 2$$

$$4A_3 + 2A_2 = 1$$

$$A_2 + A_3 = 0$$

$$A_3 = \tfrac{1}{2} \quad A_2 = -\tfrac{1}{2}$$

$$a_n = \tfrac{1}{2}n^2 - \tfrac{1}{2}n + 1$$

7.

$$P_n - P_{n-1} = 2(P_{n-1} - P_{n-2})$$

this years change

$P_n = \alpha^n$

$$\alpha^n - \alpha^{n-1} = 2(\alpha^{n-1} - \alpha^{n-2})$$

$$\alpha^2 - \alpha = 2\alpha - 2$$

$$\alpha^2 - 3\alpha + 2 = 0$$

$$(\alpha-2)(\alpha-1) = 0$$

$\alpha = 2$ or $\alpha = 1$

general solution $\quad P_n = A_1(2)^n + A_2(1)^n$

$$P_0 = A_1 + A_2 = 1$$

$$P_1 = 2A_1 + A_2 = 4$$

$$A_1 = 3$$

$$A_2 = -2$$

$$P_n = 3(2)^n - 2(1)^n$$

$$= 3 \cdot 2^n - 2$$

10. Fibonacci Relation

$$F_n = F_{n-1} + F_{n-2} \qquad F_0 = F_1 = 1$$

$$F_n = \alpha^n$$

$$\alpha^n = \alpha^{n-1} + \alpha^{n-2}$$

$$\alpha^2 = \alpha + 1$$

$$\alpha^2 - \alpha - 1 = 0$$

$$\alpha = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$$

general solution

$$F_n = A_1\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)^n + A_2\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)^n$$

$$F_0 = F_1 = 1 \Rightarrow A_1 = \frac{1}{\sqrt{5}}\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)$$

$$A_2 = \frac{1}{\sqrt{5}}\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)$$

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)^n + \frac{1}{\sqrt{5}}\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)^n$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)^{n+1} + \frac{1}{\sqrt{5}}\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)^{n+1}$$

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{\frac{1}{\sqrt{5}}\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)^{n+2} + \frac{1}{\sqrt{5}}\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)^{n+2}}{\frac{1}{\sqrt{5}}\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)^{n+1} + \frac{1}{\sqrt{5}}\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)^{n+1}}$$

$$\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right) < 0$$

$$= \lim_{n\to\infty} \frac{\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right) + \frac{\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)^{n+2}}{\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)^{n+1}}}{1 + \frac{\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)^{n+1}}{\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)^{n+1}}}$$

$$= \frac{1}{2} + \frac{\sqrt{5}}{2}$$

Section 6.4

22. $P(x) = \sum_{k=0}^{\infty} p_k x^k$

a.

$m_k = \sum_{j=0}^{\infty} j^k p_j$

exponential generating function, for m_k

$= \sum_{k=0}^{\infty} m_k \frac{x^k}{k!} = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\infty} j^k p_j\right) \frac{x^k}{k!}$

$= p_1 + p_2 + p_3 + \cdots + p_j + \cdots$

$+ \left(p_1 + p_2 2 + p_3(3) + \cdots + p_j j + \cdots\right) \frac{x^1}{1!}$

$+ \left(p_1 + p_2 2^2 + p_3(3)^2 + \cdots + p_j j^2 + \cdots\right) \frac{x^2}{2!}$

$\vdots$

$+ \left(p_1 + p_2 2^k + p_3(3)^k + \cdots + p_j j^k + \cdots\right) \frac{x^k}{k!}$

$+$

$\vdots$

change variables. fix j and let k run.

$p_j + p_j j x + p_j j^2 \frac{x^2}{2!} + \cdots + p_j j^k \frac{x^k}{k!} + \cdots$

$= p_j \left(1 + jx + \frac{j^2 x^2}{2!} + \cdots + \frac{j^k x^k}{k!} + \cdots\right)$

$= p_j e^{jx}$

j goes from 0 to ∞

$\sum_{j=0}^{\infty} p_j e^{jx} = P(e^x)$

6. Kth factorial moment

$$m_k^* = \sum_{j=k}^{\infty} \frac{j!}{(j-k)!} p_j$$

exponential generating function for m_k^*.

$$\sum_{k=0}^{\infty} m_k^* \frac{x^k}{k!} = \sum_{k=0}^{\infty} \left(\sum_{j=k}^{\infty} p_j \frac{j!}{(j-k)!} \right) \frac{x^k}{k!}$$

$$= \sum_{k=0}^{\infty} \left(\sum_{j=k}^{\infty} p_j \binom{j}{k} \right) x^k$$

$$p_0 + p_1 + p_2 + \cdots + p_j + \cdots$$

$$+ \quad p_1\binom{1}{1}x + p_2\binom{2}{1}x + \cdots + p_j\binom{j}{1}x + \cdots$$

$$+ \quad p_2\binom{2}{2}x^2 + \cdots + p_j\binom{j}{2}x^2 + \cdots$$

$\vdots$

$$+ \quad p_j\binom{j}{j}x^j + \cdots$$

$\vdots$

fix a j, $\quad$ j runs from 0 to ∞

$$p_j + p_j\binom{j}{1}x + p_j\binom{j}{2}x^2 + \cdots + p_j\binom{j}{j}x^j$$

$$= p_j\left(1 + \binom{j}{1}x + \binom{j}{2}x^2 + \cdots + \binom{j}{j}x^j\right)$$

$$= p_j\,(1+x)^j$$

$$\sum_{j=0}^{\infty} p_j (1+x)^j = P(x+1)$$

c. X number of heads when n coins are flipped $P(X=k) = p_k = \dfrac{\binom{n}{k}}{2^n}$

$$g(x) = \sum_{k=0}^{\infty} \frac{\binom{n}{k}}{2^n} e^{kx}$$

$$m_1 = g'(0) \qquad g'(x) = \sum_{k=0}^{\infty} \frac{\binom{n}{k}}{(2^n)} k \cdot e^{kx}$$

$$m_1 = g'(0) = \frac{1}{2^n}\sum_{k=0}^{\infty} k\binom{n}{k} = \frac{1}{2^n} \cdot \sum_{k=0}^{n} k\binom{n}{k}$$

$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k \qquad = \frac{1}{2^n} \cdot n2^{n-1} = \frac{n}{2}$$

$$n(1+X)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k} X^{k-1}$$

$$n(2)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k}$$

$$m_2 = g''(0) \qquad g''(x) = \sum_{k=0}^{\infty} \frac{\binom{n}{k}}{2^n} k^2 e^{kx}$$

$$m_2 = g''(0) = \sum_{k=0}^{\infty} \frac{\binom{n}{k}}{2^n} k^2 = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} k^2$$

$$n(n-1)(1+x)^{n-2} = \sum_{k=0}^{n} k(k-1)\binom{n}{k} x^{k-2}$$

$$n(n-1)2^{n-2} = \sum_{k=0}^{n} \binom{n}{k}(k^2-k) = \sum_{k=0}^{n}\binom{n}{k}k^2 - \sum_{k=0}^{n} k\binom{n}{k}$$

$$n(n-1)2^{n-2} = \sum_{k=0}^{n}\binom{n}{k}k^2 - n\cdot 2^{n-1}$$

$$n(n-1)2^{n-2} + n2^{n-1} = \sum_{k=0}^{n}\binom{n}{k}k^2$$

$$n2^{n-2}(n-1+2) = \sum_{k=0}^{n}\binom{n}{k}k^2$$

$$n(n+1)2^{n-2} = \sum_{k=0}^{n}\binom{n}{k}k^2$$

$$= \frac{1}{2^n} \cdot n(n+1)2^{n-2}$$

$$= \frac{n(n+1)}{4}$$

$$g(x) = \sum_{k=0}^{\infty} \frac{\binom{n}{k}}{2^n} (x+1)^k$$

$m_2^* = g''(0)$

$$g''(x) = \sum_{k=0}^{\infty} \frac{\binom{n}{k}}{2^n} k(k-1)(x+1)^{k-2}$$

$$g''(x) = \frac{1}{2^n} \sum_{k=0}^{\infty} k(k-1)\binom{n}{k}$$

$$= \frac{1}{2^n} \sum_{k=0}^{\infty} k(k-1)\binom{n}{k} = \frac{1}{2^n} n(n-1)2^{n-2}$$

$$= \frac{n(n-1)}{4}$$

d. X is poisson

$$P(X=k) = P_k = \frac{u^k}{k!} e^{-u}$$

$m_1 = g'(0)$ where $g(x) = \sum_{k=0}^{\infty} \frac{u^k}{k!} e^{-u} e^{kx}$

$$g'(x) = \sum_{k=0}^{\infty} \frac{u^k}{k!} e^{-u} \cdot k e^{kx}$$

$$g'(0) = \sum_{k=0}^{\infty} \frac{u^k}{k!} e^{-u} k$$

$$= e^{-u} \cdot \sum_{k=1}^{\infty} \frac{u^k}{(k-1)!} = e^{-u} \cdot u \sum_{k=1} \frac{u^{k-1}}{(k-1)!}$$

$$= e^{-u} \cdot u \cdot e^{u} = u$$

$m_1^* = g'(0)$ where

$$g(x) = \sum_{k=0}^{\infty} \frac{u^k}{k!} e^{-u} (x+1)^k$$

$$g'(x) = \sum_{k=0}^{\infty} \frac{u^k}{k!} e^{-u} \cdot k(x+1)^{k-1}$$

$$m_1^* = g'(0) = \sum_{k=0}^{\infty} \frac{u^k}{k!} e^{-u} k = e^{-u} \sum_{k=0}^{\infty} \frac{u^k}{(k-1)!}$$

$$= u e^{-u} \sum_{k=0}^{\infty} \frac{u^{k-1}}{(k-1)!}$$

$$= u e^{-u} \cdot e^{u} = u$$

Section 6.5 # 8

$$h(x) = \sum_{r=0}^{\infty} a_r x^r \qquad S_r = \sum_{k=r+1}^{\infty} a_k$$

generating function for S_r,

$$\sum_{r=0}^{\infty} S_r x^r = \sum_{r=0}^{\infty} \sum_{k=r+1}^{\infty} a_k$$

$$\begin{aligned}
= \; & a_1 + a_2 + a_3 + \dots + a_j + \dots \\
+ \; & a_2 x + a_3 x + \dots + a_j x + \dots \\
+ \; & a_3 x^2 + \dots + a_j x^2 + \dots \\
& \vdots \\
+ \; & a_j x^{j-1} + \dots
\end{aligned}$$

reverse the order of summation.

$$\sum_{j=1}^{\infty} \sum_{k=0}^{j-1} a_j x^k = \sum_{j=1}^{\infty} a_j \sum_{k=0}^{j-1} x^k$$

$$= \sum_{j=1}^{\infty}\left(a_j \frac{1-x^j}{1-x}\right) = \frac{1}{1-x} \sum_{j=1}^{\infty} a_j (1-x^j)$$

$$= \frac{1}{1-x}\left(\sum_{j=1}^{\infty} a_j - \sum_{j=1}^{\infty} a_j x^j\right)$$

$$= \frac{1}{1-x}\left(\alpha - x \cdot \sum_{j=1}^{\infty} a_j x^{j-1}\right)$$

$$= \frac{1}{1-x}\left(\alpha - x h(x)\right)$$

$$= \frac{x h(x) - \alpha}{x-1}$$

since α is a constant generating function $\dfrac{x h(x)}{x-1}$